MATHEMATICAL DICTIONARY

AND

CYCLOPEDIA

OF

MATHEMATICAL SCIENCE.

COMPRISING

DEFINITIONS OF ALL THE TERMS EMPLOYED IN MATHEMATICS—AN ANALYSIS OF EACH BRANCH, AND OF THE WHOLE, AS FORMING A SINGLE SCIENCE.

BY

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PREFACE.

THE SCIENCE OF MATHEMATICS treats of the two abstract quantities, Number and Space. Primarily, it treats of the measurements and relations of these quantities, and of the operations and processes by means of which they are ascertained: and secondarily, of the applications of the principles thus developed to the practical affairs of life.

The quantities operated upon are denoted by figures or letters, and the operations to be performed are indicated by certain characters called Signs. The figures, letters and signs, are called *symbols*, and are elements of the mathematical language.

The language of mathematics is partly technical and partly popular, being made up of symbols which either represent quantity or denote operations, and of words adopted from our common vocabulary. Both branches of this language are undergoing changes corresponding to the progress and development of the science; and hence it is, that new terms become necessary, while the significations of the old ones are modified, either by enlargement or restriction.

It is of the first importance, in prosecuting mathematical inquiries, to acquire an accurate knowledge of the office and power of every symbol, and a clear and distinct apprehension of the signification of every technical term. Most of the difficulty experienced in the study of mathematics, has arisen, we apprehend, from the use of terms in a vague or ambiguous sense; and the discussions on "controverted points," are mainly due to a misuse or misapprehension of the meaning of technical terms.

1. It is a leading object of this work, to define, with precision and accuracy, every term which is used in mathematical science; and to afford, as far as possible, a definite, perspicuous and uniform language.

2. A second object is, to present in a popular and condensed form, a separate and yet connected view of all the branches of Mathematical Science. Hence, the work has been called—"A DICTIONARY AND CYCLOPEDIA OF MATHEMATICAL SCIENCE."

3. The work has also been prepared to meet the wants of the general reader, who will find in it all that he needs on the subject of mathematics. He can learn from it the signification and use of every technical term, and can trace such term, in all its connections, through the entire science. He will find each subject as fully treated as the limits of the work will permit, and the relations of all the parts to each other carefully pointed out.

4. The practical man will find it a useful compendium and hand-book of reference. All the formulas and practical rules have been collected and arranged under their appropriate heads.

5. The chief design of the work, however, is to aid the teacher and student of mathematical science, by furnishing full and accurate definitions of all the terms, a popular treatise on each branch, and a general view of the whole subject.

In pursuing a course of mathematics, arranged in a series of Text-Books, it is often difficult, if not impossible, to understand a single branch fully until its connections with other branches shall have been traced out. The various branches of mathematics, though apparently differing widely from each other, are, nevertheless, pervaded by common principles and connected by common laws. In bringing all these branches within the compass of a single volume, an opportunity has been afforded of examining their common principles and pointing out the connections of their several parts. Hence, the Dictionary affords to the diligent and intelligent student, the means of understanding the connections of the different subjects of the mathematical science; and to such, we are confident, it will prove an efficient auxiliary in removing the obstacles which have rendered the acquisition of mathematical science a difficult and forbidding task.

The diffusion of knowledge and the employment of mathematics in the investigations of the Natural Sciences, as well as in all practical matters, have given great value to mathematical acquirement, if they have not rendered a certain amount of it absolutely necessary; hence, it would seem desirable to afford every facility for the prosecution of so useful a study.

As many of the subjects treated in this work have common parts, it became necessary either to interrupt the processes of investigation by references, or to use, occasionally, the same matter in different places. As the entire work is rather a collection of separate treatises than a single treatise on a single subject, the latter method has occasionally been adopted, though the other has been generally used.

It will not be a matter of surprise, that a work of so much labor should have been a joint production. In its prosecution, many questions have arisen in regard to definitions, methods of discussion, classification and arrangement. In deciding these points we have been guided, uniformly, by the best standards. When differences were irreconcilable we have looked to the authority of general principles.

Fishkill Landing,
June, 1855.

MATHEMATICAL DICTIONARY

AND

CYCLOPEDIA OF MATHEMATICAL SCIENCE.

A. The first letter of the English alphabet. Among the ancients it was used as a numeral denoting 500, or with a dash over it, thus, $\bar{A}$, it stood for 500,000. In Greek, Hebrew, and Arabic, it stood for 1.

In *Algebra*, it is employed to denote a known or given quantity—In *Geometry* and *Trigonometry* it often stands for an angle—In *Surveying* it is used as an abbreviation for *acre*—In *Commerce* it stands for accepted, as in the case of a bill of exchange.

AB'A-CIST, [from abacus]. One who makes arithmetical computations, a computor or calculator.

AB'A-CUS. [L. *abacus*, anything flat. Gr. *αβαξ*, a slab for reckoning on]. In architecture, a table constituting the crowning member of a column and its capital.

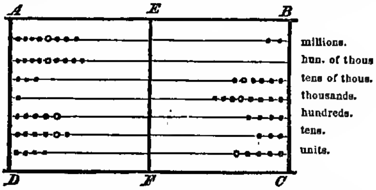
The name abacus was given to an instrument formerly used to facilitate arithmetical computations, and still retained in one of its modifications, for imparting to children a knowledge of the elements of arithmetic.

The most ancient abacus consisted of a table, surrounded by a raised border or ledge, and covered with sand, upon which diagrams were drawn, and computations made.

In later times the sanded table was replaced by an entirely different instrument, called also an abacus, and which with little change of principle, continued to be used for many centuries, by the most enlightened nations of the world.

The principle of this kind of abacus will be readily understood by an examination of the diagram.

ABCD is a wooden frame, supposed to be vertical, supporting several horizontal wires, 1, 2, 3, 4, &c.; each wire bearing nine beads of glass, wood, or ivory, which slide freely.



The vertical bar EF divides the rectangle AC into two separate compartments, but is placed far enough in front of the wires to allow the beads or counters to slide freely behind it.

We have supposed the instrument arranged according to the decimal scale, so that each bead on the lower wire denotes a simple *unit*, or unit of the first order; a bead upon the second wire, a unit of the second order, or *one ten;* one upon the third wire, a unit of the third order, or *one hundred;* &c.

Sometimes, for the purpose of diminishing the number of counters, intermediate wires are introduced; a bead upon one of them denoting five beads upon the wire next below We shall now explain the mode of recording a number by means of this abacus. The beads are all pushed along the wires into the compartment EC before the record is commenced. Let the number in question be 7.931,564 First slide four beads along the lower wire into the compartment AF, these will denote the four units; then slide six beads along the

second wire to denote the six tens, and so on, moving into the space AF a sufficient number of counters on each wire to denote the number of units of the corresponding order; when the operation is complete, the given number will be represented as in the diagram.

We have supposed the value of the unit, in passing from wire to wire, to increase according to the decimal scale; but the instrument is equally applicable when the value of the unit increases according to any scale, either uniform or varying. If the duodecimal scale be adopted, there will be required eleven counters on each wire; if any varying scale is used, the number of counters on any wire must be at least equal to the number of units of that order contained in a unit of the next superior order, diminished by 1. The method of recording a number constructed according to any scale, is entirely similar to that already explained.

During the middle ages, the abacus was used by bankers, money-changers, &c., but instead of a frame with wires and beads, they made use of a bench or *bank* covered with black cloth, divided into checks by white lines at right angles to each other. Counters placed upon the lower bar denoted pence, those on the second bar shillings, those on the third, pounds, and those on the fourth, fifth, sixth, &c., bars, denoted tens, hundreds, thousands, &c., of pounds.

Abacus Pythagoricus, or Pythagorean abacus. A table computed for the purpose of facilitating numerical calculations. It is nothing more than the multiplication table as given in ordinary treatises on arithmetic.

Abacus Logisticus, or sexagesimal canon. The same as the Pythagorean abacus or multiplication table, carried to 60, both ways.

AB-BRE'VI-ĀTE. [L. *abbrevio*, to shorten]. To reduce to a small compass, to epitomize.

AB-BRE-VI-Ā'TION, the operation of abbreviating or shortening. The abbreviation of a fraction is the operation of reducing it to lower terms; thus, if both numerator and denominator of the fraction $\frac{9}{27}$ be divided by 9, the fraction is said to be *abbreviated*, or reduced to $\frac{1}{3}$. In Algebra, an expression is said to be abbreviated when it is shortened by any algebraic process: thus, the expression

$$4a^2b^3 - 2a^3b^2,$$

may be abbreviated to the expression,

$$2a^2b^2(2b - a),$$

by simply factoring it.

An *abbreviation* is a single letter, or a simple combination of letters, standing for a word or sentence: thus, *A.* stands for *acre*, *hhd.* for hogshead, *lb.* for pound, &c.

A-BRIDGE', [Fr. *abréger*, to shorten. Gr. *βραχυς*, short]. To shorten, to contract. The abridgment of an expression in Algebra, is the operation of shortening it by substitution: thus, every equation of the second degree containing but one unknown quantity, is a particular case of the general form

$$ax^2 + bx + c = 0;$$

or, dividing both members by a,

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0;$$

which may be abridged by substituting $2p$ for $\frac{b}{a}$ and q for $\frac{c}{a}$, giving the equation,

$$x^2 + 2px + q = 0.$$

The last equation is not only easier to remember, but is also under a simpler form for discussion.

The operation of abridging may generally be resorted to with advantage, whenever complicated expressions enter into long computations. After completing the computations, we can, if necessary, substitute for the symbols introduced, their values in terms of the original quantities employed.

AB-RUPT' POINT of a curve. A point at which a branch of a curve terminates: thus, the curve whose equation is

$$y - b = (x - a)\,l\,(x - a)$$

has an abrupt point for $x = a$. See *Singular point.*

AB-SCIS'SA. [L. *abscissus*, *ab*, from, and *scindo*, to cut]. One of the elements of reference by means of which a point is referred to a system of rectilineal co-ordinate axes. If we draw two straight lines, in a plane, intersecting each other, one of them being horizontal, it has been agreed to call the horizontal one the axis of X, or the axis of abscissas, and the other one the axis of Y, or the axis of ordinates.

In such a system, the abscissa of a point is the distance cut off from the axis of X by a line drawn through it, and parallel to the axis of Y. All abscissas measured to the right, are, by convention, regarded as positive, and consequently, all at the left must be considered negative. The abscissas of all points situated on the axis of Y, are 0. In space, the term abscissa is applied in a more general sense, and may mean a distance measured parallel to either of the horizontal axes, the distance measured on a parallel to the axis of Z being always called the ordinate. It is customary to define the abscissa of a point in space, to be the distance of the point from the co-ordinate plane YZ, measured on a line parallel to the axis of X. The rule for signs is analogous to that employed in a plane system; all distances measured towards the right are considered positive, those to the left must be negative; the abscissas of points in the plane YZ are 0. When the term abscissa is applied to distances measured from the plane XZ and parallel to the axis of Y, they are considered positive when measured in front of the plane, and negative when measured behind it. When the point lies in the plane XZ, the abscissa is 0.

AB'SO-LUTE, [L. *absolutus*, *ab*, from, and *solvo*, to loose or release]. Complete in itself, independent. The absolute term of an equation, is that term which is known, or which does not contain the unknown quantity: thus, in the equation

$$ax^3 + bx^2 + cx + d = 0,$$

d is the absolute term. If we regard every term as involving the unknown quantity in some form, then the absolute term is that in which the exponent of this quantity is 0. In every entire equation, the absolute term is equal to the continued product of all the roots of the equation with their signs changed. Hence, if the absolute term of an equation is 0, one or more of the roots of the equation must be equal to 0.

In Analytical Geometry, the equations employed are indeterminate; that is, they involve more unknown quantities than there are equations, and the absolute term is that one which is independent of all the unknown quantities or variables. It may be demonstrated that when the absolute term is 0, the geometrical magnitude represented by the equation passes through the origin of co-ordinates. See *Analytical Geometry*.

ABSOLUTE SPACE, is space considered without reference to material objects. or limits.

AB'STRACT. [L. *abstractus*, to draw from or separate; from *abs*, from, and *traho*, to draw]. Separate, distinct from something else.

ABSTRACT EQUATION, is an equation expressing a relation between abstract quantities only, as,

$$3x^2 + 4x - 5 = 0.$$

ABSTRACT QUANTITY, is one which does not involve the idea of matter, but simply that of a mental conception; it is expressed by a letter, symbol, or figures: thus, the number *three* represents an abstract idea, that is, one which has no connection with material things, whilst *three feet*, presents to the mind an idea of a physical unit of measure, called a *foot*. So a "portion of space bounded by a surface, every point of which is equally distant from a point within, called the centre," is a mere conception of form. When we call it a sphere, we employ the term to express our idea of the abstract magnitude.

All numbers are abstract when the unit is abstract. Arithmetic, which treats of the relations and properties of such numbers, is *abstract arithmetic*. This embraces the whole *science* and *theory* of arithmetic: Concrete or Denominate Arithmetic being nothing more than the art of applying the principles developed in Abstract Arithmetic to Denominate Numbers.

Since Algebra differs little from ordinary Arithmetic, except in the nature of the language employed, we must regard the Science of Algebra as purely abstract. In Geometry also, the magnitudes considered, viz., lines surfaces and solids, are mere mental conceptions of extent and form, which are represented by geometrical figures. The discussion of these magnitudes in the development of their relations and properties is, therefore, necessarily confined to abstract quantities. We may, therefore, regard Geometry as an Abstract Science. And generally, all the *principles* of Mathematical Science are developed from a consideration of abstract quantities only.

What is usually termed Abstract Mathe

matics, or pure mathematics, involves the entire *science*, whilst that which is called Concrete, or Mixed Mathematics, is nothing more than the *art* of applying previously developed principles to physical objects, as suggested by the demands of society.

AB-SURD′. [L. *absurdus*, *ab*, from, and *surdus*, deaf, insensible]. A proposition is absurd, when it is opposed to a known truth. The term absurd, is used in connection with a kind of demonstration called the "*rcductio ad absurdum.*" In this kind of demonstration, a certain proposition is assumed as true, and is combined with known truths by a course of logical arguments, thus deducing a chain of conclusions until one is arrived at which disagrees with a known truth, when the original supposition or hypothesis is pronounced absurd, and its contrary is considered proved. As an example of this mode of demonstration, we may instance the proposition to show that, "If two straight lines have two points in common, they will coincide throughout." In this proposition it follows that if they do not coincide, they must enclose a spacc which is manifestly impossible; hence, the proposition that they do not coincide involvcs an absurdity, and the proposition is said to be reduced to an absurdity. This method of proof, though sometimes objected to as unsatisfactory, is, nevertheless, as strictly logical, and as conclusive as any other method. The reasoning is quite as perfect, and the conclusions equally irresistible.

A-BUND′ANT NUM′BER. A number which is less than the sum of all its aliquot parts: thus, 12 is an abundant number, because

$$12 < 1 + 2 + 3 + 4 + 6.$$

An abundant number is distinguished from a *perfcct* number, which is equal to the sum of its aliquot parts, and from a *deficient* number, which is greater than the sum of its aliquot parts.

AC-CI-DENT′AL POINT of a line. In Perspective, the point in which a line drawn through the point of sight, and parallel to the given line, pierces the perspective plane. It is a point of the indefinite perspective of the line. See *Vanishing Point*.

AC-CLIV′I-TY, [L. *acclivus*, from *ad*, to, and *clivus*, an ascent]. In Topography, the steepness or slope of rising ground, in contra-distinction to that of descending ground, which is called *declivity*. Acclivity implies *asccnt*, declivity implies *descent.*

A′CRE [L. *ager*, land. Gr. *αγρος*, a field]. A unit of measure employed in land surveying. In the United States, the standard acre contains 4840 square yards, or 43,560 square feet. In the form of a square, one side would measure about 69.5701 yards, or 208.7103 feet.

The acre contains 160 square rods, or perches. The subdivisions of the acre are roods and perches. The acre containing 4 roods, and the rood 40 perches.

There are 640 acres in a square mile, hence, an acre is the $\frac{1}{640}$th part of a square mile.

The *English statute acre* is the same as that of the United States.

The *Irish acre* contains 1 acre 2 roods $19\frac{21}{121}$ perches English.

The *Scotch acre* contains 1 acre 1 rood $3\frac{87}{121}$ perches English.

The *Welsh acre* contains about 2 English acres.

The *Strasbourg acre* is about one-half of an English acre.

The French acre or *arpent of Paris* contains 4,088 square yards, or nearly $\frac{5}{6}$ of an English acre.

The French *woodland arpent* contains 6,108 square yards, or about 1 acre 1 rood 1 perch English.

In the new decimal system of France, the *Are* contains 119.603 square yards, the *Decare* 1196.03 square yards, and the *Hecatare* 11960.3 square yards.

A-CUTE′, [L. *acutus*, sharp pointed]. Sharp as opposed to obtuse. An *acute angle*, is one that is less than a right angle. In degrees, an acute angle is less than 90°.

ACUTE-ANGLED TRIANGLE, is one that has all of its angles acute.

ACUTE CONE, is one in which the vertical angle of the meridian triangle is less than 90°, or less than a right angle.

ACUTE HYPERBOLA, is one whose asymptotes make an acute angle with each other. In it, the transverse axis is always greater than the conjugate.

ADD, [L. *addo*, from *ad*, to, and *do*, to give]. To unite or put together, so as to form an aggregate of several particulars.

AD-DI'TION, [L. *additio*, from *addo*, to give to]. The operation of finding the simplest equivalent expression for the aggregate of two or more quantities of the same kind. Such expression is called the *sum* of the quantities.

In *arithmetic*, the quantities to be added are always numbers, written either according to the decimal scale, or according to some varying scale. In the first case, the operation of adding is called Addition of Simple Numbers, in the second, Addition of Denominate Numbers. The operation in both cases are identical in principle, and may be described as follows:

Write down the numbers to be added so that units of the same order or denomination shall fall in the same column.

Add together the units of the lowest order, and divide their sum by the number of such units contained in one of the next higher order: set down the remainder, and carry the quotient to the next column. Continue the operation till the column of units of the highest order is reached, and set down the entire sum of that column.

Addition of Decimals. The rule in this case does not differ from that already given. In fact, every number written in the scale of tens is a decimal, whose value depends upon the place of the decimal point. When this point is fixed, the orders are counted from it in both directions. When numbered to the left, they are called orders of entire units; when to the right, they are called orders of decimals.

Addition of Vulgar Fractions. Reduce all the fractions to equivalent ones having a common fractional unit. Add the numerators together, and write their sum over the denominator of the fractional unit: the result will be the sum required.

Proof.—There are several methods to verify the accuracy of the operation of addition.

1. If the columns of units have been added from the bottom upwards, let them be added from the top downwards; the results should be the same.

2. Separate the numbers to be added into two or more groups, and add these groups, each by itself, and take the sum of the results; this sum should be the same as that first obtained. The principle on which this method depends is, *that a whole is equal to the sum of all its parts.*

3. There is a third method of proof, which is only applicable to numbers written in the decimal scale. It is called the method by casting out the 9's. The principle on which this method depends requires some elucidation. Since

$$10 = 9 + 1,\ 100 = 99 + 1,\ 1000 = 999 + 1$$

and so on, it follows that if a number expressed by 1 followed by any number of 0's be divided by 9, the remainder will be 1.

Again,

$$20 = 2(9 + 1),\ 200 = 2(99 + 1),$$
$$2000 = 2(999 + 1), \text{ \&c.};$$

hence, if a number expressed by 2 followed by any number of 0's be divided by 9, the remainder will be 2. Generally, if a number expressed by 3, 4, 5, 6, &c., followed by any number of 0's, be divided by 9, the remainder will be 3, 4, 5, 6, &c.

It is evident that if we divide each of the parts by 9, and then divide the sum of the remainders found, by 9, the final remainder will be the same as that which is found after dividing the entire number by 9.

Any number, as 5634, may be written

$$5000 + 600 + 30 + 4,$$

and from the preceding principle it follows that if any number be divided by 9, the remainder will be the same as that obtained by dividing the sum of its digits by 9

Upon these principles is based the following rule:

Take the sum of the digits in each number to be added, and having divided each sum by 9, set down the remainder in a column at the right. Take the sum of these remainders and divide it by 9, setting the remainder beneath. If this remainder is the same as that found by dividing the sum of the digits in the sum total by 9, the work is probably correct.

EXAMPLE.

	Excess of 9's.
4567	4
3214	1
1187	8
8968	4

The sum of the digits in the first number is 22, and the remainder found, 4. In the second number, the sum of the digits, 10, and the remainder 1. In the third, the sum of the digits is 17, and the remainder, 8. The

sum of these remainders is 13, and the remainder 4, which is also the remainder obtained by dividing 31, the sum of the digits in the sum total, by 9. Hence, we conclude that the operation of addition was correctly performed.

None of these methods of proof are strictly perfect, since it is possible that two errors might be committed which would exactly balance each other; the last one is, however, nearly free from any liability to error.

In *Algebra,* the quantities to be added are represented by symbols arranged according to the rules of algebraic Notation.

ADDITION OF ENTIRE QUANTITIES. Set them down so that similar terms, if there are any, shall fall in the same column. Add the several sets of similar terms, and to the result annex the remaining terms, giving to each its proper sign. To add similar terms, take the numerical sum of the co-efficient of the additive and subtractive terms separately: subtract the less from the greater, and give to the remainder the sign of the greater, after which write the common literal part. This will be the sum required.

ADDITION OF FRACTIONS. The rule is the same as that already given for the addition of arithmetical fractions.

ADDITION OF RADICALS. Reduce them, if possible, to equivalent radicals which shall be similar. Add the co-efficients, and to this sum annex the common radical part. This will be the sum required. If the given radicals cannot be reduced to equivalent similar radicals, the addition can only be indicated.

When the quantities are written by means of exponents, reduce them, if possible, to equivalent expressions having the same exponent. Add the co-efficients for a new co-efficient, after which write the common part. The result will be the sum required.

ADDITION OF RATIOS, is the same as the addition of fractions.

ADD'I-TIVE. A quantity is additive when it is preceded by a positive sign. If it is not preceded by any sign, the sign + is always understood.

AD-FECT'ED. Compounded, that is, made up of terms involving different powers of the unknown quantity; thus,

$$ax^3 + bx^2 + cx + a = 0$$

is an adfected equation, containing terms which involve different powers of x. See *Affected.*

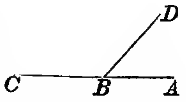
AD IN-FI-NI'TUM. [L.] To endless extent, according to the same law. When a series is given, and a sufficient number of terms are written to indicate the law of the series, the words *ad infinitum* are added to show that there are an infinite number of succeeding terms, connected by the same mathematical law, with those already given.

Ad infinitum sometimes means to the limit. For example, if a regular polygon be inscribed in a circle, and the arcs subtended by the sides be severally bisected, and the points of bisection be joined by chords with the adjacent vertices of the polygon, a new regular polygon will be formed, having double the number of sides, and approaching more nearly to an equality with the circle. If the operation be then repeated, we shall have a polygon still nearer in area to the circle, and so on. If the operation be repeated *ad infinitum* we shall reach the limit, that is, the inscribed polygon will coincide with the circle.

AD-JA'CENT. [From *ad,* to, and *jaceo,* to lie]. Contiguous to, or bordering upon.

ADJACENT ANGLES, in a plane, are those which have one side in common, and their other sides in the prolongation of the same straight line.

Thus, the angles ABD and DBC are adjacent.



Two diedral angles are adjacent when they have a common face, and their other faces lying in the same plane produced.

Two spherical angles are adjacent when they have one side in common, and their other sides arcs of the same great circle.

The sum of the two adjacent angles, in each case, is equal to two right angles.

AD-JUST'MENT. [From *ad,* to, and *just us,* just.] The operation of bringing all the parts of a mathematical instrument into their proper relative positions. When the parts have these positions, the instrument is said to be in adjustment, and is fit for use. When several independent steps have to be taken.

each step is often called an adjustment: thus, in the theodolite we say there are four adjustments.

1 To bring the intersection of the cross hairs into the axis of the Y's.

2. To make the axis of the upper level parallel to this axis.

3. To make the axes of the lower levels perpendicular to the axis of the instrument.

4. To make the axis of the vertical limb perpendicular to the axis of the instrument.

These separate steps, strictly speaking, make up but a single adjustment.

For an account of the method of adjusting particular instruments, see the articles referring to those instruments respectively.

AD-MEAS'URE-MENT — AD-MEN-SU-RĀ'TION. The same as measurement and mensuration, which see.

Ā-Ē'RI-AL, [L. *aërius*, belonging to the air]. Appertaining to the air or atmosphere.

Aerial Perspective. That branch of perspective which relates to the shading of a picture, by weakening the tints in proportion to their distance from the point of light. It treats also of giving the proper colors and shades of colors, so that the picture shall appear in color and tint like the object itself. This branch of Perspective is not properly considered mathematical, except so far as connected with linear perspective.

AF-FECT'ED, [*ad*, to, and *facio*, to make]. More used than *adfected*. It means made up of terms involving different powers of the unknown quantity: thus,

$$x^2 + px = 9,$$

is an affected equation of the second degree. When a quantity is preceded by the sign $+$ or $-$, it is said to be affected with a positive or negative sign. Also, when an exponent or index of a quantity is positive or negative, we say that it is affected with a positive or negative exponent or index.

The term affected, is sometimes even applied to the numerical co-efficients, in which case the literal parts are said to be affected with positive or negative co-efficients. In this last case the term is improperly applied.

AF-FIRM'A-TIVE. [L. from *ad*, to, and *firmo*, to make firm]. In Algebra, an affirmative quantity is one that is to be added, in contradistinction to one which is to be subtracted. The term implies that the quantity is essentially positive, that is, of such a nature that when added to another quantity, the latter will be increased.

Affirmative Sign. The same as the sign of addition or *plus*, denoted thus $+$. When placed before a quantity, it signifies that the quantity is to be considered in a sense directly opposed to what it would have been had it been preceded by the sign *minus*. The two signs are perfectly antagonistic to each other, and every quantity whatever must be affected with one or the other.

It is customary to regard quantity considered in a certain sense as positive, whence it immediately follows, from the nature of the case, that it must be regarded as negative when considered in a contrary sense. For example, if it is agreed to call time to come positive, time past must be represented by a negative expression.

If it is agreed to call distance estimated in one direction positive, then distance estimated in a contrary direction must be negative, and so on. This view of the case disposes of all difficulty in explaining the nature and use of the two symbols $+$ and $-$, about which so much discussion has been had.

AF'FIX. [L. *affigo*, from *ad*, to, and *figo*, to fix]. To unite at the end: thus, to affix 0's to a number, is the same as to annex them, or to write them after it.

A FOR-TI-O'RI. [L.] For a more apparent reason.

AG'GRE-GĀTE. [L. from *ad*, to, and *grex*, a herd or band]. An assemblage of parts to form a whole. An aggregate of several particulars, is equivalent to their sum.

AL'GE-BRĀ. [From the Arabic words *al* and *gabron*, reduction of parts to a whole]. That branch of analysis whose object is to investigate the relations and properties of numbers by means of symbols. The quantities considered are generally represented by letters, and the operations to be performed on these are indicated by signs. The letters and signs are called symbols. Algebra embraces all the operations of Addition, Subtraction, Multiplication, Division, raising to powers denoted by constant exponents.

and extraction of roots indicated by constant indices; it also includes the discussion of the nature and properties of all equations in which the relations between the known and unknown quantities can be expressed by the ordinary operations of Algebra. Such equations are called *algebraic*.

Higher or Transcendental Algebra treats of those quantities which cannot be exactly expressed by a finite number of algebraic terms, and which are therefore called transcendental. It also investigates the nature of transcendental equations, that is, all which are not algebraic. Under this branch of Algebra also falls the treatment of logarithms, formation and laws of series, and all that class of problems which arise in the investigation of Analytical Trigonometric formulas.

These two branches form what may be called the Science of Algebra; besides these, a third might be added, having for its object the practical application of the principles deduced, to the solution of all kinds of problems, whether abstract or concrete, which come within the range of algebraic analysis. It also includes the formation of rules for many of the higher arithmetical operations, as Interest, Annuities, Alligation, &c.

For an account of the several processes of Algebra, the reader is referred to the several articles, Addition, Subtraction, Multiplication, Division, Equations, &c., under their appropriate headings.

The most ancient Treatise extant on the subject of Algebra, is that of Diophantus, who wrote about the year 350. His work consists principally of a collection of solutions of problems relating to properties of numbers, and more particularly to the properties of square and cube numbers, of which some account may be found in the article on Diophantine Analysis. The science was cultivated by the Arabians, and from them a knowledge of its principles was derived by the Italians, about the beginning of the 13th century. Many improvements were introduced, and many new processes discovered by Ferreas, Cardan, Tartalea, and others of the Italian school, amongst the most important of which may be mentioned the method of solving cubic equations.

No great advances, however, were made in systematizing the science till after the introduction of a concise system of notation, the foundation of which was laid by a German named Stifel, or Stifelius, who wrote about the middle of the 16th century.

From this period, improvements, both in the methods of notation, and in the generalization of processes, were rapidly made by such mathematicians as Robert Recorde, Vieta, Albert Girard, Harriot, and many others, by whose labors the science was advanced, as far as its general outline is concerned, to nearly its present condition.

In the year 1637, Descartes published his great work on the application of the principles of Algebraic Analysis to the investigation of geometrical truths, and besides opening an entirely new field of mathematical research, contributed much to the advancement and perfection of pure Algebra. Since his time there has been no great revolution in Algebra, as a science, but it has been vastly improved in its details, and greatly extended in its applications. The theory of Series has been successfully developed by Euler, Wallis, the Bernouillis, Newton, De Moivre, Simpson, and others. The composition of equations has been investigated, and the methods of approximating to their roots systematized and reduced to order.

Amongst the more recent laborers in the field of Algebra, may be mentioned Taylor, M'Laurin, Clairaut, Euler, Legendre, Arbogast, Gauss, Bourdon, and many others.

Perhaps the work containing the most complete exposition of the present state of the science, is the recent edition of L'Algèbre de M. Bourdon.

AL-GE-BRĂ'IC — AL-GE-BRĂ'IC-AL. Appertaining to Algebra: thus, we say algebraic solutions, algebraic symbols, algebraic characters, &c.

ALGEBRAIC CURVE. A curve such that the relation between the co-ordinates of all its points can be expressed by the ordinary operations of Algebra. They are sometimes called geometrical curves, because their different points may be constructed by the operations of Elementary Geometry. The name algebraic is used in contra-distinction to transcendental.

ALGEBRAIC EQUATION. One in which the relation between the known and unknown

quantities is expressed by the ordinary operations of Algebra.

AL-GE-BRĀ'IST. One learned or skilled in Algebra.

AL'GO-RITHM. The art of computing in any particular way. We speak of the algorithm of numbers, surds, imaginary quantities, &c. The word is of Arabic origin, and properly means the art of numbering readily and correctly.

AL'I-QUANT PART. [L. *aliquantum*, a little]. In arithmetic, is such a part of a number as will not exactly divide it. Or, it is a part such that being taken any number of times, the result will be either greater or less than the given number: thus, 4 is an aliquant part of 10, because, being taken twice, the result is 8, a number less than 10, and being taken three times, the result is 12, a number greater than 10. Again, 6 shillings is an aliquant part of a pound, made up of the two aliquot parts 4 shillings and 2 shillings. The term is used in contra-distinction to *Aliquot part.*

AL'I-QUOT PART. Such a part of any number or quantity as will exactly divide that number or quantity. Thus, 2 is an aliquot part of 4, 6, or any even number; and 1 is an aliquot part of any whole number whatever. To find all of the aliquot parts of any number: Divide it by the least number except 1, that will exactly divide it; then divide the quotient by its least divisor, except 1; and so on, always dividing the last quotient by its least divisor except 1, till 1 is found as a quotient; the several divisors, together with 1, are the prime aliquot parts. If we next form every possible product of these divisors, taken in sets of two, in sets of three, and so on, in sets of $n - 1$, n being the number of divisors, the products thus formed, taken with the original divisors, will make up all the aliquot parts of the number. To find all the aliquot parts of 30: We divide it by 2, which gives a quotient 15; we next divide 15 by 3, which gives a quotient 5, which, on being divided by 5, gives a quotient 1; hence, 1, 2, 3 and 5 are the prime aliquot parts; but by multiplying these factors together, two and two, we find 6, 10, and 15 for the compound aliquot parts. Hence, all the aliquot parts of 30 are 1, 2, 3, 5, 6, 10, and 15. In like manner any number may be resolved into factors, and its aliquot parts found. The idea of aliquot parts seems to exclude that of fractions forming any aliquot part of a whole number; still, in the case of denominate numbers, there is an apparent exception; as, for example, we say that 2*s*. 6*d*. is an aliquot part of a pound, being one-eighth of it; 1*s*. 4*d*. is also an aliquot part of a pound, being one-twelfth of it. An aliquot part should not be confounded with a *commensurable part*, for although every aliquot part of a number is commensurable with it, every commensurable part is not an aliquot part. Thus 40 is a commensurable part of 60, but it is not an aliquot part.

AL-LI-GĀ'TION. [L. From *ad*, to, and *ligo*, to bind]. A rule of practical Arithmetic relating to the compounding or mixing of ingrediënts. The rule is named from the method of connecting or tying together the terms by certain ligature-like signs.

The rule is divided into two parts: *Alligation medial*, and *alligation alternate.*

Alligation Medial teaches the method of finding the price or quality of a mixture of several simple ingredients whose prices or qualities are known.

Alligation Alternate teaches what amount of each of several simple ingredients, whose prices or qualities are known, must be taken to form a mixture of any required price or quality.

As an example of a problem in alligation medial, we take the following:

Having a mixture of 30 bushels of wheat, worth 150 cents per bushel, 72 bushels of rye, worth 90 cents per bushel; and 60 bushels of barley, worth 60 cents per bushel; required the price of a bushel of the mixture.

30 bush. of wheat,	at 150 cts.,	worth	4500 cts.
72 " " rye,	" 90 "	"	6480 "
60 " " barley	" 60 "	"	3600 "
162 bushels of the mixture,		worth	14580

Whence, 1 bushel is worth $\frac{1}{162}$ of 14580 cents, or 90 cents.

Again: Suppose a goldsmith to mix gold as follows: 6*oz*. of 22 carats, with 4*oz*. of 17 carats; required the quality of the mixture.

6*oz*. of 22 carats gives,	132
4*oz*. of 17 " "	68
10	200

If, now, we divide 200 by 10, the whole number of ounces in the mixture, we shall find 20 carats for the quality of the mixture. The principle in the last example is in no wise different from that in the former, the apparent difference lying entirely in the language employed in stating the proposition. We may in this example regard 24 as the value of pure gold per ounce; then 22 and 17 will be the respective values of each specimen mixed, and we shall find, as before, 20 for the value of an ounce of the mixture, that is, an ounce will contain $\frac{20}{24}$ths of pure gold.

We may then write this rule for solving all questions in alligation medial:

Rule.—*Multiply the price or quality of a unit of each simple by the number of such units; take the sum of their products, and divide it by the whole number of units; the quotient will be the price or quality of a unit of the mixture.*

Alligation Alternate, as may be seen from the definition, gives rise to the solution of an indeterminate problem in Algebra. According as fewer or more restrictions are imposed, the solutions will be more or less numerous. There will be several cases. We shall first discuss the general one, in which it is required to find the amount of each simple of known value, which must be mixed so that each unit of the mixture shall have a given value.

Let there be three simples of the respective values of a, b, and c; let x, y, and z denote the number of units taken from the respective simples to form m units of the mixture; and let d denote the price or quality of a unit of the mixture. Then, from the conditions of the question, we shall have

$$ax + by + cz = md \ldots . (1),$$
$$x + y + z = m \ldots \ldots (2);$$

two equations of condition, which can, by the elimination of m be reduced to a single equation:

$$(a - d)x + (b - d)y + (c - d)z = 0 \ldots (3).$$

This equation must be satisfied, in all cases, and any set of values of x, y, and z, which will satisfy it, will give a true answer to the question, considered in its most general sense.

Ordinarily, negative solutions are rejected, and the results are required to be integral; these restrictions greatly limit the generality of the problem.

Since there are three simples, there are three unknown quantities, any two of which may be assumed at pleasure, and the value of the third deduced from equation (3).

If there are n simples, the equation of condition will contain n unknown quantities, and $(n - 1)$ of them may be assumed at pleasure.

In order to deduce a practical rule for solving questions in alligation alternate, let us begin with the case where there are but two simples. Denote the price or quality of a unit of the mixture by a; let $a + b$ and $a - c$ be the respective values of a unit of each simple, and let x and y denote, as before, the number of units of these simples that are taken. We shall have, as before,

$$(a + b)x + (a - c)y = a(x + y) \quad (1).$$

Whence, by reduction, we find

$$bx - cy = 0 \quad (2).$$

Or, $\frac{x}{y} = \frac{c}{b}$, a relation which shows that any two values of y and x which are to each other as b is to c, will fulfill the required condition, hence, $y = b$ and $x = c$, are answers of the question, as well as any equi-multiples of b and c.

From a consideration of the notation employed, it appears that b is the excess of the value of a unit of the first simple over that of a unit of the mixture, and c is the excess of value of a unit of the mixture over that of a unit of the second simple. The above discussion indicates the following rule, when there are but two simples:

Write down the values of a unit of each simple beginning with the greatest, and link them together by a bracket; write on their left the value of a unit of the mixture; subtract the last value from the first value given, and set the difference opposite the second; subtract the second value from the last, and set the difference opposite the first: these differences, or any equi-multiples of them, will be answers to the question proposed.

1. Required the number of bushels of oats at 50 cents per bushel, and of wheat at 120 cents per bushel, that must be mixed, so that

a bushel of the mixture shall be worth 75 cents.

			bush.	
75	120	...	25	of wheat.
	50	...	45	of oats.

Hence, 25 bushels of wheat and 45 bushels of oats, are the quantities required. Any equi-multiples or proportional parts of 25 and 45, will also satisfy the conditions of the problem, as 5 and 9, 50 and 90, 20 and 30, and so on for an infinite number of pairs of numbers.

The result may be easily verified: Taking 25 bushels of wheat at 120 cents, gives 3000 cents, and 45 bushels of oats at 50 cents, 2250 cents. Hence we see that 70 bushels of the mixture is worth 5250 cents, and one bushel 75 cents, as was required.

By an analagous train of reasoning, we may, when there is any number of simples of different prices, establish the following general

RULE.—*Write down the prices of the simples under each other, and the price of the mixture at the left-hand; link the prices of the simples two and two, so that each price greater than that of the mixture may be linked with one less than it, and the reverse.*

Subtract the price of the mixture from each greater price of the simples, and write the difference opposite the price or prices with which it is linked; subtract each less price of the simples from that of the mixture, and write the difference opposite the price or prices with which it is linked; then, the number or sum of the numbers written opposite each price, will express the amount of that simple which is to be taken.

Any equi-multiples of these numbers will also satisfy the conditions of the problem.

It is to be observed that the quantities may be linked in many different ways, but the answers in all cases will be true.

1. Required the number of pounds of tea of the respective values of 2*s*., 3*s*., 4*s*., 6*s*., and 8*s*. per pound, which must be taken so that the mixture may be worth 5*s*.

1*st. Method of Linking.* *Verification.*

			Ans.	
5	8	... 3	... 3*lb.*	$3\times8=24$
	6	... 2+1	... 3*lb.*	$3\times6=18$
	4	... 1	... 1*lb.*	$1\times4=\ 4$
	3	... 1	... 1*lb.*	$1\times3=\ 3$
	2	... 3	... 3*lb.*	$3\times2=\ 6$
			11	)55(5.

2*d. Method of Linking.* *Verification.*

			Ans.	
5	8	... 3+2	... 5*lb.*	$5\times8=40$
	6	... 1	... 1*lb.*	$1\times6=\ 6$
	4	... 1	... 1*lb.*	$1\times4=\ 4$
	3	... 3	... 3*lb.*	$3\times3=\ 9$
	2	... 3	... 3*lb.*	$3\times2=\ 6$
			13	)65(5.

3*d. Method of Linking.* *Verification.*

			Ans.	
5	8	... 1+2	... 3*lb.*	$3\times8=24$
	6	... 3	... 3*lb.*	$3\times6=18$
	4	... 3	... 3*lb.*	$3\times4=12$
	3	... 3	... 3*lb.*	$3\times3=\ 9$
	2	... 1	... 1*lb.*	$1\times2=\ 2$
			13	)65(5.

And so on, many other ways of linking can readily be conceived, and the number of ways becomes greater as the number of simples is increased.

The following are cases that may arise:

1. When the amount of one simple is given.

Solve the general problem by the rule already given; divide the amount of the given simple by the amount opposite to its price found by the rule; multiply the amount opposite the price of each of the other simples by this ratio and the products will be the respective amounts required.

For example, in the last case, solved by the first method of linking, let it be required to form a mixture which shall contain 4 pounds of tea at 8*s*. The ratio found is $\frac{4}{3}$; there will, therefore, be

			Verification
4*lbs.*	at	8*s.*	$4\times8=32$
4*lbs.*	at	6*s.*	$4\times6=24$
$\frac{4}{3}$*lbs.*	at	4*s.*	$\frac{4}{3}\times4=\frac{16}{3}$
$\frac{4}{3}$*lbs.*	at	3*s.*	$\frac{4}{3}\times3=\ 4$
and 4*lbs.*	at	2*s.*	$4\times2=\ 8$
$\frac{44}{3}$)			$\frac{220}{3}$(5.

2. When the amount of the mixture is given.

Solve the general problem as before, and take the sum of the results; divide the amount of the mixture given by this amount, and multiply the amount opposite the price of each simple by the ratio found; the products will be the respective amounts required.

For example, take the case already considered, and let it be required to form a mixture of 65 pounds. Then the ratio will be 65

divided by 13, which is equal to 5, and the several amounts will be as follows:

	25*lbs.*	at	8*s.*
	5*lbs.*	at	6*s.*
	5*lbs.*	at	4*s.*
	15*lbs.*	at	3*s.*
and	15*lbs.*	at	2*s.*

A result which may be verified as before.

It is evident, from a review of the preceding discussion, that we may assume the amount of all the simples except one, and the amount of that one can then be found from the equation of condition. If the amount so found is positive, the answer will be true; if it is negative, we must vary our assumption till a positive result is found.

Many other problems than those already discussed may arise; in fact their number is infinite, but an attentive consideration of the principles discussed will readily present the proper mode of procedure for their solution.

AL'MA-CAN-TAR. See *Almucantar.*

AL'MA-GEST. A collection of problems in astronomy and geometry, drawn up by Ptolemy. The same name has been given to other works of a like kind

AL'MU-CAN-TAR [Arabic]. A circle of the celestial sphere, whose plane is parallel to the horizon. Since every point of an almucantar has the same altitude, it is often called a circle of equal altitude.

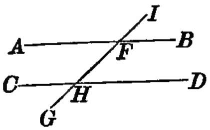
AL'MU-CAN-TAR STAFF. An instrument having an arc of about 15°, used for observing the sun or a star when near the horizon, to find the amplitude or the variation of the needle.

AL'TERN-ATE [L. *alternatus*, by turns]. Succeeding each other by turns.

Alternate Alligation In arithmetic. See *Alligation.*

Alternate Angles. In elementary geometry, if two parallel straight lines are intersected by a third, the two inner angles, on opposite side of the cutting line, are called *alternate interior angles:* also, the two outer angles, on opposite sides of the third line, are called *alternate exterior angles.* If two parallel planes are intersected by a third, the analogous angles are called by the same names.

The angles AFH and FHD, also FHC and BFH, are alternate interior angles.



The angles IFB and CHG, also AFI and GHD, are alternate exterior angles.

Alternate Proportion. Quantities are in proportion alternately or by alternation, when antecedent is compared with antecedent and consequent with consequent. Thus,

$$a \;:\; b \;\cdot\; c \;:\; d,$$

if we change the order of the terms so as to read

$$a \;:\; c \;::\; b \;:\; d,$$

the comparison is said to be made by alternation.

AL'TERN-Ā'TION. Sometimes used in Algebra and Arithmetic for *permutation*, to express the changes in the order of the quantities considered.

AL-TIM'E-TER. [L. *altus*, high, and Gr. *μετρον*, measure]. An instrument for measuring altitudes, as a quadrant, sextant, or theodolite.

AL-TIM'E-TRY. The art of measuring altitudes by means of an altimeter, and the application of geometrical principles.

AL'TI-TUDE. [L. *altus*, high]. The third dimension of a body, or its height.

Altitude of a Triangle. The perpendicular distance from the vertex of the triangle to the base, or base produced. Either side of a triangle may be regarded as a base, and then the vertex of the opposite angle is the *vertex of the triangle.* The side which appears horizontal, in viewing the figure, is generally considered as the base, unless some other side is especially pointed out. In the right angled triangle, the base is always one of the sides about the right angle, the other one being the measure of the altitude.

If either angle at the base is obtuse, the line on which the altitude is measured will fall upon the base produced in the direction of the obtuse angle.

Altitude of a Trapezoid. The perpen

dicular distance between its parallel sides, which are then called bases.

Altitude of a Parallelogram is the perpendicular distance between any two parallel sides taken as bases.

Altitude of a Cone or Pyramid is the perpendicular distance of the vertex from the plane of the base.

Altitude of a Frustrum of either a cone or pyramid, is the perpendicular distance between the planes of its bases.

Altitude of a Parallelopipedon is the distance between the planes of any two parallel faces taken as bases.

Altitude of a Spherical Segment, or Zone, is the distance between the planes of the circles which constitutes its bases. If the segment or zone has but one circular base, the altitude is the distance between the plane of that base and a plane drawn parallel to it and tangent to the surface of the segment or zone.

In Leveling, the altitude of one point above another, is the difference of their distances from the centre of the earth.

In Surveying, the altitude of an object is the distance between two horizontal planes, drawn one through the highest, and the other through the lowest point of the object, or through the position of the observer. Such altitudes are divided into two classes, *accessible* and *inaccessible.*

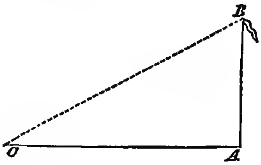
Accessible Altitude is the altitude of an object whose base is accessible, so that the surveyor may measure the distance from his station to it.

Inaccessible Altitude is the altitude of an object such, that the surveyor cannot measure the distance from his station to its base, by direct measurement, on account of some intervening obstacle.

I. *To measure an accessible altitude.*

There are three principal methods:

1. Let it be required to determine the altitude of an object AB.

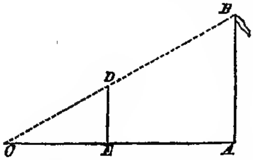


Select any convenient point C, on a horizontal line through A, and from C measure with any suitable instrument, the angle BCA, and also the distance CA. Then from the right-angled triangle CAB, we have

$$BA = CA \tan BCA,$$

from which the value of BA may readily be computed.

2. When no means for measuring angles are at hand. Select the point C as before, and measure the distance AC; then measure off a distance CE towards the object, and at E



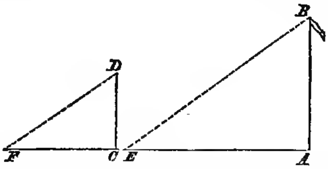
set up a vertical stake; from C, sight to the top of the object, and note the point D where this line of sight cuts the stake, and then measure DE. We shall have, from similar triangles,

$$CE : DE :: CA : BA;$$

whence, $$BA = \frac{DE \times CA}{CE}.$$

Hence the altitude becomes known. The last method does not require that the line AC should be horizontal, though it is better that it should be.

3. The altitude of an object which is accessible, may be determined by means of its shadow.



Let AB represent the object whose altitude is to be determined, and let CD represent a staff planted vertically, whose length above the ground is known. At any moment of time note the point E where the shadow of the point B falls, and also the point F where the shadow of the point D falls. Measure FC and AE; then from the similar triangles FCD and EAB, we shall have

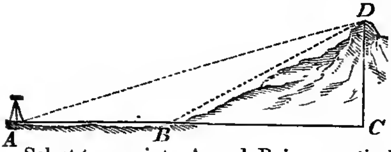
$$FC : CD :: EA : AB,$$

whence, $\quad AB = \dfrac{CD \times EA}{FC}$.

This method does not require that the plane AF should be horizontal.

II. *To measure an inaccessible altitude.*

1. Let it be required to determine the altitude of the point D above the horizontal plane AC.



Select two points A and B in a vertical plane through D, and measure the distance AB between them. At the points A and B, with some suitable instrument, measure the angles of elevation DAC and DBC. Then, in the triangle DAB, since the sine of DBC is equal to the sine of DBA, we have

$$\sin ADB \;.\; \sin DBC \; :: \; AB \; : \; AD,$$

whence, $\quad AD = \dfrac{AB \times \sin DBC}{\sin ADB}$.

But $\quad ADB = DBC - DAB$;

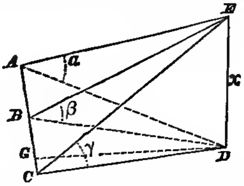
hence, $\quad AD = \dfrac{AB \times \sin DBC}{\sin(DBC - DAB)}$;

having found AD by this formula, we have, from the right-angled triangle ADC,

$$DC = AD \sin DAC;$$

from which the required altitude may readily be found.

2. Having selected three points, A, B and C, situated in the same horizontal straight line, measure the distances AB and BC, and also the angles of elevation at each of the points.



Denote the distances measured by a and b, and the angles of elevation by α, β and γ, and the required altitude DE, above the horizontal plane through AC, by x.

From the right-angled triangles, in the figure, we have

$AD = x \cot \alpha$, $BD = x \cot \beta$, $CD = x \cot \gamma$.

If we now conceive a straight line DG to be drawn from D perpendicular to AC, we shall have, from a known principle of geometry,

$$AD^2 = AB^2 + BD^2 + 2AB \cdot BG \text{ and}$$

$$CD^2 = BC^2 + BD^2 - 2BC \cdot BG,$$

whence, by substituting the expressions for the several distances already deduced, and the values of AB and BC, we have

$$x^2 \cot^2 \alpha = a^2 + x^2 \cot^2 \beta + 2a \cdot BG,$$

$$x^2 \cot^2 \gamma = b^2 + x^2 \cot^2 \beta - 2b \cdot BG.$$

Eliminating BG from these equations, and finding the value of x, we have

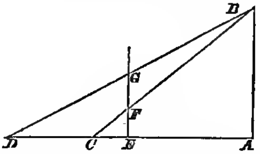
$$x = \sqrt{\frac{ab\,(a+b)}{b \cot^2 \alpha + a \cot^2 \gamma - (a+b) \cot^2 \beta}}.$$

If the distances AB and BC are equal,

$$x = \sqrt{\frac{a}{\frac{1}{2} \cot^2 \alpha + \frac{1}{2} \cot^2 \gamma - \cot^2 \beta}}.$$

This method enables us at the same time to determine the distances from the stations to the object, by simply multiplying the altitude formed by the cotangents of the angles of elevation.

3. When no suitable instrument can be had for measuring angles, the altitude may be determined as follows: let AB represent



the required altitude, and denote it by x. Select two points C and D in a vertical plane through B, and measure the distance between them and call it c. At some point E, between C and A, plant a vertical staff, and having measured the distance CE, call it f. From C sight to B, and note the point F; also from D sight to B, and note the point G: Denote the distance AC by y, the distance EF by a, and the distance FG by b. From the similar triangles CEF and CAB we have

$$\frac{a}{f} = \frac{x}{y} \quad \cdots \quad (1),$$

and from the similar triangles DAB and DEG we have

$$\frac{a+b}{c+f} = \frac{x}{y+c} \quad (2).$$

From equation (1) we find

$y = \frac{fx}{a}$, which in (2) gives $\frac{a+b}{c+f} = \frac{ax}{fx+ac}$, whence we deduce

$$x = \frac{a^2 c + bac}{a(c+f) - (a+b)} \quad (3).$$

There are other methods of determining inaccessible altitudes, but those already given suffice to indicate the general manner of proceeding. The horizontal distance from any selected point to the object, may be determined by the methods given in the article *Distance*, and then the altitude may be determined by the first method given. Wherever angles have to be measured it is important to be very accurate, since small variations of the angles may give rise to great errors in the altitude. More attention is requisite in measuring vertical than horizontal angles, because the instruments employed have not the necessary arrangements for repetition and accurate reading that are furnished in those used for measuring horizontal angles.

In order to determine what effect a small error in measuring the vertical angle will have upon the determination of an altitude, let us consider the first case of determining an accessible altitude.

Let us denote the altitude AB by h, the base AC by b, and the angle of elevation BCA by a; then we shall have

$$h = b \tan a;$$

whence, by differentiating,

$$dh = b\,\frac{da}{\cos^2 a},$$

and by substituting for b its value $\frac{h}{\tan a}$ and reducing, we have

$$dh = \frac{hda}{\sin a \cos a};$$

or, since $\sin a \cos a = \frac{1}{2} \sin 2a$,

$$dh = \frac{2hda}{\sin 2a}.$$

In this expression da is the small arc described with the radius 1, which measures the error committed in measuring the angle of elevation, and dh is the corresponding error in finding the altitude.

It is plain, from the expression for the error in altitude, that it will be the least possible when $\sin 2a$ is the greatest possible; that is, when $2a = 90$, or when $a = 45°$; this shows the advantage of taking the station C, so that the angle of elevation shall be as near 45° as possible.

To explain the use of the above formula, let us take the angle $a = 45°$, and suppose that there was an error in it of one minute of arc. Since there are 10800 minutes of arc in a semi-circumference, we have the proportion

$$10800 : 3.1416 :: 1 : da,$$

whence

$$da = .0002909 \text{ whence } dh = .0005818.h;$$

that is, the error in the computed altitude when the error in angle is one minute, is about $\frac{1}{1719}$th part of the entire altitude. If the angle of elevation differs from 45°, the error will be greater.

For the method of determining the altitude of one point above another in leveling, see *Leveling*.

A favorite method of determining the approximate altitude of one point above another, is by the use of the barometer. The following is Bailey's Formula for making the computation.

$$H = 60345.51 \left\{1 + .001111\,(t + t' - 64)\right\} \times \log\left\{\frac{h}{h'} \times \frac{1}{1 + .0001(T - T')}\right\} \times (1 + .002695 \cos 2l),$$

in which

H denotes the altitude required in English feet.
l denotes the latitude of the place in degrees.
h " height of mercurial column. } at lower station, and
t " temperature of air in deg.
T " " of mercury in deg.
h' " height of mercurial column. } at the upper station.
t' " temperature of air in deg.
T' " " of mercury in deg.

This formula gives good results when the atmosphere is calm and the observations are made contemporaneously at the two stations. It is also to be observed that the accuracy of the determination depends somewhat upon the proximity of the two stations.

When the stations are near each other, and but one set of observations can be made at a time, it will be found advantageous to make two sets of observations at one of the stations at equal intervals before and after the time of making the observations at the other station, and taking a mean of the observed heights and temperatures.

For the more convenient application of the above formula, tables have been constructed, by means of which the arithmetical operations are much facilitated.

Three separate tables are constructed; the first of which gives

$$\log\left[60345.51\left\{1+.00111(t+t'-64^\circ)\right\}\right]$$

for every value of $t+t'$, from 1° to 180°; the second gives

$$\log\left\{1+.0001(T-T')\right\}$$

for every value of $(T-T')$ from 0° to 59°; and the third gives the value of

$$\log(1+.002695\cos 2l)$$

for every value of l, at intervals of 5° from 0° to 90°. In any particular case, if we denote the first log by A, the second by B, the third by C, and assume

$$D=\log h-(\log h'+B),$$

we have

$$\log H = A + C + \log D.$$

Altitude of a Point, above the level of the sea, may be determined by observing the zenith distance of the sea horizon when it is visible from the station.

The formula for computation given by Bégat in his Géodésie, is as follows:

$$\log H=\log\frac{N}{2}\left(\frac{\sin 1''}{1-r}\right)^2+\log(\delta-90^\circ)^2$$
$$+\frac{M}{4}\left(\frac{\sin 1''}{1-r}\right)^2(\delta-90^\circ)^2$$

in which

H denotes altitude required in English feet.
N " normal of earth's mer. at the place.
r " coefficient of terrestrial refraction, the mean value of which may be taken at 0.08, being about 0.06 in summer, and 0.10 in winter.
M " modulus of common system of logarithms.
δ " observed zenith dist. of the horizon.

To insure accuracy, the zenith distance should be observed for several days, and a mean of the whole taken; the state of the tide should also be observed.

Where great accuracy is not required, N may be taken equal to the normal of the meridian at latitude 45°, in which case

$$\log N = 7.3213623,$$

and the last term may be rejected.

Altitude and Azimuth Instrument. An instrument used in geodesy for measuring horizontal and vertical angles. It differs but little in principle from the Theodolite, which see. It is much larger than the Theodolite, more complicated in its construction, and, on account of its want of portability, is not much used, except in the operations of practical Astronomy.

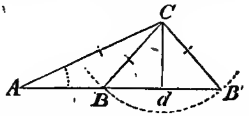
Circles of Altitude. See *Circle*.

AM-BIG'E-NAL HYPERBOLA, [L. *ambo*, both, and *genu*, knee]. An hyperbola of the third order, one of whose infinite branches is tangent to the asymptote within, and the other without the angle which the asymptotes form with each other. See *Hyperbola*.

AM-BIG'U-OUS, [L. *ambiguus*, ambiguous]. Having two meanings, or admitting of two interpretations. The double sign $\pm$, written before an expression, is sometimes called the ambiguous sign, the true meaning of which is generally that the quantity has both signs. For example, the square root of a^2 is $\pm a$; or, in other words, there are two quantities, $+a$ and $-a$, whose second powers are equal to a^2. Generally, if an even root of any quantity is to be extracted, two real quantities will be found equal with contrary signs, which will fulfill the conditions of the problem. Thus

$$\sqrt[2n]{a^{2n}}=\pm a.$$

Ambiguous Case, in Trigonometry, is the case in which two sides are given, and an acute angle opposite one of them.



In this case there are in general two correct solutions: in one case the angle opposite the other given side is acute, and in the other obtuse, they being supplements of each other. Thus, if the sides AC and BC are given, and the angle opposite BC is acute, then will there be two triangles as in the figure, which satisfy the conditions of the problem. The two solutions arise from the fact that the sine of an angle is equal to the sine of its supplement

If the distance CB is exactly equal to the

perpendicular Cd, there will be but one solution. If CB is less than Cd, there will be no solution, and the problem becomes impossible.

AM'BIT, [L. *ambitus*, a circuit]. The perimeter or periphery of a plane figure. Not much used.

AM'BLY-GON, [Gr. αμβλυς, obtuse, and γωνια, an angle]. An obtuse angle, triangle, or other polygon.

AM-BLYG'ON-AL. Having one obtuse angle.

AM'I-CA-BLE NUMBERS, [L. *amicabilis*, from *amicus*, a friend, and *amo*, to love]. Pairs of numbers, each of which is equal to the sum of all the aliquot parts of the other.

Thus, 220 and 284, are amicable numbers The aliquot parts of the first are 0, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110, and their sum is 284. The aliquot parts of the second are 1, 2, 4, 71, and 142, and their sum is 220.

The pairs of numbers 6232 and 6368, 17296 and 18416, 9363584 and 9437056, are also amicable numbers. The formulas for finding amicable numbers are

$$A = 2^{n+1}d \quad \text{and} \quad B = 2^{n+1}bc,$$

in which n is a whole number, and b, c, and d, are prime numbers, satisfying the following conditions:

1st. $3 \times 2^n - 1 = b$, 2d. $6 \times 2^n - 1 = c$, and 3d. $18 \times 2^n - 1 = d$.

If we make $n = 1$, we find $b = 5$, $c = 11$, and $d = 71$, which, being substituted in the formulas above give

$$A = 4 \times 71 = 284 \text{ and } B = 4 \times 5 \times 11 = 220,$$

the first pair of amicable numbers. Again, if $n=3$ we find $b=23$, $c=47$, and $d=1151$, all of which are prime numbers. Substituting these in the formulas, we obtain

$$A = 16 \times 1151 = 18416$$

and $$B = 16 \times 23 \times 47 = 17296,$$

the second pair of amicable numbers. Had we made $n=2$, we should have found $d=287$, which is not prime; hence there is no pair of amicable numbers corresponding to that supposition. In a similar manner other pairs of amicable numbers may be determined. Four pairs only are known at the present time.

A-MOUNT'. In Arithmetic the sum or sum total of two or more numbers. Thus, the amount of 5 and 7 is 12.

AM'PLI-TUDE, [L. *amplitudo*, from *amplus*, large]. The angular distance of a heavenly body at the time of its rising or setting from the true east or west points of the horizon. It is the same as the angle included between the prime vertical and a vertical plane through the centre of the body, and is equal to the complement of the azimuth of the body.

The magnetic amplitude is measured from the magnetic east and west points, and differs from the true amplitude by the amount of the variation of the needle.

AN'A-LEM-MA, [Gr. αναληµµα, *altitude*]. The orthographic projection of the sphere on the plane of the meridian. If the projection is made upon the plane of the solstitial colure, the equinoctial colure and all circles of latitude will be projected into limited straight lines, respectively equal to their diameters; all the meridians, except the colures, will be projected into ellipses.

The name *Analemma* is also applied to an instrument of brass or wood, composed of a plate upon which this projection of the sphere is made, having a horizontal fitted to it. It was formerly much used in solving problems in astronomy, such as finding the time of the sun's rising and setting, the length of the longest day in any latitude, the hour of the day, &c.

A-NAL'O-GY, [Gr. αναλογια, from ανα and λογος, ratio, proportion]. An agreement or likeness between things in certain respects: thus, we say that there is an analogy between the hyperbola and ellipse, because we can deduce most of the properties of the one from the analytical expressions for those of the other, by simply changing the sign of r^2, and interpreting the results obtained. See *Hyperbola*.

Analogous Properties. Those properties of different things by virtue of which they resemble each other: for example, in the ellipse, the squares of the ordinates of any two points are to each other as the rectangles of the corresponding segments into which they divide the transverse axis; in the hyperbola the squares of the ordinates of any two points are to each other as the rectangles of the corresponding distances from the foot of each ordinate to the extremities of the transverse axis. These properties in the

two curves, are called *Analogous Properties*.

When we are led to form a conclusion with respect to one thing from our knowledge with respect to a similar thing, we are said to *reason from analogy*.

Such a conclusion, though possessing a certain degree of probability, cannot be admitted as *proof* in the exact science of mathematics. Accordingly, every species of reasoning from analogy is, and ought to be rejected, in a course of rigid demonstration. It is not, however, to be inferred that analogical reasoning is entirely useless in mathematics; on the contrary, it is often of the highest utility in suggesting facts which may afterwards be verified by rigorous investigation. Indeed it is to this source that some of the most brilliant discoveries of science may be traced.

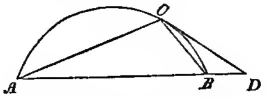
A-NAL'Y-SIS, [Gr. *αναλυσις*, compounded of *ανα*, and *λυσις*, loosening or resolving]. A term, which in its most general signification embraces all of that portion of the science of mathematics in which the quantities considered are denoted by letters, and the operations to be performed upon them are indicated by signs,

Writers on this subject have drawn a distinction between the *ancient* and *modern* analysis, and we shall consider the two separately:

1. Ancient Analysis. A very good idea of what is meant by ancient analysis, may be gathered from the following extract taken from a work by Pappus, one of the most celebrated of the ancient analysts.

"Analysis is the course of reasoning, which, setting out from the thing sought, and which for the moment is taken for granted, conducts by a succession of consequences to something already known." * * * "By this method, therefore, we ascend from a truth or proposition to its antecedents, and we call it *analysis* or *resolution*, as indicating an inverted solution. In synthesis, on the contrary, we set out with that proposition which comes last in the analysis, and proceed by arranging, according to their nature, the antecedents which present themselves as consequents in the analytical method, and combining them together till we arrive at the conclusion sought." This is the true idea of analysis according to the derivation of the term and the usages of the ancients. To illustrate more fully the meaning of the term as above defined, we subjoin an example of the method, taken from the works of the author above mentioned.

"Let it be required to draw two straight lines AC and AB through the extremities of a given arc of a circle, and meeting each other on the circumference, so that they shall bear to each other the given ratio of F to G.



"Analysis. Suppose the thing done, or that the point C is found, and draw CD tangent to the circle at C, and meeting AB produced at D. By the hypothesis,

$$AC : BC :: F : G; \text{ and}$$

$$AC^2 : BC^2 : DA : DB,$$

which may be proved thus: DC is tangent to the circle and BC cuts it; therefore the angle BCD is equal to the angle BAC; the angle at D is common to both the triangles ACD and BCD: hence they are similar, and

$$DA \cdot DC :: DC : DB,$$

multiplying term by term by the proportion.

$$DA : DC :: DA : DC,$$

$$DA^2 . DC^2 :: DA : DB.$$

Moreover,

$$DA : AC :: DC : CB \text{ and}$$

$$DA \cdot DC \cdot: AC : CB, \text{ or}$$

$$DA^2 \cdot DC^2 :: AC^2 : CB^2;$$

and therefore, by equality,

$$AC^2 . BC^2 \cdot AD : DB.$$

But the ratio of AC^2 to BC^2 is given, since the ratio of AC to BC is given, and, consequently, that of AD to DB is known. Then, since the ratio of DA to AB is known, DA is given in magnitude, and may be constructed."

"Synthesis. Construction. Make

$$F^2 : G^2 .: AD : DB,$$

which may be done, as AB is given, by making

$$F^2 - G^2 : G^2 :: AB : DB,$$

and then, by composition.

$$F^2 : G^2 :: AD : DB;$$

then, from the point D, thus found, draw a tangent to the circle; from the point of contact C drawing CA and CB, and the thing is done."

From an attentive consideration of the above discussion, it will appear that the ancient Analysis was a kind of reasoning which stood opposed to the *synthetic method* Certain propositions were assumed, and the reasoning carried on till a result was arrived at which was known to be either true or false. If the ultimate consequence was true, the assumed proposition must have been true; if the ultimate consequence was false, the proposition assumed was also false, and the method came under what has been called "*reductio ad absurdum.*"

In case of a problem, the reasoning was carried on till some proposition was arrived at which was comprised amongst what the old geometers called *data*. Then the construction was made by reversing the order of the analysis.

2. Modern Analysis. In the modern acceptation of the term, analysis implies the *means* made use of for mathematical investigation, rather than the peculiar method of reasoning employed. It is used in contradistinction to the geometrical method, so that every mathematical process in which symbols are employed, and which is not geometrical, is analytical. In it the quantities considered are represented by simple characters, operations are indicated by concise symbols, demonstrations are reduced to certain rules, and are carried on by systematic processes, so that results are often arrived at in a few lines which would have required pages, if not volumes, by the geometrical method. Indeed, many of the results of analysis are entirely beyond the reach of even the most refined geometrical processes. Analysis is the great instrument of invention, and to its successful cultivation may be ascribed the immense improvement which has taken place in mathematics, and the vast range of discoveries which have been made in Philosophy during the last two centuries. The application of analysis not only serves to simplify ordinary geometrical processes, but it also generalizes the results, and opens a wider field of discovery; it throws light upon the more abstruse intellectual operations of geometrical reasoning, making them more easy to apprehend, and more interesting to follow. The adoption of the analytical method of investigation has entirely revolutionized the whole range of Physics, so that by its aid we now acquire, in a short time, a knowledge of science, which could hardly have been acquired in many years, by the ancient method.

Analysis, in some branches of mathematics, particularly in Arithmetic and Descriptive Geometry, is a name given to the synopsis or exposition of the principles to be employed in demonstrating a proposition, or solving a problem.

AN'A-LȲST. A person skilled in analysis.

AN-A-LȲT'IC-AL. Something that belongs to, or partakes of the nature of analysis: thus we say, an analytical demonstration, an analytical method, &c.

Analytical Geometry is that branch of analysis which has for its object the analytical investigation of the relations and properties of geometrical magnitudes.

It is generally divided into two parts, *Determinate* and *Indeterminate*.

1. Determinate Geometry, is that which has for its object the solution of determinate problems, that is, problems in which the conditions given, limit the number of solutions. It is called determinate geometry, because the equations employed to express the relation between the known and unknown elements of the problem are always equal in number to the required parts. For this reason their solution gives determinate values for these parts.

This part is comparatively of little importance, being nothing more than the application of the principles of algebra to the solution of geometrical problems.

The following rule for the solution of determinate problems will sufficiently indicate the nature of this branch of the subject.

Conceive the problem solved, and draw a figure, the different parts of which shall represent all of the given and required parts of the problem. Then draw such other lines as will enable us to establish the necessary relations between them. Denote all the known parts by the leading letters of the alphabet, and

the unknown parts by the final letters. Express the relations existing between the known and unknown parts by equations, and by solving these equations determine the values of the required parts.

The results obtained will indicate the necessary constructions.

The equations determined are called the equations of the problem, and by examining them we are enabled to pronounce upon the nature of the problem. If the number of independent equations found is just equal to the number of required parts, the problem is *determinate;* if the number of equations is less than the number of required parts, the problem is *indeterminate;* if the number of independent equations exceeds the number of required parts, the problem is *impossible.*

The operations for deducing the general formulas of Analytical Trigonometry, may be referred to the determinate branch of Analytical Geometry, since they are nothing more than the application of algebra to the discovery of geometrical relations.

2. Indeterminate Geometry is that part of analytical geometry which has for its object the determination and discussion of the general properties and relations of lines and surfaces. In it the relative positions of the points of lines and surfaces is determined by referring them to a sufficient number of fixed objects of reference, by means of certain elements called co-ordinates. The relations between these variable elements are expressed by means of equations, the number of which must, in every case, be less than the number of co-ordinates employed. These equations are called the equations of the magnitudes, and by suitably transforming them and interpreting the results, the relations and properties of the magnitudes are made known. Since the number of equations is always less than the number of unknown quantities employed, they are always indeterminate, and it is from this circumstance, that this part of Analytical Geometry is called *indeterminate.*

It may be divided into two separate parts, *Elementary* and *Transcendental.*

The *first* part embraces all investigations in which the relations between the co-ordinates of the points of the magnitudes considered can be expressed by the ordinary operations of algebra; and the *second part,* includes those investigations in which the relations between the co-ordinates cannot be thus expressed. The *first part* includes a complete discussion of the nature and properties of the straight line, the conic sections, and all surfaces of the first and second orders. It also considers algebraic lines, and surfaces of a higher order than the second, so far as these magnitudes can be discussed, without the aid of the Calculus.

The *second part* treats of a great variety of transcendental magnitudes, such as the cycloid, logarithmic curve, curve of sines, tangents, &c., the cissoid, conchoid, spirals, &c., with their corresponding surfaces. A complete discussion of these magnitudes, however, requires the aid of the Calculus, and they are usually treated of under that head.

Indeterminate geometry, as we have above defined it, was first cultivated as a science by Descartes, about the beginning of the 17th century; Determinate geometry, or the application of algebra to geometry, was used at an earlier period—Descartes also contributed much towards the improvement of the last mentioned branch of analysis.

Although we have classed Determinate and Indeterminate geometry together, as constituting the science of Analytical Geometry, it will readily be seen, that aside from the fact that both have for their object to develop geometrical truths, they have little in common. Indeed the two methods are so radically different from each other, that they might well be separated and treated as distinct branches of mathematics; but we have chosen to retain them under the same heading as has been customary heretofore, and to content ourselves with pointing out the logical difference between the two systems.

In determinate geometry we denote the magnitudes themselves by letters, and then, from known geometrical relations, we proceed to establish the equations of the problem. Having established these equations, we cease to consider the magnitudes, and by the application of the rules of algebra, we transform these equations so as to deduce those results

which, when interpreted, will give the solution of the problem in question. Throughout this process, we are actually engaged in reasoning upon the *magnitudes themselves*, or upon their direct representatives, although the reasoning may have been conducted by the aid of algebraic formulas of thought, instead of the more complex ones of ordinary language.

In indeterminate geometry, the basis upon which we reason is entirely different. Instead of reasoning upon the magnitudes themselves, we consider the *relative positions of the points* of which they are composed, and from a knowledge of these positions we ultimately arrive at a knowledge of the magnitudes themselves. The fundamental principle upon which the system rests, is, *that as the form of a magnitude determines the relative positions of all its points, so will the relative positions of its points determine the form of the magnitude.* Therefore, in order to represent any magnitude analytically, it is simply necessary to express the law which governs the relative positions of its points; and to investigate the nature of the magnitude, it is only necessary to discuss this law of relation. Such is the beautiful conception on which Descartes founded his system of geometry. a system in which every geometrical conception is capable of being represented by a simple algebraic expression, and in which every geometrical process is reduced to the application of the known rules of algebra.

Analytical Trigonometry. That branch of trigonometry which has for its object the analytical investigation of the general relations existing between the trigonometrical functions of arcs or angles. See *Trigonometry*.

AN-A-LYT'IC-AL-LY. In the analytical manner, after the manner of analysis.

AN-A-LYT'ICS The science of analysis; any branch of mathematics analytically considered.

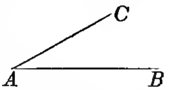
AN'A-LYZE. To investigate analytically.

AN-A-MORPH'O-SIS, [Gr. *ανα*, and *μορφωσις*, formation]. In perspective, a drawing which, when viewed in the common way, presents a *monstrous* or *distorted* image of the thing represented, or else presents an image of some different thing; but when viewed from a particular point, or on being reflected from a curved mirror, presents a correct view of the object.

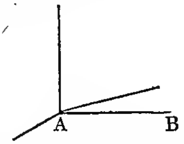
AN'GLE, [L. *angulus*, a corner, Gr. *αγκυλος*]. A portion of space lying between two lines, or between two or more surfaces, meeting in a common point. There are four kinds of angles: *Plane*, *Spherical*, *Diedral*, and *Polyedral.*

A Plane Angle is a portion of a plane lying between two straight lines, meeting in a common point. The two straight lines are called *sides* of the angle, and the common point, the *vertex*.

Thus, the part of the plane lying between AB and AC, is a *plane angle;* AB and AC are its *sides*, and A is its *vertex*.



A plane angle is a species of geometrical magnitude, entirely independent of the length of its sides. but depending upon their opening or inclination.



To acquire an idea of this species of *angular quantity*, let us suppose AB to be a fixed straight line, extending from the point A indefinitely to the right: let us also suppose another straight line, coinciding at first with AB, to be revolved uniformly about the point A until it returns to a coincidence with AB. The space swept over by the moving line will be constantly proportional to the *amount of turning*. The space described by the moving line whilst making one-quarter of a revolution, is called a *right angle*, and is assumed as a unit of measure for all plane angles. The space swept over during any portion of a revolution, is a *plane angle*, and when expressed in terms of the assumed unit, it is entirely independent of the length of the revolving line, so that we may, if we choose, regard this line as infinite in length. This is the true geometrical idea of a plane angle, and this method of viewing the subject enables us to explain what is meant by an angle greater than *four right angles*. For, if after the line has completed one revolution, it sets out on a second, the whole space passed over

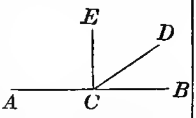
from the beginning of the motion will be greater than four units, and the angle greater than four right angles. If the motion be continued, there is no limit to the value of the angle which may be described.

If we conceive the revolving line to be turned in a contrary direction to that already considered, we shall have the geometrical idea of a negative angle.

Since the path described by any point of the revolving line is proportional to the area described, we may take an arc of a circle whose centre is at the vertex, as the *measure* of an angle.

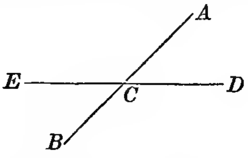
This method of measuring plane angles is the one usually adopted, in consequence of its simplicity and ready application to the methods of trigonometrical computation. For the purpose of comparing angles in this system of measurement, the entire circumference is divided into 360 equal parts, called *degrees*, each degree into 60 equal parts, called *minutes*, and each minute into 60 equal parts, called *seconds*. The right angle contains 90 degrees. The radius of the measuring circle is generally taken equal to the linear unit, or one, when the terms *angle* and *arc* may be used for each other. Such is the general custom.

Plane Angles, in Elementary Geometry, are divided into two classes: *Right Angles* and *Oblique Angles*.



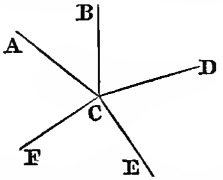
If a straight line meet another straight line, so that the two adjacent angles formed are equal to each other, both are called *right angles*, as ACE and ECB; all other angles are *oblique*, as ACD and DCB.

If an oblique angle is less than a right angle, it is said to be *acute*, as DCB; if it is greater than a right angle, it is *obtuse*, as DCA.

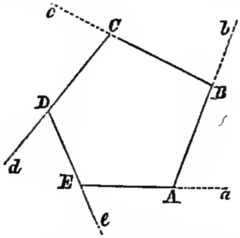


If two straight lines intersect each other, four angles are formed about the common point, which have received different names with respect to their relative position. Those which lie on the same side of one of the lines, but on opposite sides of the other, are said to be *adjacent*. Thus ACE and ECB are adjacent, also ACD and DCB. Those which lie on opposite sides of both lines are said to be *opposite* or *vertical angles*, as ACD and ECB, also DCB and ACE. The sum of any two adjacent angles is equal to two right angles; any two opposite angles are equal to each other; if the two lines are perpendicular to each other, all the angles are *right angles*.

Contiguous Angles are those which have their vertex, and one side in common; if the sum of two contiguous angles is equal to two right angles, they are adjacent. Thus, DCB and BCA, &c., are contiguous angles.



In polygons, an angle lying between two adjacent sides, and within the polygon, is said to be *interior*: thus ABC, BCD, &c., are interior angles.



The sum of the interior angles of any rectilineal polygon, is equal to two right angles, taken as many times as the polygon has sides, less two. In a quadrilateral, the sum of the interior angles is equal to four right angles; in a pentagon, it is equal to six right angles; in a hexagon, it is equal to eight right angles, and so on.

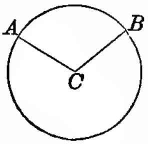
The angle lying between any side and the prolongation of an adjacent one, is called an *exterior angle*: thus, aAB, bBC, &c., are exterior angles. If all of the sides of the polygon are prolonged in the same direction, going round the polygon, the sum of the exterior angles is equal to four right angles.

In both these cases the polygon is supposed to be salient, that is, of such a nature, that the prolongation of any side cannot intersect any other side of the polygon. In this case, all of the interior angles are said to be *salient*.

INTERIOR ANGLE of a polygon is said to be *salient*, when it is less than two right angles; when it is greater than two right angles, it is called *re-entrant*, and the polygon is also re-entrant.

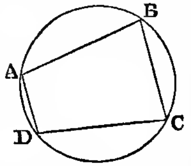
HOMOLOGOUS ANGLES, are those which are like placed in two similar polygons.

In a circle, an angle is said to be *at the centre*, when its vertex is at the centre, and its sides radii of the circle.

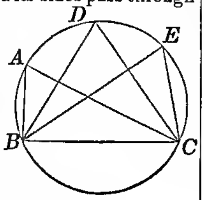


Thus, the angle ACB is an angle at the centre.

An angle is *at the circumference*, or is an *inscribed* angle, when its vertex is on the circumference, and its sides are chords of the circle: thus, the angles DAB, BCD, ABC, &c., are inscribed angles.

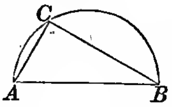


An angle is inscribed in an arc when its vertex is on the arc, and its sides pass through the extremities of the arc: thus, the angles BAC, BDC, BEC, are inscribed in the arc BDC. All angles inscribed in the same arc are equal to each other; that is,

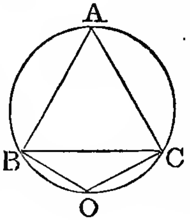


$$BAC = BDC = BEC, \text{ \&c.}$$

If the arc is a semicircle, the angles inscribed are right angles, as BAD.



If the arc is less than a semicircle, the inscribed angle is obtuse, as BOC. If the arc is greater than a semicircle, the angle is acute, as BAC.



In *Surveying*, plane angles are distinguished as *vertical*, *horizontal* and *oblique*. A *vertical angle*, is one whose plane is *vertical*; a *horizontal angle*, has its plane *horizontal*; and an *oblique angle*, has its plane *oblique*.

Two kinds of vertical angles are of importance in surveying.

ANGLES OF ELEVATION, which are vertical angles, having one side horizontal, and the inclined side above it, and

ANGLES OF DEPRESSION, which are vertical angles, having one side horizontal, the inclined side being below it.

In *Shades and Shadows*, there are two kinds of angles, which require definition.

ANGLE OF INCIDENCE, is an angle included between a ray of light incident or falling upon a surface, and the normal to the surface at the point of incidence.

ANGLE OF REFLECTION, is the angle lying between the normal at the point of incidence, and the ray reflected from that point.

VISUAL ANGLE, in *Perspective*, is an angle whose vertex is at the eye, or point of sight.

ANGLE OF THE RHUMB, OR LOXODROMIC ANGLE, in *Navigation*, is the constant angle which a rhumb line or loxodromic curve makes with the meridians which it crosses.

In *Analytical Geometry*, in the rectilineal system, the co-ordinate plane is divided into four angles by the two rectilineal axes which have been numbered as follows:

1*st Angle*. Above the axis of abscissas, and at the right of the axis of ordinates.

2*d Angle*. Above the axis of abscissas, and at the left of the axis of ordinates.

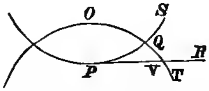
3*d Angle*. Below the second.

4*th Angle*. Below the first.

In the *polar* system, in a plane, the *variable angle* is the angular element of reference, lying between the initial line and the radius vector.

Besides the plane angles already enumerated, there are various angular shaped portions of planes, which have received names as angles, though they are not angles within the true meaning of the term. These names are principally met with in the old works on geometry, now not much used. They are of two kinds, *Mixtilineal*, that is, formed by a straight line and curve, and *Curvilineal*, that is, formed by curves.

Horned Angle is the angular space between a straight line (whether secant or tangent) and the circumference of a circle, as QPR or RVT.



Cissoid Angle is the inner angular space formed by the intersection of two arcs whose convexities are turned in opposite directions, as OQP. The outer angle, under the same circumstances, is called a *sistroid angle*, as SQV.

Lunular Angle is the angular space included between two arcs of circles intersecting each other, and having their convexities turned in the same direction.

Angle of Contact is the angular space between two tangent lines at their point of contact, and it may belong to any one of the four preceding varieties, according to the nature of the lines considered.

Whenever two curve lines intersect, either in a plane or in space, the angle which they make with each other at the common point, is the same as the angle between two straight lines, one drawn tangent to each curve at this point.

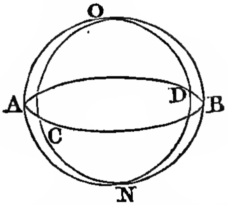
Two straight lines in space are supposed to make an angle with each other, although they do not intersect; which angle is equal to that formed by drawing lines from a given point respectively parallel to the two lines.

The angle formed by a straight line and plane, is the same as that formed by the line and its projection on the plane.

Spherical Angle is the angle included between the arcs of two great circles of a sphere.

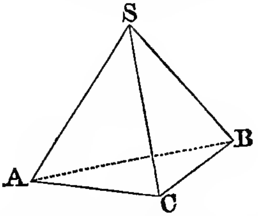
A spherical angle is equal to the diedral angle formed by the planes of its sides, and as has already been explained, is the same as that included between two rectilineal tangents drawn to the arcs at the point of intersection. Spherical angles, like plane angles, are distinguished as *right* and *oblique*, the oblique being subdivided into *acute* and *obtuse*.

Thus OCA is a right angled spherical triangle, AOC is an acute angled spherical triangle, and COB an obtuse angled spherical triangle.



Diedral Angle is the angular space lying between two planes which meet each other, the space between two planes at right angles being taken as the unit. The planes are called *faces* of the angle, and their line of meeting is called the *edge*. The measure of a diedral angle is the same as that of a plane angle formed by two straight lines, one lying in each face, and both perpendicular to the edge at the same point. Diedral, like plane and spherical angles, may be either *right*, *acute*, or *obtuse*, and under like circumstances.

The angle between the two planes ASC and CSB is a diedral angle, whose measure is the same as that of the angle ACB.



Polyedral Angle is the angular space included between several plane angles lying in different planes meeting at a common point, as the space bounded by the three plane angles ASC, ASB, and BSC. The polyedral angle bounded by three right angles, is taken as a unit of measure of this kind of angle.

The plane angles are called *faces* of the polyedral angle, the sides of the plane angles are *edges* of the polyedral angle, and their common vertex is the vertex of the polyedral angle.

Since an arc of a circle has been adopted as the measure of a plane angle, so may a portion of the surface of a sphere be taken as the measure of a polyedral angle. If the vertex of the angle be the centre of a sphere, that portion of the surface lying within the faces of the angle may be taken as the measure of the angle, the tri-rectangular triangle

being regarded as the unit. Every polyedral angle, whose measure is greater than the trirectangular triangle, is *obtuse*, and every one whose measure is less than the trirectangular triangle is *acute*.

AN-GUIN'E-AL HYPERBOLA, [*anguineal*, L. *anguis*, a snake]. A name given by Newton to a hyperbola of a serpentine form, given by the equation

$$xy^2 + ey = -ax^3 + bx^2 + cx + d.$$

AN''GU-LAR. Relating to angles, or having angles, as an angular figure.

AN''GU-LAR SECTIONS. A name given to that part of mathematics which treats of the division of angles into equal parts. The bisection of an angle is readily accomplished by the aid of the principles of Elementary Geometry, and since each of these portions may be again bisected, and so on, *ad infinitum*, it follows that any angle may be divided into a number of equal parts, denoted by any power of 2. The trisection of an angle requires the aid of Higher Geometry, by means of which it can be effected in a variety of ways. The general problem of dividing an angle into any number of equal parts, has not yet been solved by Elementary Geometry.

AN-NEX', [L. from *ad* and *necto*. to tie or connect]. To write after. Thus, to annex 0's to a number, is to write them after it.

AN-NU'I-TY, [L. from *annus*, a year]. A sum of money payable yearly, to continue for a given number of years, for life, or forever; an annual income charged on the person of the grantor; an annual allowance. The term is also applied to any rents or interests payable at regular intervals, whether yearly, half yearly, quarterly, or otherwise.

An annuity payable for a definite length of time is a *certain annuity;* if payable for an uncertain length of time, as during the life of one person, or during the life of several persons, it is a *contingent annuity;* an annuity not to be entered upon immediately, but after a certain period of time, is called a *deferred annuity;* when it is not to be entered upon till after the death of a certain person, it is a *reversionary annuity;* when it is to be entered upon at once, it is an *annuity in possession;* when it is to continue during the life of one or more persons, it is a *life annuity;* when it is to continue for a certain number of years, provided a certain person survives the period mentioned, it is a *contingent life annuity;* when it is to continue forever, it is a *perpetual annuity*. An annuity may be *in arrears*, that is, the payment may not have been made when due, in which case it is said to be *forborne*.

The subject of annuities is one of great importance in the practical affairs of life, and the raising of money for present use, or the investing of ready money so as to produce a regular income for future wants, is a matter of every day occurrence. The accurate determination of the present value in various kinds of annuities, is therefore of much importance. We shall, for this reason, endeavor to explain the principles on which the computation is made, and apply them to those cases which most frequently arise.

I. Certain Annuities. The present value of an annuity of this kind depends only upon the amount of the annuity, the length of time that it is to continue, and the rate of interest of money. Let us denote the annual payment by a, the number of years it is to continue by t, and the rate per cent of interest by r. At the end of the first year the payment a will be due; its present value is evidently such a sum as being put at interest for one year will produce a, hence it is $\frac{a}{1+r}$; at the end of the second year a second payment a is due, and its present value is such a sum as being placed at compound interest for two years, will produce a, or such a sum as being placed at interest for one year, will produce $\frac{a}{1+r}$, hence it is $\frac{a}{(1+r)(1+r)} = \frac{a}{(1+r)^2}$; in like manner the present value of the third payment a is such a sum as being placed at interest for one year, will produce $\frac{a}{(1+r)^2}$; hence, it is $\frac{a}{(1+r)^3}$, and so on as indicated below.

Present value of first payment is $\frac{a}{1+r}$.

" " second " $\frac{a}{(1+r)^2}$.

Present value of third payment is $\dfrac{a}{(1+r)^3}$.

" " fourth " $\dfrac{a}{(1+r)^4}$.

&c. &c. &c. &c.

" " tth " $\dfrac{a}{(1+r)^t}$.

Hence, the aggregate present value of all the t payments is equal to

$$a\left(\frac{1}{1+r}+\frac{1}{(1+r)^2}+\frac{1}{(1+r)^3}\right.$$
$$\left.+\frac{1}{(1+r)^4}+\text{\&c.}+\frac{1}{(1+r)^t}\right).$$

Or, denoting the present value by p, and summing the geometrical series within the parenthesis, we have.

$$p=\frac{a}{r}\left(1-\frac{1}{(1+r)^t}\right).$$

If we make $a=1$, we shall have

$$p=\frac{1}{r}\left(1-\frac{1}{(1+r)^t}\right)\cdots\cdot(A);$$

which is an expression for the present value of an annuity of one dollar per annum for t years at r per cent. This value of p may be calculated by giving to r suitable values, corresponding to the usual rates of interest, as .04, .045, .05, &c., and attributing to t every value from 1 up to any given number. Such a table is called an annuity table, and shows by inspection the present value of an annual annuity of one dollar, or one pound, or any other unit of money, for any number of years at the ordinary rates of interest.

The following is such a table:

Present value of an Annuity of $1.

Yrs.	At 3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	0.97087	0.96154	0.95238	0.94340
2	1.91347	1.88610	1.85941	1.83339
3	2.82861	2.77509	2.72325	2.67301
4	3.71710	3.62990	3.54595	3.46511
5	4.57971	4.45182	4.32948	4.21236
6	5.41719	5.24214	5.07569	4.91732
7	6.23028	6.00205	5.78637	5.58238
8	7.01969	6.73274	6.46321	6.20979
9	7.78611	7.43533	7.10782	6.80169
10	8.53020	8.11090	7.72173	7.36009
11	9.25262	8.76048	8.30641	7.88687
12	9.95400	9.38507	8.86325	8.38384
13	10.63496	9.98565	9.39357	8.85268
14	11.29607	10.56312	9.89864	9.29498
15	11.93794	11.11839	10.37966	9.71225
16	12.56110	11.65230	10.83777	10.10590
17	13.16612	12.16567	11.27407	10.47726
18	13.75351	12.65930	11.68959	10.82760
19	14.32380	13.13394	12.08532	11.15812
20	14.87747	13.59033	12.46221	11.46992
21	15.41502	14.02916	12.82115	11.76408

To use the table, look at the top for the heading which corresponds to the given rate of interest; then, in that column opposite the number corresponding to the number of years, will be found a number expressing the present value of one unit for the given time, at the given rate. Multiply this number by the number of units in the annual payment, the product will be the present value required.

1. To find the present value of an annuity of $100, to continue 21 years at 6 per cent.

Under the heading, 6 per cent., and opposite the number 21, we find 11.76408, which, multiplied by 100, gives for a result $1176,41 for the required present value.

From what has been stated, it will be easy to find the present value of a *deferred certain annuity.*

It is evident that the present value will be found by finding the present value, as though it were to commence immediately, and then finding the present value up to the time at which it is to be entered upon, and taking the difference between them. Thus, if it were required to find the present value of an annuity deferred 5 years, and then to continue 5 years, at 5 per cent., the annual payment being $100, we find the present value of the annuity for 10 years, commencing at once, to be $772,17, and for 5 years, $432,94: hence, the present value of the deferred annuity is $339,23.

If the annuity is perpetual $t=\infty$, in equation A, and $p=\dfrac{1}{r}$, which shows that the value of a perpetual annuity is equal to the annual payment divided by the rate per cent.

If a perpetual annuity is deferred, its present value may be found in the same manner as in the case of a *deferred certain* annuity. If we denote the number of years which the annuity is deferred by T, the present value till the end of that time is, from equation A, equal to

$$\frac{1}{r}\left(1-\frac{1}{(1+r)^T}\right);$$

and since the present value of a perpetual annuity is $\dfrac{1}{r}$, we have for the present value of the deferred annuity the following formula:

$$p=\frac{1}{r}\cdot\frac{1}{(1+r)^T}\cdots\cdot(B).$$

II. To find the value of an annuity in ar-

rears, or which has been forborne for t years. There may be two cases: 1st. When the computation is made at *simplc interest:* 2d. At *compound interest.*

1. *At simple interest.* At the end of the first year, a payment a will be due, at the end of the second year, a second payment a will be due, together with ar, the interest on the first payment, and so on, as indicated below.

At the end of	1st. year	the sum due is	a
"	2d.	"	$a+ar$
"	3d.	"	$a+2ar$
"	4th.	"	$a+3ar$
&c.	&c	&c.	&c.
"	t^{th}	"	$a+(t-1)a.$

Hence, if we denote the sum of this by S, we shall have

$$S=at+ar\left\{1+2+3+\cdots+(t-1)\right\};$$

or, by summing the series within the parenthesis,

$$S=a\left\{t+\frac{r.t(t-1)}{2}\right\}.$$

2. *At compound interest.* At the end of the first year, the payment a becomes due; at the end of the second year, the payment a becomes due, and the interest ar on the first payment; at the end of the third year, the payment a is due, and the interest $r(2a+ar)$ upon the accumulated capital at the end of the second year, and so on as indicated below.

Whole amount due at the end of	1st. year,			$a.$
"	2d.	"	"	$2a+ar=a+a(1+r).$
"	3d.	"	"	$a+a(1+r)+a(1+r)^2.$
"	4th.	"	"	$a+a(1+r)+a(1+r)^2.$
&c.	&e.			$+a(1+r)^3$ &c.
"	t^{th}			$a\left\{1+(1+r)+(1+r)^2+(1+r)^3\cdots+(1+r)^{t-1}\right\};$

or, summing the series and denoting the sum by S,

$$S=\frac{a}{r}\left\{(1+r)^t-1\right\}, \text{ or making } a=1,$$

$$p=\frac{1}{r}\Big((1+r)^t-1\Big)\cdots\cdot(C).$$

We have hitherto supposed the annuity payable annually, but the principles which have been employed will be equally applicable to the case in which payments are made semi-annually, quarterly, or at any regular period of time.

To modify equation (A) so as to apply to a case in which payments are made m times per year, we have only to recollect that the present value of such an annuity is the same as that of an annual annuity for mt years at a rate per cent. equal to $\frac{r}{m}$; substituting these values for t and r, equation (A) becomes

$$p=\frac{m}{r}\left(1-\frac{1}{\left(1+\frac{r}{m}\right)^{mt}}\right);$$

$$\text{or,}\quad p=\frac{m}{r}\left\{1-\left(\frac{m}{m+r}\right)^{mt}\right\}\cdots\cdot(A'),$$

and equation (C) becomes

$$p=\frac{m}{r}\left((1+\frac{r}{m})^{mt}-1\right)$$

$$=\frac{m}{r}\left(\left(\frac{m+r}{m}\right)^{mt}-1\right)\cdots\cdot(C').$$

III. Life Annuities. When the annuity is to cease with the life of a certain individual or certain individuals, the computation becomes more complicated. It then becomes necessary to combine the results already obtained with the probabilities of the individuals, on the duration of whose lives it depends, surviving any given period.

Now it has been shown, in discussing the theory of probabilities, that the measure of the probability of any event occurring, is the quotient obtained by dividing the number of favorable chances by the whole number of chances, both favorable and unfavorable. If, then, we denote the number of persons of a given age, who are living at a given period, by n, and the number of these persons who are living at the end of one year by k', the probability that any one of these will survive the year is $\frac{k'}{n}$; if we denote the number who survive till the end of the second year by k'', the probability that any one will survive two years, is $\frac{k''}{n}$; and, in like manner, if k''', k'''', &c., $k^{m'}$ denote the number surviving at the end of the third, fourth, &c, to m years, then will $\frac{k'''}{n}$, $\frac{k''''}{n}$, &c., $\frac{k^{m'}}{n}$ denote the probabilities that any one will survive three, four, . . . m years.

The values of k', k'', &c., are taken from extensive tables of mortality which have been prepared to show the ratio of the number of

individuals who enter upon any given year to the number who survive to the end of it.

1. *Where the annuity depends upon the continuance of a single life.* We have before shown that the present value of a payment a, certainly due at the end of the year, is $\frac{a}{1+r}$, but the probability of receiving it in the case considered is $\frac{k'}{n}$; therefore, the present value, taking into consideration the chance of life, is $\left(\frac{a}{1+r}\right) \times \frac{k'}{n}$. In like manner, the present value of a payment a, certainly due at the end of two years, is $\frac{a}{(1+r)^2}$, and the chance of its being received is $\frac{k''}{n}$; hence, the present value, taking into consideration the chance of life, is $\frac{a}{(1+r)^2} \times \frac{k''}{n}$, and so on. Now if we denote the present value of the life annuity by P, we shall have

$$P=\left(\frac{a}{1+r}\right) \times \frac{k'}{n} + \frac{a}{(1+r)^2} \times \frac{k''}{n} + \&c. + \frac{a}{(1+r)^m} \times \frac{k^{m'}}{n} + \&c.,$$

the series being continued till the last term is equal to 0, or till it is so small that it may be neglected without error.

There is no way of computing the value of this series except by finding from the data given the value of each term separately and then taking their sum.

However, as the object in general is not to determine the value of an annuity at any particular age, but to construct a table showing this value at every age, there is a method of deducing the value at one age in terms of the value at another age, which was discovered by Euler, and which serves to abridge the operations when such a table is to be calculated. To explain this method, let us consider the case in which the life annuity depends upon the life of an individual A years of age. If we denote the present value of such an annuity by P, we shall have, from what has just been shown,

$$P=\frac{a}{n}\left\{\frac{k'}{1+r} + \frac{k''}{(1+r)^2} + \&c. + \frac{k^{m'}}{(1+r)^m} + \&c.\right\} \cdot (1).$$

Let us designate by P' the present value of an annuity which depends upon the life of an individual aged $A+1$ years.

Since we have denoted the whole number of n persons who survived at the end of the first year by k', and of those who survived at the end of two years by k'', &c., the probability that a person aged $A+1$ years will survive one year is $\frac{k''}{k'}$, that he will survive two years is $\frac{k'''}{k'}$, &c., that he will survive $(m-1)$ years, $\frac{k^{m'}}{k'}$ &c.; hence

$$P' = \frac{a}{k'}\left\{\frac{k''}{1+r} + \frac{k'''}{(1+r)^2} + \&c. + \frac{k^{m'}}{(1+r)^{m-1}} + \&c.\right\} \quad (2)$$

Multiplying both members of Equation (2) by $\frac{k'}{n} \times \frac{1}{1+r}$, we have

$$\frac{P'}{n} \times \frac{k'}{1+r} = \frac{a}{n}\left\{\frac{k''}{(1+r)^2} + \frac{k'''}{(1+r)^3} + \&c. + \frac{k^{m'}}{(1+r)^m} + \&c.\right\} \cdot\cdot (3)$$

and subtracting Equation (3) from (1), member from member,

$$P - P'\frac{k'}{n(1+r)} = \frac{a}{n} \times \frac{k'}{1+r},$$

whence

$$P = \frac{k'}{n(1+r)}\left\{a + P'\right\}$$

and if we make $a = 1$,

$$P = \frac{k'}{n(1+r)}\left\{1 + P'\right\} \cdot \text{(D)}.$$

Hence, to find the present value of an annuity of 1 dollar, or other unit, which depends upon the life of an individual aged A years, knowing that of one depending the life of an individual aged $A+1$ years, *add* 1 *to the last value and multiply the sum by the probability of the life* A *lasting one year, into the present value of a unit due at the end of* 1 *year.*

By the aid of this rule, extensive tables of life annuities have been calculated for every possible age. By the aid of these tables, a variety of problems in annuities may be solved.

To find the present value of a deferred life annuity: suppose, for example, that the person on whose life the annuity depends, is 30 years of age, and that the annuity is deferred

7 years. After 7 years, if the person be then alive, the value of the annuity for the remainder of his life, will be equal to that on the life of a person aged 37 years, which may be found from the tables. Let this be denoted by P''. The present value of one dollar, certainly due at the end of 7 years, may easily be computed; designate this by P'''; denote the probability that a person aged 30 will live to be 37, by K, which number may be found from the tables of mortality, and it is evident that $K.P'''$ will denote the present value of 1 dollar due 7 years hence, taking into account the chances of the life in question. If, therefore, we designate the present value of the deferred annuity by Q, we shall have

$$Q = K \times P'' \times P''.$$

The present value of a temporary life annuity, to run n' years, may be found by adding together n' terms of the series in the second member of Equation (1); or, it may be found by taking the present value of the whole life annuity, and then subtracting from it the present value of the annuity in the same life deferred n' years, since the sum of the temporary life annuity and of the deferred annuity which constitutes the remainder of the life annuity, makes up the entire life annuity.

2. *When the annuity depends upon the joint continuance of two lives.* If we denote the probabilities that A will survive 1, 2, 3, &c. years, by k', k'', k''', &c., and that B will survive 1, 2, 3, &c. years, by h', h'', h''' &c., then from the theory of probabilities, the probability that both will survive 1, 2, 3, &c. years, will be $k'h'$, $k''h''$, $k'''h'''$, &c., and from the principles already employed, we shall have for the present value of a joint life annuity, depending upon these lives

$$P = a\left\{\frac{k'h'}{(1+r)} + \frac{k''h''}{(1+r)^2} + \&c. + \frac{k^{m'}h^{m'}}{(1+r)n} + \&c.\right\}$$

in which formula k' is the same as $\frac{k'}{n}$ in equation (1), and h' has an analagous value with respect to the life of the second individual, &c.

3. *When the annuity depends upon the life of the survivor of two individuals.* Let P denote the present value of the annuity, were it dependent only on the life of A, and P' its value were it dependent only on the life of B. Let p denote the probability that A will live more than n years, and q the probability that B will live more than n years; then since a certainty is equal to 1, we shall have $1 - p$ to denote the probability that A will die before the end of n years, $1 - q$ to denote the probability that B will die before the end of n years, and $(1 - p)(1 - q)$ to denote the probability that both will die before the end of n years; hence, $1 - (1 - p)(1 - q)$, will denote the probability that both will not die before the end of n years. Reducing and denoting the last probability by k, we have,

$$k = p + q - pq.$$

This expression is the measure of the probability that a payment will be received at the end of the n^{th} year. The present value of a payment a, due certainly at the end of the n^{th} year is $\frac{a}{(1+r)^n}$, which multiplied by the value of k, gives

$$\frac{a}{(1+r)^n}(p+q-pq) = \frac{ap}{(1+r)^n} + \frac{aq}{(1+r)^n} - \frac{apq}{(1+r)^n};$$

if $a = 1$, the expression becomes

$$\frac{p}{(1+r)^n} + \frac{q}{(1+r)^n} - \frac{pq}{(1+r)^n}.$$

If now we make n equal to 1, 2, 3, &c., successively, and take the sum of the results, this will be the value of the annuity to the survivor. Making the substitutions, and denoting the sum by Q, and the corresponding values of p and q, by p', q', &c., we find

$$Q = \left(\frac{p'}{1+r} + \frac{p''}{(1+r)^2} + \&c.\right) + \left(\frac{q'}{1+r} + \frac{q''}{(1+r)^2} + \&c.\right) - \left(\frac{p'q'}{1+r} + \frac{p''q''}{(1+r)^2} + \&c.\right)$$

But the first term of the second member is the present value of an annuity on the single life of A, the second term of an annuity on the single life of B, and the third the value of an annuity on the joint lives of A and B.

Hence, *the present value of an annuity on the surviving life of two, is equal to the sum of the annuities on each of the single lives, diminished by the annuity on their joint lives.*

Many other problems may arise in the dis-

cussion of annuities, but the principles indicated above are sufficient to show the method of solving them. When there are more lives than two, the number of different cases that may arise becomes very great. A complete discussion of them would require more space than can be given to this subject.

AN'NU-LAR. [L. *annulus*, a ring]. Something which has the form of, or which resembles a ring. Thus, if a square be revolved about a straight line parallel to one of its sides as an axis, it will generate an annular solid.

AN'NU-LUS. A portion of a plane included between the circumferences of two concentric circles.

AN'SWER. A solution; the result of a mathematical operation. The term is chiefly used in arithmetic and algebra.

AN-TAG-O-NIST'IC. [Gr. *αντι*, against, and *αγωνιστης*, a champion]. Acting against each other; as the antagonistic screws in the level and theodolite.

ANT-ĂRC'TIC CIRCLE. [Gr. *αντι*, against, and *αρκτος*, the bear]. A small circle of the celestial and terrestrial spheres, which passes through the southern pole of the ecliptic. It is distant from the equator about $66\frac{1}{2}°$. It takes its name from being opposite to another circle, which passes through the north pole of the ecliptic, called the Arctic circle.

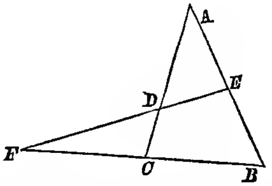
AN-TE-CĒD'ENT. [L. *ante* and *cedo*, to go before]. Of *a ratio*, is the first of the two terms which are compared together. As its name implies, it forms the standard of comparison, since it must be known before the value of the consequent can be expressed. The measure of the ratio of the antecedent to the consequent is, therefore, the quotient which arises from dividing the latter by the former.

AN-TI-CLĪ'NAL LINE. [Gr. *αντι*, against, and *κλινω*, to incline]. In topography, a line from which the surface dips in both directions at right angles to it. The crest of a hill or ridge is an anticlinal line.

AN-TIM'E-TER. [Gr. *αντι* and *μετρον*, measure]. An optical instrument for measuring angles more accurately than can be done by means of the sextant.

AN-TI-LOG'A-RITHM. [L. *anti* and *logarithm*]. Is a number corresponding to any given logarithm. Thus, 100 is the antilogarithm of 2 in the common system, because 2 is the logarithm of 100 in that system. According to the most recent notation, the antilogarithm is represented by the symbol $\log^{-1}$; thus, in the instance above given, $\log^{-1}2 = 100$, which is read *the number whose logarithm is* 2 *is* 100. The term antilogarithm has been often used to designate the arithmetical compliment of a logarithm. In this sense the term is now but little used.

AN-TI-PAR'ALLELS, in Geometry, are straight lines which make equal angles with two given straight lines, but in contrary order



Thus, if AC and AB are two given straight lines, and the two straight lines CB and ED are so situated as to make the angle DEA equal to the angle ACB, and the angle EDA equal to the angle B: then are the last two lines antiparallels with respect to the first two, and conversely, the first two are antiparallels with respect to the last two. See *Sub-contrary*.

AN-TIP'O-DĒS. [Gr. *αντι*, *against*, and *ποδος*, *foot*]. Two points on the earth's surface at the extremities of the same diameter. They have the same latitude, the one north, and the other south, and are distant from each other 180° in longitude.

Ā'PEX. [L. *apex*]. The vertex, top, or summit of any thing. The apex of a cone, or pyramid, is the same as its vertex.

AP'O-THEM of a regular polygon, is the perpendicular distance from the centre to one of the sides of the polygon. If n denote the number of sides of the polygon, r the radius of the circumscribed circle, and a the apothem, we shall have

$$a = r \cos \frac{360°}{2n}.$$

If the number of sides is infinite, $\frac{360°}{2n}$ becomes 0, and we have $a = r$. Hence, in the case of the circle, regarded as a regular polygon, having an infinite number of sides, the apothem is equal to the radius, and we may also infer that the radii are perpendicular to the elements of the curve.

A-POT'O-ME. [Gr. αποτεμνω, to cut off]. A name given by ancient writers to the difference between two incommensurable quantities: thus, the difference between the diagonal of a square and one of its sides is an apotome. The term is much used by Euclid, who distinguished several kinds of apotomes:

1. When the greater number is rational, and the difference of the squares of both is a perfect square; as, $3 - \sqrt{5}$. The difference of the squares of these quantities is 4, which is a perfect square.

2. When the lesser quantity is rational, and the square root of the difference of the squares of the two quantities will exactly divide the greater quantity; as, $\sqrt{18} - 4$; then the square root of the difference of the squares is $\sqrt{2}$, and since $\sqrt{18} = 3\sqrt{2}$, the quotient is 3.

3. When both quantities are irrational, and the greater is exactly divisible by the square root of the difference of the squares of the two quantities; as, $\sqrt{24} - \sqrt{18}$, then the greater quantity is equal to $2\sqrt{6}$, and the square root of the difference of the squares is $\sqrt{6}$; hence, the quotient is 2.

4. When the greater quantity is rational, and is not divisible by the square root of the difference of the squares of the two quantities; as, $4 - \sqrt{3}$. Here, $\sqrt{13}$ will not exactly divide 4.

5. When the lesser quantity is rational, and the greater is not exactly divisible by the square root of the difference of the squares of the two quantities; as, $\sqrt{6} - 2$. Here, the $\sqrt{6}$ is not exactly divisible by 2.

6. When both quantities are irrational, and the greater is not exactly divisible by the square root of the difference of the squares of the two quantities; as, $\sqrt{6} - \sqrt{2}$. Here, the $\sqrt{6}$ is not exactly divisible by 4.

AP-PĀR'ENT LEVEL. In Leveling, the line of level indicated by the axis of the telescope when made horizontal. The true level is a line every point of which is equally distant from the centre of the earth; hence, a line of apparent level at any point, is tangent to the line of true level through the same point; or more strictly speaking, the plane of apparent level at any point is tangent to the surface of true level passing through the same point.

In the practical operations of leveling, the reading of the leveling staff indicates the distance from the point in which the plane of apparent level through the axis of the telescope cuts the staff, to the foot of the staff.

To find the distance from the point in which the surface of true level cuts the staff, to the foot of the staff, it is necessary to subtract a certain *correction* from the record made. The formula for this correction is

$$c = \frac{d^2}{2r},$$

in which c is the correction, d the distance from the instrument to the staff, and r the radius of the earth, all expressed in feet.

This correction is called the correction for curvature, and has been found for one mile equal to about two-thirds of a foot.

Hence, within the limit of the distances usually considered in practical operations, since the correction varies as the square of the distance d, we may employ the following rule for determining the correction.

The correction for curvature in feet, is equal to two-thirds of the square of the number of miles from the level to the staff.

If the distances considered are less than one mile, they must be expressed decimally, in terms of a mile, and the rule will apply.

A simple conversion of the above rule will, in some cases, enable us to determine the approximate distance to an object when its height is known. For example, knowing the height of a light-house, the summit of which is just visible to the eye, situated at the level of the sea; its distance in miles may be found by multiplying the number of feet in its height by $\frac{3}{2}$, and extracting the square root of the product: thus, if a light-house is 96 feet high, we have

$$\sqrt{\tfrac{3}{2} \times 96} = \sqrt{144} = 12,$$

hence, it is visible to the eye, at the level of the sea, 12 miles distant. If the eye of the observer is elevated above the level of the sea, the same rule must be applied to this elevation, and also to the elevation of the light-house, and the sum of the results will be the required result: thus, if a light-house is 96 feet high, and the eye of the observer 24 feet high, the light-house is visible at a distance equal to

$$\sqrt{\tfrac{3}{2} \times 96} + \sqrt{\tfrac{3}{2} \times 24}, \text{ or } 12 + 6, \text{ or } 18 \text{ miles.}$$

This rule only gives approximate results, since it does not take into account the effect of refraction which operates to increase the range of vision. See *Leveling*.

AP'PLI-CATE. A chord which is bisected by a diameter. If a curve is referred to a diameter, and a line parallel to the chords which it bisects, then an applicate is the same as the double ordinate through any point of the diameter.

Applicate Numbers are the same as concrete numbers.

Applicate Ordinate. An applicate with reference to an axis of the curve. It is the same as the double ordinate, perpendicular to an axis of the curve.

AP-PLI-CĀ'TION [L. *applicatio*]. The operation of applying one thing to another, or of comparing one thing with another by bringing them together: thus, the length of a line is determined by applying to it some unit of measure, and determining the number of times which it contains the unit.

In this sense, application is nearly synonymous with division. In Arithmetic, the term is employed to denote the use of the principles of science in the solution of practical problems. In Geometry, one figure is applied, or conceived to be applied to another, for the purpose of determining whether they are equal or unequal. In this manner, many of the fundamental principles of Geometry have been proved: thus, it is proved, that if two triangles have two angles and the included side of the one, equal to two angles and the included side of the other, each to each, they will be equal. It is also proved, by the method of application, that the diameter of a circle divides the circle into two equal parts.

The application of one branch of mathematics to another, or of one science to another, consists in using the principles developed in one, for the purpose of developing or illustrating the principles of the other. For this purpose, algebra has been applied to geometry, geometry to algebra, and both to mechanics, astronomy, navigation, &c.

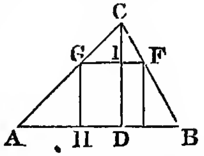
Application of Algebra to Geometry, consists in applying the rules and principles of algebra to the solution of geometrical problems, or the demonstration of geometrical propositions. Instances of this kind of investigation occur in the works of the earliest mathematicians, as Diophantus, Tartalea, &c., as well as in those of more recent date.

The algebraic solution of a geometrical problem, consists of three parts: 1st, expressing the conditions of the problem in algebraic language by means of equations. 2d. Combining these equations by means of known rules, so as to develop the relations between the required and known parts; and 3d Interpreting the results, and making the necessary constructions thus indicated.

The general method of proceeding has already been indicated in treating of determinate analytical geometry.

A few examples will serve to illustrate the rule there given, as well as the different methods of translating the conditions, and interpreting the results, which cannot be reduced to any fixed and invariable rules.

1. Having given a triangle, let it be required to find the sides of an inscribed rectangle, such that its adjacent sides shall bear to each other a given ratio. Let ABC be the given triangle, and suppose the required rectangle to be inscribed.

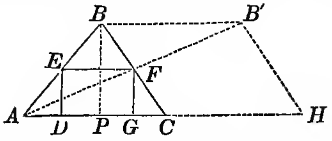


Denote the base of the given triangle by b, its altitude by h. Denote the side of the rectangle perpendicular to the base by x, and the adjacent side by nx, n being the given ratio. Then, from the figure,

$$b : h :: nx : h - x \quad \therefore x = \frac{bh}{b + nh}.$$

To construct this value of x, produce the

base AC, and on the prolongation lay off CH′ equal to nh; through H′ draw H′B′



parallel to CB, and through the vertex B draw BB′ parallel to the base AC. Join B′A, and through F draw FG perpendicular to AC, and FE parallel to AC, and complete the rectangle FD, which will be the required rectangle: for, from the figure,

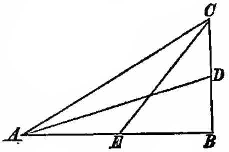
$$AH' : AC :: BP : FG, \text{ or}$$
$$b + nh : b :: h : FG;$$

$$\therefore FG = \frac{bh}{b + nh} = x.$$

Hence, FG is equal to the side designated by x, and FE to the side designated by nx. The construction is therefore verified.

2. In a right-angled triangle, having given the lengths of two lines drawn from the vertices of the acute angles to the middle points of the opposite sides, to find the sides of the triangle.

Let ABC represent the triangle, and AD and CE the given lines. Denote AD by a, CE by b, AB by $2x$, and CB by $2y$. Then, from known principles of geometry, we have



$CE^2 = CB^2 + BE^2$, or $b^2 = 4y^2 + x^2$, and
$AD^2 = AB^2 + BD^2$, or $a^2 = 4x^2 + y^2$.

Combining these equations, we find

$$x = \sqrt{\frac{4a^2 - b^2}{15}}, \text{ or } 2x = 2\sqrt{\frac{4a^2 - b^2}{15}}, \text{ and}$$

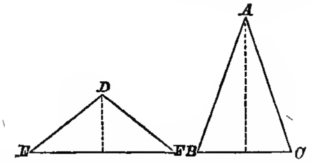
$$y = \sqrt{\frac{4b^2 - a^2}{15}}, \text{ or } 2y = 2\sqrt{\frac{4b^2 - a^2}{15}}.$$

Hence, the hypothenuse is equal to

$$\sqrt{4x^2 + 4y^2} = 2\sqrt{\frac{3a^2 + 3b^2}{15}}.$$

For the method of constructing these expressions, see *Construction*.

3. Having given an isoceles triangle, to find a second isoceles triangle, which shall



have an equal area and an equal perimeter. Let ABC represent the given triangle, and DEF the required triangle. Denote the area of the given triangle by A, and its perimeter by p; denote the base of the first triangle by $2a$, and one of its equal sides by b; then will its altitude be equal to

$$\sqrt{b^2 - a^2}.$$

Denote the base of the required triangle by $2x$, and one of its equal sides by y, then will its altitude be denoted by

$$\sqrt{y^2 - x^2}.$$

Since the area of the second triangle is

$$x\sqrt{y^2 - x^2}, \qquad \text{we have}$$

$$A = x\sqrt{y^2 - x^2}, \qquad \text{and}$$

$$\tfrac{1}{2}p = x + y, \quad \text{or,} \quad y = \tfrac{1}{2}p - x;$$

whence

$$y^2 = \tfrac{1}{4}p^2 - px + x^2;$$

which, substituted for y^2, gives, after reduction,

$$A^2 = \tfrac{1}{4}p^2x^2 - px^3. \qquad \text{or,}$$

$$A^2 + px^3 - \tfrac{1}{4}p^2x^2 = 0.$$

Substituting for A and p their values $a\sqrt{b^2 - a^2}$ and $\frac{1}{2}(a + b)$, we shall find, after clearing of fractions,

$$2(a + b)x^3 - (a + b)^2x^2 + a^2(b^2 - a^2) = 0;$$

whence, by factoring,

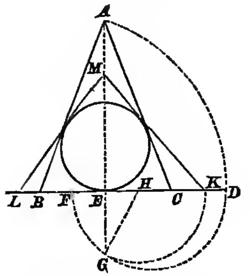
$$(x - a)\,[2(a + b)x^2 - (b^2 - a^2)x - a(b^2 - a^2)] = 0.$$

Putting the factors separately equal to 0, we find, for the first factor, $x = a$, which gives the first triangle; and for the second factor,

$$x + \frac{b - a}{4} \pm \sqrt{\frac{a}{2}(b - a) + \left(\frac{b - a}{4}\right)^2},$$

which will give the second.

To construct the last value of x.



First, construct the given triangle BAC, then, with B as a centre and with BA as radius, describe the arc AD cutting BC produced in D; let fall the perpendicular AE, and bisect BE in F; then is EF equal to $\frac{a}{2}$, and ED is equal to $(b-a)$. Erect EG perpendicular to FD till it intersects the semicircle described on FD as a diameter; then will EG^2 be equal to $\frac{a}{2}(b-a)$. Make EH equal to one-fourth of ED, or equal to $\frac{b-a}{4}$, and draw GH; then

$$GH = \pm\sqrt{\frac{a}{2}(b-a) + \left(\frac{b-a}{4}\right)^2}$$

Make HE equal to HG: then

$$EK = \frac{b-a}{4} + \sqrt{\frac{a}{2}(b-a) + \left(\frac{b-a}{4}\right)^2} = x.$$

Hence, K is one of the vertices at the base, and, by laying off EL = EK, we find a second vertex. Now, let a circle be inscribed in the first triangle, and through the points K and L draw tangents to it, forming the triangle KLM; this will be the triangle required. If the given triangle is equilateral, the construction will give only the triangle itself.

The second value of x corresponds to a second construction, which would give a triangle lying below the given triangle, which corresponds to the algebraic enunciation of the problem. It may easily be constructed.

Nothing but long experience can enable the student to seize upon the relations of the parts of the problems presented so as to give the simplest solutions. It is, therefore, well to solve every given problem in every manner possible, so as to see the relative advantages of each method.

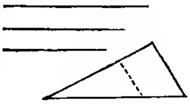
2. Application of Geometry to Algebra, consists in applying the principles of geometry to the elucidation of algebraic formulas and principles. Higher geometry, and sometimes elementary geometry, may be usefully applied to the purpose of investigating the nature of the roots of equations, and also to determine the value of those roots by geometrical construction. It is also of use in the investigation of trigonometrical formulas. It is said that the Arabians discovered the rule for solving complete equations of the second degree, by the aid of geometry; and also that by the same means Tartalea and Cardan deduced and demonstrated the rules for solving cubic equations, employing for that purpose the principles of solid geometry.

The method of proceeding in this kind of investigation is to construct a figure such that each part shall represent one of the given quantities in the expression, and such that the relation between these parts shall be the same as that expressed by the algebraic expression; then from the known geometrical properties of the figure, to deduce the required relations. We annex the geometrical method of constructing the roots of equations of the first, second, third, and fourth, and higher degrees.

1. *Equations of the First Degree.* Let us take $ax - b = 0$, which gives

$$x = \frac{b}{a}, \text{ whence } a : b :: 1 : x.$$

Draw any two straight lines AE and AB, intersecting at A.



Lay off from A on AB, the distance $AB = a$, and from A on AE, the distance $AE = b$; draw EB; lay off from A on AB, the distance $AC = 1$, and draw CD parallel to BE; then is AD the representation of the value of x. For, from the figure,

$$AB : AE :: AC . AD;$$

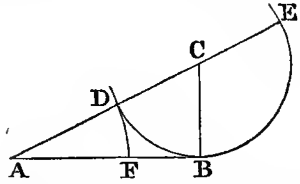
or $a : b :: 1 : AD \therefore AD = \frac{b}{a}$,

or $AD = x$.

2. *Equations of the Second Degree.* Every equation of the second degree, containing

but one unknown quantity, can be reduced to one of the forms below, in which p is essentially positive.

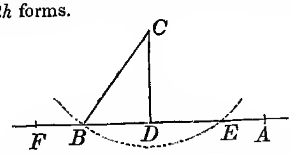
$$\left.\begin{array}{ll}(1). & x^2+2px=\ \ q^2\\ (2). & x^2-2px=\ \ q^2\\ (3). & x^2+2px=-q^2\\ (4). & x^2-2px=-q^2\end{array}\right\}\text{whence}\left\{\begin{array}{l}x=-p\pm\sqrt{p^2+q^2}\\ x=+p\pm\sqrt{p^2+q^2}\\ x=-p\pm\sqrt{p^2-q^2}\\ x=+p\pm\sqrt{p^2-q^2}\end{array}\right.$$



To construct the roots of the first form. Construct a right angled triangle ABC, right angled at B, in which $CB = p$ and $AB = q$; then will AC be equal to $\sqrt{p^2+q^2}$. With C as a centre and CB as a radius, describe a circumference of a circle cutting AC at D and AC produced at E. Then will AD represent the first root and $-$ EA the second root.

To construct the roots of the *second* form, draw a figure as before, and AE will represent the first root, and $-$ AD the second root.

To construct the roots of the *third* and *fourth* forms.



Draw an indefinite right line FA, and at any point as D, erect a perpendicular DC equal to q; from C as a centre, with a radius CB equal to p, describe an arc of a circle cutting FA in B and E; then is

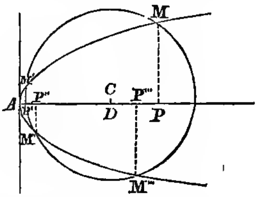
$$BD = \sqrt{p^2 - q^2}.$$

From D lay off on FA, in both directions, the distances DF and DA respectively, equal to p. The lines $-$ AE and $-$ AB will represent the first and second roots of the *third* form, and the lines FB and FE the first and second roots of the *fourth* form.

If $p = q$ the circle BE is tangent to FA at D, and the roots of the two forms are all numerically equal, each being equal to the line FD or $-$ AD.

If $p < q$ the circle does not cut FA, the construction fails, and the roots of both forms are imaginary.

3. *Equations of the third and fourth Degrees.* The construction of the roots of these equations requires the aid of the higher geometry.



Construct a parabola whose axis is AP, and whose parameter is equal to $2p$. Lay off on the axis a distance $AD = a$, and at D erect a perpendicular equal to b; from its extremity C as a centre, and with a radius CM equal to r, describe the circumference of a circle cutting the parabola in the points M, M′, M″, and M'''; from each of these points let fall a perpendicular upon the axis, and these perpendiculars will be roots of an equation of the fourth degree.

If one of these points of intersection fall at A, the perpendiculars will be roots of a cubic equation.

The following considerations will serve to determine proper values for a, b, $2p$, and r, in any given case.

The equation of the parabola is $y^2 = 2px$, and of the circle $(x-a)^2 + (y-b)^2 = r^2$. If we combine these equations, and eliminate x, the values of y in the resulting equation, will represent the four perpendiculars PM, P′M', P″M'', and P‴M'''. Performing the combination, and eliminating, we have, after reduction,

$$\begin{aligned} y^4 - (4pa - 4p^2)y^2 - 4bp^2y & \\ + 2(a^2 + b^2 - r^2)p = 0 & \ldots (1). \end{aligned}$$

In any given case, we reduce the equation to the form of equation (1), by depriving it of its second term, and making the coefficient of the first term 1. Then equate the remaining coefficients and absolute term with the corresponding coefficients in equation (1). Three equations will thus be found containing a, b, $2p$, and r, from which we may, after assuming a value for either one, deduce corresponding values for the other. The con-

struction can afterwards be made as indicated above.

If in equation (1), $a^2 + b^2 = r^2$, the circle will pass through the vertex A and the equation, after dividing both members by y, will become

$$y^3 - (4pa - 4p^2)y - 4bp^2 = 0,$$

and the corresponding construction will give the roots of a cubic equation. In the equation of the fourth degree, if two of the roots are imaginary, that fact will be indicated by the circle only cutting the parabola in two points.

Let it be required to construct the roots of the equation

$$y^4 + 2y^2 - 4y - 2 = 0,$$

which is of the required form. Equating the coefficients of the like powers of y in this and in equation (1), we have

$$2 = 4p^2 - 4pa = 2p(2p - 2a), \quad -4 = -4bp^2$$

and $\quad -2 = 2p(a^2 + b^2 - r^2).$

Let us assume $p = 1$: we deduce

$$a = \tfrac{1}{2} \quad b = 1 \quad \text{and} \quad r = \sqrt{2\tfrac{1}{4}}.$$

Which data enable us to make the construction.

There are other methods of constructing roots of equations of the third and fourth degrees, such as using an auxiliary ellipse, conchoid, or cissoid.

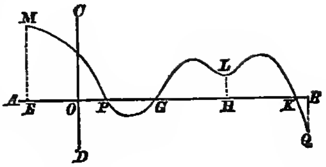
4. The construction of the roots of equations of a higher degree than the fourth is an operation which can only be approximately performed. We shall simply indicate the general method of proceeding without making any application of the principles developed. Let us take an equation of the form

$$x^m + Bx^{m-1} + Cx^{m-2} + \&c. + Nx + R = 0 \,..\, (1),$$

in which B, C, &c., N, R, are known numbers. Find by the known rules of algebra the superior and inferior limits of the real roots of the equation. Now let a second equation be formed by placing y equal to the first member of the given equation, and from the principles of analytical geometry, the resulting equation

$$y = x^m + Bx^{m-1} + Cx^{m-2} + \&c. + Nx + R \,..\, (2).$$

will be the equation of a curve, which may be constructed approximately by points as follows:



Draw two lines AB and CD at right angles to each other, and set off on the line AB the distances $-$OE and $+$OB, respectively equal to the inferior and superior limits of the real roots of the given equation. If the inferior limit is negative, as we have supposed in the figure, the distance OE will be laid off to the left; if it is positive, it must be laid off to the right. Assume, in succession, a sufficient number of values for x between the limits already determined, and substitute these separately for x in equation (2), and deduce the corresponding values of y. Each assumed value of x with the corresponding deduced value of y, will be the co-ordinates of a point which may be constructed by laying off the assumed value of x from O on the line AB, to the right, when positive, and to the left when negative. From the extremity of the distance laid off erect a perpendicular to AB, and lay off on this perpendicular, from AB, the deduced value of y, observing that it must be laid off upwards if the value of y is positive, and downwards if it is negative. In this manner a succession of points may be determined, and a curve MPLQ traced through them.

The distances from O to the points in which this curve cuts the line AB, will be the real roots of the equation. The reason is apparent, for when the curve whose equation is equation (2) cuts the line AB, y must be equal to 0, and equation (2) for that value becomes equation (1), and these distances therefore represent the real roots. When the curve approaches the line AB, and then recedes from it without cutting it, as at L, such change of direction indicates a pair of imaginary roots. To insure as much accuracy as possible, great care should be taken in constructing the curve in the neighborhood of the points in which it cuts the line AB.

Analogous methods may be employed for finding the values of the unknown quantities, when there are two equations containing two

unknown quantities, the equations being numerical and of any degree whatever.

Assume a pair of rectangular axes as before, and in like manner construct two curves by points. The first equation will be the equation of one curve, and the second equation will be that of the other curve.

Having constructed the curves, draw through the points of intersection straight lines parallel to the assumed axes. Then for each point there will be a pair of distances which will represent the simultaneous values of the unknown quantities. If the quantities are represented by y and x, as we have supposed, the distance to the axis AB will represent the value of y, and the distance to the axis CD will represent the simultaneous value of x. The number of points will determine the number of real solutions of the equations, and the accuracy of the values determined will depend upon the accuracy of the construction of the curves.

If the given equations are both of the second degree, the curves to be constructed will be conic sections, and their construction may be more readily effected by some of the methods for constructing these lines.

3. Application of Geometry and Algebra to Trigonometry. The subject of trigonometry is nothing more than a development of the results of applying the principles of geometry and algebra to determine the relation between angles and their functions. This subject is more fully discussed under the head of Trigonometry. The various applications of Mathematics to the physical sciences, engineering, &c., do not come within the scope of this work. The term application, in this sense, is used to denote the use which is made of the principles of mathematics in improving and developing these sciences.

Application of a Rule or Formula, consists in performing the operations prescribed by the rule, or indicated by the formula. Thus, the application of the rule for solving equations of the second degree, in any given case, consists in solving the particular equation by following the different steps prescribed in the rule, so as to determine the roots of the equation.

The application of the binomial formula in any case, consists in attributing to the letters in the formula such values as will bring the particular case under the general one expressed in the formula, and then deducing the results. This will be the general formula to a particular binomial.

The development of formulas and the deduction of rules, constitute the Science of Mathematics: the application of these to particular cases constitutes the Art of Mathematics. Most of the arts are little else than the application, either directly or indirectly, of the principles of science.

AP-PROX'I-MĀTE. [L. *ad*, to, and *proximus*, next]. In mathematics, an approximate result is one which is very near the true result; thus, the approximate value of a radical quantity is the result obtained by applying the rule for extracting the indicated root of the quantity under the radical sign, and continuing the operation to any desired extent. From the nature of the case, the true root cannot be obtained; but the longer the rule is applied, the more nearly will the result approximate to the true root. In short, the error may be reduced to less than any assignable quantity. The process of approximation is one of frequent use in all practical operations. See *Approximation*.

AP-PROX-I-MĀ'TION. In mathematics, a method of calculation, by which we obtain an approximate value of a quantity which cannot be found accurately, either on account of the nature of the quantity itself, or on account of the imperfection of our mode of operation.

The method of finding the ratio of the diameter of a circle to its circumference, or the length of the circumference of a circle whose diameter is 1, affords an instance of geometrical approximation. It is a principle of Geometry, that the arc of a circle is greater than its chord, however small the arc may be. Now, if we suppose a regular polygon, say of 64 sides, to be inscribed in a circle whose diameter is 1, it is evident that the length of the perimeter of the polygon will be an approximate value of the length of the circumference, though it will differ sensibly from it. If now we suppose that a regular polygon, of twice as many sides, is to be inscribed, the perimeter of the new polygon will approximate still more closely to the length of the circumference. If we continue to double the number of sides of the regular inscribed po-

lygon, we shall continue to approximate to the length of the circumference; but whatever may be the number of sides of the polygon, its perimeter will never be exactly equal to the circumference of the circle. The difference between the length of the perimeter and the circumference may be made less than any assignable line, but it can never be made equal to 0.

In analysis, the attempt to express radical quantities in entire terms, affords an example of approximation, as also some of the methods of solving numerical equations of a higher degree than the fourth.

In Arithmetic, the operation of converting certain vulgar fractions into equivalent decimal expressions, is one of approximation; thus $\frac{1}{3} = 0.333333 \cdots\cdots 3 \cdots$ *ad infinitum.* Here, no matter how far the division be carried, the result will not express the exact value of $\frac{1}{3}$, but for each decimal place added, the result will be a nearer approximation to its true value. In practical applications, the operation of approximation is one of great importance, as it gives results sufficiently accurate for the ordinary purposes of art. The various methods of finding approximate results will be fully described under the appropriate headings.

ÄR'BI-TRA-RY. [L. *arbitrarius*, uncertain, independent]. An arbitrary quantity in analysis is one to which we may assign any reasonable value at pleasure. In Analytical Geometry, the arbitrary quantities are generally styled *arbitrary constants*, to distinguish them from the variables which are in a certain sense arbitrary. Thus, in the general equation of the circle

$$(x - a)^2 + (y - b)^2 = r^2,$$

x and y are variables, and a, b and r are arbitrary constants.

In the equation, a and b denote the co-ordinates of the centre, and by attributing to them suitable values, we may place the centre at any point of the co-ordinate plane. Since r denotes the radius, such a value may be assigned to it as to give the circle any desired extent. In this case, therefore, the constants serve to determine the position and extent of the circle, with respect to the co-ordinate axes.

Since there may be an infinite number of circles, there may be an infinite number of sets of values of a, b, and r; but for a *given* circle, and a *given system of axes*, a, b, and r become known, and are absolutely fixed in value.

Not so, however, with the variables x and y. Whatever circle we choose to consider, they will represent the co-ordinates of any point of its circumference at the same instant, and of every point in succession; that is, for any one set of values of the arbitrary constants, there is an infinite number of sets of values for the variables, which will satisfy the equation.

What has been shown in this case, is in general true for all other cases; hence, the distinction between arbitrary constants and variables is this: *given values may be attributed, at pleasure, to the arbitrary constants, provided they will satisfy the conditions of the problem, giving a particular case for each set of values. The variables, on the contrary, admit of every possible value which will satisfy the equation, in each and all the particular cases, determined by attributing given values to the constants.*

The use of arbitrary constants is to cause the equation under consideration to fulfill certain conditions. The number of conditions which may be imposed, is, in general, equal to the number of arbitrary constants. For example: in the case already considered, we may cause the circle to pass through any three points. The method of determining the values of a, b, and r, so that the circle shall pass through three given points, is to substitute, separately, for x and y, in the equation of the circle, the co-ordinates of each point. We thus obtain three equations of condition, which contain a, b, r and known quantities; by combining these, we can find values of a, b and r, which, being substituted in the given equation, will make it the equation of a circle passing through the three given points.

In the Integral Calculus, the constant, added to every integral obtained by applying the rules for integration, is *arbitrary* in its nature, and serves to cause the integral to fulfill any reasonable condition. The method of using it for this end, is, to make such suppositions upon the integral, as will cause it to fulfill the required condition. We thus ob-

tain an equation from which we may deduce the value of the constant, which being substituted in the integral, will make it fulfill the required condition.

For example,

$$\int y\,dx = X + C$$

expresses a plane area included between the curve of the axis of X and any two ordinates whatever. If now we wish that any given ordinate should limit the curve in one direction, we have simply to substitute 0 for the integral, since that integral is to commence at a given ordinate, and also to substitute for x, in X, a value a, corresponding to the given ordinate. This gives

$$0 = (X)_{x=a} + C,$$

whence,

$$C = -(X)_{x=a},$$

and

$$\int y\,dx = X - (X)_{x=a}$$

expresses the same area as before, but commencing at the ordinate whose abscissa is a. These instances are sufficient to illustrate the nature and use of arbitrary quantities in mathematics.

ĂR-BI-TRĀ'TION OF EXCHANGE, is the operation of converting the currency of one country into that of another, through the medium of one or more intervening currencies.

When there is but one intervening currency, it is called *simple arbitration;* when there is more than one it is called *compound arbitration.* The following is the rule for *compound arbitration,* and will answer also for simple arbitration:

Multiply the sum to be converted by the following quotients, after canceling common factors, viz: A certain amount at the second place divided by its equivalent at the first; a certain amount at the third place divided by its equivalent at the second; a certain amount at the fourth place divided by its equivalent at the third place, and so on to the last place.

In the above rule, the amounts named are supposed to be expressed in the currency of the place from which the remittance is made. If they are expressed in the currency of the place to which the remittance is made, the terms of the multipliers must be inverted.

Example. A merchant in New York wishes to remit $4888,40 to London through Paris. He finds that he can remit to Paris at 5 francs 15 centimes to the dollar, and the exchange from Paris to London 25 francs and 80 centimes for £1 sterling. What will the remittance be worth in London?

$$4880{,}40 \times \frac{5.15}{1} \times \frac{1}{25.80} = 975.7852;$$

hence the amount is £975 15*s.* 8¼*d.*

Since 5.15 francs are equal to $1, the first multiplier is $\frac{5.15}{1}$, and because 25.80 francs are equal to £1, the second multiplier is $\frac{1}{25.80}$.

ĂRC. [L. *arcus*, a bow]. A part of the circumference of a circle or other curve. When the term arc is used without any explanation, an arc of a circle is in general understood.

As we have already explained, under *angle*, arcs of circles are employed as the measures of angles, in which case the centre of the arc is taken at the vertex of the angle. Where the radius is 1, the arc intercepted between the sides of the angle is taken as the measure of the angle; when the radius is not 1, the ratio of the radius to the intercepted arc is taken. There are various methods of expressing the values of angles by the aid of arcs of circles. Sometimes a portion of a circle, generally a quadrant, is assumed as a unit and all other arcs are expressed numerically in terms of this as a standard; sometimes the whole circumference is divided into 360 equal parts, each of which is divided into 60 equal parts, which in turn are subdivided into 60 equal parts. These parts are called, respectively, *degrees*, *minutes*, and *seconds*, and the arcs are expressed in terms of these parts. There is no difference between these methods, except in the magnitude of the unit, and the manner of subdividing it. In expressing the magnitude of arcs, the radius is often taken as the unit, and since the circumference in that case is equal to 2π, we may find the expression for any portion of a circumference, already expressed in degrees and fractions of a degree, by the following proportion:

$$n : 180 :: l : 3.1416,$$

in which n denotes the number of degrees in the arc, and l its length in terms of the radius as 1.

It is often convenient to express the length of an arc in terms of its sine or tangent; this can only be done by means of series. The most useful ones are subjoined:

$$\sin^{-1} a = a + \frac{a^3}{2.3} + \frac{3a^5}{2.4.5} + \frac{3.5a^7}{2.4.6.7} + \&c.$$

$$\tan^{-1} a = a - \frac{a^3}{3} + \frac{a^5}{5} - \frac{a^7}{7} + \frac{a^9}{9} - \frac{a^{11}}{11} + \&c.$$

For small values of a these formulas give very good approximate results by using only a few of the leading terms.

To express the length of an arc in terms of the chords of the arc and half arc,

$$a = \frac{8c - C}{3},$$

in which a denotes the length of the arc, C its chord, and c the chord of half the arc. This gives only an approximate result.

CONCENTRIC ARCS are those which have a common centre.

SIMILAR ARCS are those which subtend equal angles at the centre. The lengths of two similar arcs are to each other as their radii.

To find an expression for the length of any arc of a plane curve, when its equation is given, we have the following formula:

$$Z = \int \sqrt{dx^2 + dy^2},$$

in which Z represents the length, and x and y are the co-ordinates of its points. To employ the formula in any case, differentiate the equation of the curve, and from the given equation and its differential equation, find the value of dy in terms of x and dx, and substitute it in the formula. Integrate the result between the proper limits, and the result obtained will express the length of the arc required.

AR-CHI-ME'DES' SPIRAL. See *Spiral.*

ĂRC'O-GRAPH. [L. *arcus*, a bow, and Gr. γραφω, to describe]. An instrument used to describe an arc of a circle, without having its centre given. The simplest form is that used by carpenters for striking arcs for the top of doors, windows, &c. Three nails being driven to mark three points of the circle, two pieces of board are nailed together, forming an angle, so that their vertex shall be at the middle nail, and the two sides against the extreme ones. If now the two pieces be moved so as constantly to touch the two outer nails, the vertex of the angle will trace out the arc of a circle between them.

ĂRC'TIC. [Gr. αρκτος, a bear]. The Arctic Circle is a circle of the sphere, whose plane passes through the north pole of the ecliptic. It is about 66½° distant from the equator.

ĀRE. [L. *area*, an open surface]. In the decimal system of French measures the *are* is a square, the side of which is 10 metres in length. In contains 100 square metres or about 119.60 square yards.

Ā'RE-Ă. [L. *area*, an open surface]. In geometry is the superficial contents of any surface expressed in terms of some given surface assumed as a unit or standard of comparison. The unit of measure is generally a square, one of whose sides is a linear unit in length. For different purposes, the area may be expressed in different terms.

In land surveying, the areas of fields may be expressed in acres, the area of states may be given in square miles, whilst masons' and carpenters' work is generally expressed in square yards or square feet.

In all cases, the arithmetical expression of an area is the ratio of some assumed surface to the surface in question. When surfaces are similar, they are to each other as the squares of their homologous lines.

The most general formula for an area bounded by a plane curve, by the axis of X, and by any two ordinates, is

1. $S = \int y\,dx;$

for an area of a surface generated by revolving a plane curve about the axis of X, the formula is

2. $S = \int 2\pi y \sqrt{dx^2 + dy^2};$

and for any geometrical curved surface

3. $S = \int^2 dx\,dy \sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}.$

To apply the first formula:

Find from the equation of the curve, the value of y *in terms of* x, *and substitute it in the formula; then perform the integration indicated between any two limits, and the result will give the area contained between the curve, the axis of* X, *and the two ordinates taken as limits.*

In this manner the *area of the circle* is found to be equal to πr^2, in which $\pi = 3.1416$, and $r =$ the radius of the circle.

The *area of the ellipse* is $\pi a b$, in which a and b are the semi-axes.

The *area of any portion of the common parabola* estimated from the vertex and included between the curve, the axis and any ordinate, is $\frac{2}{3} x y$, in which x and y are the co-ordinates of the extreme point.

To apply the second formula:

Differentiate the equation of the meridian curve, and combine the resulting equation with the equation of the curve so as to deduce expressions for y *and* d y *in terms of* x *and* d x; *substitute these in the formula, and integrate between any given limits. The result will express the area of the surface contained between the two limits.*

In this manner, the *area of the surface of a sphere* has been found equal to $4 \pi r^2$, in which r is the radius of the sphere.

The area of the surface of a right cone, is equal to $\pi r h$, in which r is the radius of the base, and h the slant height.

The *area* of a surface generated by revolving a cycloid about its base, is equal to $\frac{64}{3}$ of the area of the generating circle of the cycloid.

To apply the third formula;

Differentiate the equation of the surface in question with reference to each of the variables x *and* y; *combine these partial differential equations with the equation of the surface, and find expressions for* $\frac{dz}{dx}$ *and* $\frac{dz}{dy}$, *and substitute them in the formula; integrate first with respect to* x, *between any two limits, and then integrate the result with respect to* y, *between any two limits; the final result will express the area of that portion of the surface embraced within the assumed limits.*

These formulas serve to determine a great number of useful areas, and their application presents little difficulty. The rules for finding various plane areas will be given in the article on Mensuration. It has been shown that the area of the projection of any plane area is equal to the area itself, multiplied by the cosine of its inclination to the plane of projection.

The area of a field, in plane surveying, may be determined as follows: Having determined the bearings and lengths of the several courses which bound the field, take from a traverse table, or a table of natural sines and tangents, the latitude and departure of each course; balance these, that is, distribute the error so that the sum of the eastings shall equal the sum of the westings; and the sum of the northings equal the sum of the southings. Compute the double meridian distance of each course, and multiply the double meridian distance of each course by its northing or southing, observing that like signs give plus, and unlike signs minus; then take the sum of all the positive products, and of all the negative products, and subtract the numerically less from the greater, and half the difference will be the area of the field. If chains and links were employed to express the courses, point off five decimal places to the right, and the area will be expressed in acres and decimals of an acre.

A-RITH'ME-TIC. [Gr. αριθμέω, to number]. That branch of mathematics which treats of the properties and relations of numbers when expressed by the aid of figures, or combinations of figures. It is divided into two parts. The *first*, explains the methods of representing and reading numbers by means of figures, together with the fundamental operations, which are Addition, Subtraction, Multiplication, Division, raising to powers, and extracting roots of numbers, whether the units be entire or fractional. It also treats of the transformation of numbers from one scale to another, in which the fundamental unit may be different, or in which the scale of place may be different. This comprises all of the *science* of arithmetic.

The *second* part, consists in the application of the principles of the science to the practical wants of life. It embraces rules for performing a great variety of practical operations upon numbers, such as the rule of three, analysis, percentage, interest, alligation, equation of payments, &c.

The particular operations of arithmetic depend for their details upon the manner of representing numbers. The system which we employ is based upon the decimal scale, and though numbers are used in other scales, the processes are all referred to those which depend upon that system of notation. Other

scales might, undoubtedly, have been employed with quite as great facility as the decimal; some, indeed, think that the duodecimal scale would have been preferable to it, inasmuch as the number twelve is a multiple of more numbers than ten. Leibnitz invented a binary system, using only two characters, 1 and 0, to express all numbers. In this system, 1 is represented as in the common system, 1; *two*, 10; *three*, 11; *four*, 100; *five*, 101; *six*, 110; *seven*, 111; *eight*, 1000; *nine*, 1001; *ten*, 1010, &c. A ternary system, or one in which three characters are employed, has also been developed, but these are regarded rather as matters of curiosity than of practical utility.

Arithmetic has received different names, according to the different systems of notation employed, or according to the purpose to which it is applied.

Decimal Arithmetic, is that in which numbers are expressed according to the scale of tens, either increasing or decreasing. This is the ordinary system.

Duodecimal Arithmetic, is that in which numbers are expressed according to the scale of twelves. This system is used by carpenters, bricklayers, and artificers generally, for computing their work, being adapted to the measures of feet and inches.

Sexagesimal Arithmetic, is that in which numbers are expressed according to the scale of sixties. This system is principally used in trigonometrical computations, being specially adapted to the subdivisions of the circumference into degrees, minutes and seconds.

Universal Arithmetic, is that which treats of the general properties of numbers, independent of the particular method of expressing them. This differs but little from elementary algebra.

Palpable Arithmetic, is that in which the operations are performed by the sense of feeling, and is used by the blind. In this system, instruments are employed which in principle resemble the abacus in some of its forms. Indeed, all of the operations performed by the aid of the abacus, belong to palpable arithmetic.

Instrumental Arithmetic, is that in which operations are performed by the aid of instruments prepared for the purpose; such as Napier's rods, Babbage's calculating machine, &c.

Tabular Arithmetic, is a name given to that class of operations which are performed by the aid of tables computed for the purpose, such as Hutton's tables, &c.

Political Arithmetic, is the application of the principles of arithmetic to researches connected with civil government, such as determining the number of inhabitants of a country, and classifying them according to sex, age, place of birth, &c.; determining the amount of imports, exports, &c, the distribution of taxes, laying of imposts, &c.

A-RITH-MET'IC-AL. Appertaining to arithmetic; according to the rules and processes of arithmetic: thus, an arithmetical result, is a result which arises from the application of some arithmetical process.

Arithmetical Complement of a logarithm, is the remainder found by subtracting the logarithm from 10: thus,

$$10 - 9.274687 = 0.725313;$$

hence, 0.725313, is the arithmetical complement of the logarithm 9.274687. It may be written at once, by commencing at the left-hand figure and subtracting each figure from 9 till we reach the last figure which is not 0; this must be taken from 10. The arithmetical complement is used in computations to avoid the trouble of subtraction. When two logarithms are to be added together, and a third logarithm is to be taken from their sum, the whole operation may be reduced to one of addition, by taking the sum of the first two, and the arithmetical complement of the third, and then rejecting 10 from the result obtained.

The ease with which the arithmetical complement may be obtained from the tables renders this method of proceeding not only more concise, but also more elegant than the other method.

Arithmetical Mean of any number of quantities, is the quotient obtained by dividing their sum by the number of quantities. It is the same as their average value: thus, the arithmetical mean of 3, 5 and 7, is 5. The arithmetical mean of any number of terms of an arithmetical progression, is equal to the half sum of the extreme terms

When we know the arithmetical mean of any number of quantities, the sum of all of

the quantities may be found by multiplying it by the number of quantities.

ARITHMETICAL PROGRESSION, is a series of terms, each of which is derived from the preceding one by the addition of a constant quantity, called the *common difference*; in every such series, the whole number of terms may be continued *ad infinitum*. When the common difference is greater than 0, or *positive*, the progression is said to be *increasing*. When it is less than 0, or *negative*, the progression is *decreasing*. The first and last terms *considered*, are called the *extremes*, and the intermediate terms are called *means*. An arithmetical progression is sometimes called a progression by differences.

If we denote the first term of an arithmetical progression by a, the last term by l, the common difference by d, the number of terms *considered* by n, and their sum by S, the following formulas will serve to determine any two of these elements, when the other three are given; viz.:

1. $l = a + (n-1)d;$

$$S = \tfrac{1}{2}n[2a + (n-1)d];$$

2. $n = \dfrac{l-a}{d} + 1;$

$$S = \frac{(l+a)(l-a+d)}{2d};$$

3. $n = \dfrac{d - 2a \pm \sqrt{(d-2a)^2 + 8dS}}{2d};$

$$l = a + (n-1)d;$$

4. $S = \tfrac{1}{2}n(a+l);$

$$d = \frac{l-a}{n-1};$$

5. $d = \dfrac{2(S-an)}{n(n-1)};$

$$l = \frac{2S}{n} - a:$$

6. $n = \dfrac{2S}{a+l};$

$$d = \frac{(l+a)(l-a)}{2S-(l+a)};$$

7. $a = l - (n-1)d;$

$$S = \tfrac{1}{2}n[2l - (n-1)d];$$

8. $a = \dfrac{2S - n(n-1)d}{2n};$

$$l = \frac{2S + n(n-1)d}{2n};$$

9. $n = \dfrac{2l + d \pm \sqrt{(2l+d)^2 - 8dS}}{2a};$

$$a = l - (n-1)d;$$

10. $a = \dfrac{2S}{n} - l;$

$$d = \frac{2(nl-S)}{n(n-1)}.$$

The first members of each pair of formulas above given, represent the elements to be determined, and the second members are expressed in terms of the known elements. The use of these formulas is evident from their arrangement.

Any number of means may be determined, so that, when inserted between two given quantities in their proper order, the whole shall constitute an arithmetical progression, by means of the following

RULE. *Subtract the first quantity given from the last, and divide the remainder by the number of means plus* 1; *add this quotient to the first quantity for the first mean, add it to the first mean for the second mean, and so on till the whole number of means is found.*

ARITHMETICAL SCALE. A conventional arrangement for writing numbers by means of figures, so that the same figure shall express different numbers according to its position or place. The order of arrangement may be symbolically expressed thus:

. . . .	n^{th} order.	(n^{-1}) order.	(n^{-2}) order.	(n^{-3}) order.	. . . . .	4^{th} order.	3^{d} order.	2^{d} order.	1^{st} order.	. . . .
.	0	0	0	0	. . . . .	0	0	0	0	

in which the position of each 0 indicates the place of an order of units. If a figure, 1 for example, be written in the place of the first 0 on the right hand, it indicates a unit of the first order; if it be written instead of the second 0, from the right, it indicates 1 unit of the second order; and generally, if it be written in place of the n^{th} 0 from the right, it indicates 1 unit of the n^{th} order. In like manner, if 2, 3, 4, &c., be written in the place of the n^{th} 0 from the right, they will indicate 2, 3, 4, &c. units of the n^{th} order.

The law which determines the relation between the values of two consecutive units of different orders, beginning with the lowest, is called the *ratio of the scale*, and is found by dividing the second unit by the first; and

since there may be an infinite number of such laws, there is an infinite number of scales. They may, however, all be separated into two classes, *uniform* scales, and *varying* scales. A scale is uniform when the values of a unit, in each of the different orders, from the first upward, form a geometrical progression. All other scales are *varying* scales.

1. UNIFORM SCALES. It is plain that in every uniform scale, the number of orders, upwards, is infinite; and if we place a point to indicate the 0 order, there will also be an infinite number of orders estimated downwards, in which the values of a unit of the different orders downwards, counting from the point, form a decreasing progression, so that we may express the entire scale, as follows:

Ascending.											*Descending.*								
n^{th} order.	$(n-1)^{th}$ order.	$(n-2)^{th}$ order.	$(n-3)^{th}$ order.		4^{th} order.	3^{d} order.	2^{d} order.	1^{st} order.		1^{st} order.	2^{d} order.	3^{d} order.	4^{th} order.		$(n-3)^{th}$ order.	$(n-2)^{th}$ order.	$(n-1)^{th}$ order.	n^{th} order.
0	0	0	0	...	0	0	0	0	..	0	0	0	0	...	0	0	0	0

The name of the scale depends upon the value of r, the ratio of the progression. If the value of r is 2, the scale is called *the binary scale;* if it is 3, it is called *the ternary scale;* if it is 4, it is called *the quarternary scale;* if 5, the *quinary scale;* if it is 10, it is called the *decimal scale*, or *the common scale;* if it is 12, it is called *the duodecimal scale;* if it is 60. it is called *the sexagesimal scale*, and so on.

To illustrate the method of writing numbers according to a uniform scale, we shall consider the *common* or *decimal scale.*

According to the principles already indicated, the point stands for 0, and a unit of the first ascending order, is simply 1; a unit of the second ascending order is 10; of the third order, 100; of the fourth order, 1000; of the fifth, 10000, and so on, ad infinitum. A unit of the first descending order is $\frac{1}{10}$; of the second, $\frac{1}{100}$; of the third, $\frac{1}{1000}$; of the fourth, $\frac{1}{10000}$, and so on, ad infinitum.

If we take for example the number *five hundred and sixty seven thousand three hundred and twenty-nine, and seven hundred and fourteen thousandths*, we see that it is equivalent to *five hundred thousands* plus *six tens of thousands* plus *seven thousands*, plus *three hundreds*, plus *two tens*, plus *nine*, plus *seven-tenths*, plus *one-hundredth*, plus *four-thousandths:* it may therefore be expressed in the scale of tens, thus,

Ascending.								*Descending.*			
...	6^{th} order.	5^{th} order.	4^{th} order.	3^{d} order.	2^{d} order.	1^{st} order.		1^{st} order.	2^{d} order.	3^{d} order.	....
.	5	6	7	3	2	9	..	7	1	4	....

In a similar manner, any number may be written according to the scale of tens.

The manner of writing a number according to any other uniform scale, is entirely similar. Let it be required to write the number *two hundred and eighty-nine and forty-four hundredths* in the quinary scale.

In this scale, as in all others, the value of the base, or unit of the first ascending order, is 1; of the second order, 5; of the third, 25; of the fourth, 125; and so on.

The number in question, 289.44, contains 2 units of the fourth order in the quinary scale, $2 \times (125) = 250$, and a remainder 39.44; this remainder contains 1 unit of the third order, $1 \times (25)$, and a remainder, 14.44; this remainder contains 2 units, the second $2 \times 5 = 10$, and a remainder 4.44, which contains 4 units of the first order, 4×1, and a remainder, .44; this remainder contains 2 fractional units, $2 \times (\frac{1}{5}) = .4$, and a remainder 4 hundredths, which contains the fractional unit of the second order $\frac{1}{25} = .04$, 1 time. Hence, the number in question may be written,

$$2 \times (125) + 1 \times (25) + 2 \times (5) + 4 \times (1) + 2 \times (\tfrac{1}{5}) + 1 \times (\tfrac{1}{25});$$

hence, it may be written in the quinary scale as follows:

Ascending.						*Descending.*		
...	4^{th} order.	3^{d} order.	2^{d} order.	1^{st} order.		1^{st} order.	2^{d} order.	....
..	2	1	2	4	.	2	1	......

In general, a number may be written either exactly or approximately in any given uniform scale.

Having any integral number written in the

common scale, to write it in any other uniform scale, we have the following

RULE.—*Divide the number by the ratio of the new scale; the remainder will be the number of units of the first order; divide this quotient by the ratio, and the remainder will be the number of units of the second order; divide the new quotient by the ratio, and the remainder will be the number of units of the third order, and so on; continue the operation till a quotient is found less than the ratio, and this will be the number of units of highest order in the new scale.*

1. Express the number 7843 in the quinary scale.

```
5 | 7843
 5 | 1568 .  . 3 1st. remainder.
  5 | 313 . . . 3 2d.      "
   5 | 62   . . 3 3d.      "
    5 | 12 . . . 2 4th.    "
         2 . . . 2 5th.    "
```

hence, the expression is 222333.

In the duodecimal system twelve characters are necessary to express all numbers; and, in general, the number of characters necessary to express all numbers is, in any system, equal to the ratio of the scale according to which the system of numbers is written. In the duodecimal system, let π stand for 10, and ϕ stand for 11.

2. Express the number 844371 in that system.

```
12 | 844371
 12 | 70364  . . 3 1st. remainder.
  12 | 5863  .   8 2d.      "
   12 | 488 . .  7 3d.      "
    12 | 40 . .  8 4th.     "
          3   . . 4
```

hence, the expression is 348783.

3. Express 17987 in the duodecimal system.

$$
\begin{array}{llll}
12 \mid 17987 & & & \\
\quad 12 \mid 1498 & 11 = \phi & \text{1st. remainder.} & \\
\quad\quad 12 \mid 124 \ldots & 10 = \pi & \text{2d.} & " \\
\quad\quad\quad 10 = \pi & 4 = & \text{3d.} & "
\end{array}
$$

hence, the expression is $\pi 4 \pi \phi$.

If a number expressed in the common system by decimals, according to the descending scale, it will be expressed in any other system also, according to the descending scale. To find the expression:

Multiply the given decimal by the ratio, and point off in the product according to the rule for multiplication of decimals: the number on the left of the decimal point will express the number of units of the first order.

Multiply the decimal part of the product as before, and continue the operation till a sufficient number of orders of figures is obtained, or until the decimal part becomes 0.

1. Express .15734 in the quinary system.

```
 .15734
      5
0.78670   0 is the first figure.
      5
3.93350   3   "   second  "
      5
4.66750   4   "   third   "
      5
3.33850   3   "   fourth  "
      5
1.69250   1   "   fifth, &c.
```

hence, the expression is .03431 +.

2. To express the same number in the duodecimal system.

$$
\begin{array}{lllll}
12 \times & .15734 & & & \\
12 \times & 1.88808 \cdot\cdot 1 & \text{is the first figure.} & & \\
12 \times & 10.65696 \cdot\cdot \pi & " & \text{second} & " \\
12 \times & 8.08352 \cdot\cdot 8 & " & \text{third} & " \\
12 \times & 1.00224 \cdot\cdot 1 & " & \text{fourth} & " \\
 & 0.02688 \cdot\cdot 0 & " & \text{fifth, \&c.} &
\end{array}
$$

hence, the expression is $.1\pi 810$.

If the given number is expressed in whole numbers and decimals, the whole number is transformed by the first rule given, and the decimal part by the last, and the two results written in the new system.

A number written in any system can be reduced to the decimal system *by multiplying the number of units in each order by the value of a unit of that order, and taking the sum of the products.*

To pass from any system, other than the decimal, to a second system, not the decimal system, pass first to the decimal system by the rule just given, and then by the previous rules pass to the required system. The subject of arithmetical scales is chiefly interesting, as serving to illustrate the great advantage of our adopted system of arithmetical notation.

2. VARYING SCALES. These are comparatively few in number, and correspond for the most part to the subdivision of currency, weights, and measures. They are only employed for writing concrete numbers, and often

require the aid of the common scale for their complete expression.

For example, the units of the different orders in *liquid measure*, are: 1 gill, 1 pint, 1 quart, 1 gallon, 1 barrel, 1 hogshead, 1 pipe, and 1 tun. Here the whole number of orders in the scale is eight; a unit of the second order is equal to 4 units of the first; one of the third to two of the second; one of the fourth to four of the third; one of the fifth to thirty-one and a half of the fourth; one of the sixth to two of the fifth; one of the seventh to two of the sixth; and one of the eighth to two of the seventh. The law of the scale is expressed by writing these ratios in their order, beginning with, at the right hand. Thus, the law of the varying scale for liquid measure, is

$$2 . 2 . 2 . 31\tfrac{1}{2} . 4 . 2 . 4.$$

The absolute value of a unit of any order is equal to the continued product of all the ratios, from the first to that order inclusive. In *dry measure*, the units are: pint, quart, peck, bushel, and chaldron, and the scale,

$$36 . 4 . 8 . 2.$$

In *measures of time*, the units are: 1 second, 1 minute, 1 hour, 1 day, 1 week, 1 month, 1 year, and 1 century, and the scale is

$$100 . 13 . 4 . 7 . 24 . 60 . 60.$$

In *circular measure*, the units are: 1 second, 1 minute, 1 degree, 1 sign, and 1 circle, and the scale,

$$1230 . 30 . 60 . 60.$$

When the law of the scale is known, any number may be written in it according to the rules already given, but to avoid confusion in using *varying scales*, the particular name of the order is written over it, as for example in *avoirdupois weight*,

T.	*cwt.*	*qr.*	*lb.*	*oz.*	*dr.*
5	16	3	19	12	13

Arithmetical Triangle. A name given to a table of numbers arranged in a triangular manner, and formerly employed in arithmetical computation. It is equivalent to a multiplication table.

A-RITH-ME-TĬ′′CIAN. One versed in arithmetic; one skillful in numbers.

ĂR′PENT. The old French name for acre. See *Acre*.

AR-RĀNGE′MENTS. [F. *arranger*, *ad*, to, and *ranger*, to set in order]. Are the different ways in which m letters can be written when taken in sets of n, n being less than m. If Y denote the whole number of arrangements, we have the formula,

$$Y = m(m-1)(m-2)\ldots.(m-n+1).$$

If, in this formula, we suppose $m = n$, and denote the corresponding value of Y by X, we shall have

$$X = m(m-1)\ldots.2.1,$$

or $$X = 1.2.3.4\ldots.(m-1)m,$$

which is the formula for the number of permutations of m quantities. The formulas for the number of arrangements and permutations of m letters is used in demonstrating the binomial theorem; they are also extensively employed in the investigation of natural science, music, and more particularly in the theory of probabilities.

AR-RĒARS. [F. *arrière*, behind]. An annuity is said to be in arrears when one or more payments are due. See *Annuity*.

ÄRT. [L. *ars*, art]. Skill in the application of the rules and principles of science, so as to meet the practical demands of life. The entire range of subjects classed under the head of mathematics may be separated into two parts; 1st. The *science* which investigates principles and deduces general rules. 2d. The *art* which explains the method of applying these rules and principles to every particular case that may arise to which they are applicable.

AS-CEND′ING SERIES. [L. *ascendo*, to ascend]. A series in which each term is greater than the preceding one.

Ascending Scale of numbers is that in which the ratio is greater than *one*.

AS-SĪGN′A-BLE. [L. *ad*, to, and *signo*, to allot, to mark out]. That may be allotted or pointed out; that may be specified. An *assignable magnitude* is any finite magnitude that can be expressed or denoted. An *assignable ratio*—a ratio that can be exactly expressed or denoted.

AS-SŪME′. [L. *ad*, to, and *sumo*, to take]. To take, to take for granted; thus, axioms and postulates are assumed for granted. In making any demonstration, we assume the truth of all axioms, postulates, and previous propositions.

AS-SŪR'ANCE. [F. *assurer*, to assure]. A contract for the payment of a certain sum, on the occasion of an event, usually death. The term is nearly synonymous with Insurance, but modern usage applies the term *assurance* to life contingencies, whilst *insurance* applies to all other contingencies, such as loss by fire, water, &c.

Assurances on lives are contracts for the payment of a certain sum of money on the death of one or more persons, in consideration of a certain immediate payment, or more often of an annual payment, to be continued during the existence of the life or lives assured. Although any individual life is exceedingly precarious, yet nothing is more certainly established than the great uniformity in the average duration of human life. Voluminous tables of the lengths of individual lives have been accumulated, and from these the most accurate calculations of the average duration of life, after any given age, have been made; upon these data, and upon the rate of interest of money, the established rates of payment or premium are based.

There is a great analogy between the method of computing the value of life Assurances and Annuities, the principles being nearly the same in both. In both, a table of mortality is selected, from which we deduce the probability of a given life surviving 1, 2, 3, &c., years, and in both, a rate of interest of money is assumed. In computing life assurances, the following is the process.

Let us denote the probabilities that a person of any given age will live

1, 2, 3, 4, 5, &c. years,

by k', k'', k''', k'''', k^v, &c.,

determined as described under the article *Annuity*. Let r denote the assumed rate of interest, and make, for simplicity, $\frac{1}{1+r} = v$, and suppose that the sum assured is to be paid at the end of the year in which the life expires.

We shall first consider the case in which the payment for assurance is immediate. The present value of \$1, payable at the end of one year is $\frac{1}{1+r}$ or v, but it will not be paid if the life survives the year. The probability of the life surviving the year is k', hence the probability that it will not survive the year is $(1 - k')$; therefore the present value of \$1, payable on the contingency of death, within the year, is $v(1 - k')$. The present value of \$1, payable at the end of two years, is $\left(\frac{1}{1+r}\right)^2$ or v^2; the probability that the life will continue one year being k' and that it will continue two years being k'', the probability that it will expire during the second year is $(k' - k'')$; hence the present value of the assurance for the second year is $(k' - k'').v^2$; and in like manner, the present value of the assurance for the third, fourth, &c. years, is $(k'' - k''')v^3$, $(k'''' - k''')v^4$, &c. And since the present value of the entire assurance is equal to the sum of all its values for the different years, if we denote the present value by P, we shall have

$$P = (1 - k')v + (k' - k'')v^2 + (k'' - k''')v^3 + (k'''' - k''')v^4 + \&c., \text{ or}$$

$$P = v(1 + k'v + k''v^2 + k'''v^3 + k''''v^4 + \&c.) - (k'v + k''v^2 + k'''v^3 + k''''v^4 + \&c).$$

But it has been shown in the article *Annuity* that the series

$$k'v + k''v^2 + k'''v^3 + k''''v^4, \&c.,$$

is the present value of \$1 for the life under consideration; denoting this by p, we have

$$P = v(1 + p) - p;$$

or substituting $\frac{1}{1+r}$ for v,

$$P = \frac{1}{1+r}(1 + p) - p;$$

or finally,

$$P = \frac{1}{1+r}(1 - rp) \quad (1).$$

The second member shows the amount which must be paid down to secure the payment of \$1 at the end of the life considered.

It is, however, more common to require an annual payment, or premium, the first to be made immediately, and others at the end of each successive year. Let the annual premium be denoted by P'; then the payments after the first will constitute an annuity, and is consequently equal to $P' \times p$. Hence, the present value of all the premiums is $P' + P' \times p$, or $P'(1 + p)$, and this is equal to the

amount to be paid immediately, as deduced above; hence,

$$P'(1+p) = \frac{1}{1+r}(1-rp) = v(1+p) - p;$$

whence,

$$P' = v - \frac{p}{1+p} = \frac{1}{1+r} - \frac{p}{1+p};$$

from which the value of the annual premium can be readily computed, when we have an annuity table. The value of P′ is the amount that a person ought to pay annually to have $1 assured to his heirs on his decease.

Temporary Assurances are contracts for the payment of a certain sum on the contingency of death happening within a certain number of years.

The value of the annual premium may be easily found from the preceding principles. To explain the method of procedure, let us consider the ease of a person aged 30 years, and let it be required to find the immediate payment that must be made in order that $1 may be received if the individual die within seven years. Let Q be the present value of $1, to be paid on the death of a person aged 30, and Q' the present value of $1 to be paid on the death of a person aged 37. Seven years hence the present value of an assurance of $1 on the life of the person now aged 30, will be Q'. The present value of $1, payable certainly at the end of seven years, is v^7. And the probability that the person will not die in seven years, is k^{vii}; hence, the present contingent value of Q' is

$$v^7\, k^{\text{vii}}\, Q';$$

subtracting this from Q, and we have

$$P = Q - v^7\, k^{\text{vii}}\, Q',$$

in which P denotes the immediate payment to secure the assurance of $1 for seven years. The amount may be obtained by the following rule:

Multiply the assurance on a life seven years older than the given life by the present value of $1, payable seven years hence, and by the probability that the given life will survive seven years; subtract the product from the assurance on the given life and the remainder will be the immediate payment necessary to secure an assurance of $1, or the given life for seven years.

To determine the annual premium that must be paid for an assurance on the same life for seven years. The first payment must be made immediately, consequently all the payments after the first are equivalent to a temporary annuity for six years. Designating the present value of such an annuity by A', and one of the required annual payments by P''; the present value of all the payments is $P'' + P''A'$, which must be equal to the expression for the immediate payment just deduced; hence,

$$P''(1 + A') = Q - v^7\, k^{\text{vii}}\, Q',$$

or,

$$P'' = \frac{Q - v^7\, k^{\text{vii}}\, Q'}{1 + A'}.$$

In like manner, the value of an assurance on any number of joint lives may be determined. It is only necessary to replace A in the above formulas, by the value of an annuity on the joint lives. Thus, let M designate the value of an annuity to continue during the joint lives of A and B, then

$$v(1 + M) - M$$

is the value of an assurance to be paid at the end of the year in which the first one dies, and the corresponding annual premium is

$$v - \frac{M}{1 + M}.$$

AS'TRO-LĀBE. [Gr. ἀστηρ, a star, and λαβειν, to take]. An instrument formerly used for taking observations on the altitude of the sun and stars at sea. Also the stereographic projection of the circles of the sphere on the plane of the equator or of a meridian.

A-SYM'ME-TRY. [Gr. α, priv., and συμμετρια, symmetry]. The want of symmetry of a magnitude. The term has sometimes been used as synonymous with incommensurability.

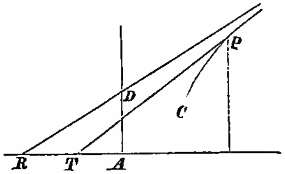
AS'YMP-TOTE. [Gr. α, priv., and συν, with, and πτοω, to fall, not meeting, or coinciding]. A line which continually approaches a curve line, and becomes tangent to it at an infinite distance. Asymptotes are of two kinds, rectilinear and curvilinear.

1. Rectilinear Asymptotes. The equation of a tangent to any plane curve referred to any rectangular axis, is

$$y - y'' = \frac{dy''}{dx''}(x - x'') \cdot (1);$$

in which x'' and y'' denote the co-ordinates

of the point of contact, and $\frac{dy''}{dx''}$ denotes what the general expression for the differential co-efficient of the ordinate of any point of the curve becomes, when for x and y we substitute x'', y''. If in this equation we make $y = 0$, and deduce the corresponding



value of x, it will represent the distance $A\,T$ from the origin to the point in which the tangent cuts the axis of X; in like manner making $x = 0$, and deducing the corresponding value of y; this will represent the distance AD.

Making these substitutions, we deduce

$$A\,T = x'' - y''\frac{dx''}{dy''} \text{ and } A\,D = y'' - x''\frac{dy''}{dx''}.$$

If we have a curve given, we can find the expressions for $A\,T$ and $A\,D$, by differentiating its equation, finding the differential co-efficient of the ordinate and its reciprocal, substituting in these x'' and y'', for x and y, and then substituting the results in the formulas for $A\,T$ and $A\,D$. Then, to ascertain whether the curve has any rectilineal asymptotes, we have simply, in these expressions, to give to x'' and y'' such values as will place the point of contact at an infinite distance; if the values of $A\,T$ and $A\,D$ are not both infinite, there will be at least one asymptote, and the number of asymptotes will be denoted by the number of pairs of real values found for $A\,T$ and $A\,D$.

If the values of both $A\,T$ and $A\,D$ become finite, they may be laid off on the axes, and a straight line drawn through the points R and H, thus determined, will be the asymptote required. If one is finite and the other infinite, there will be an asymptote parallel to the axis on which the distance cut off is infinite, and it may be constructed by laying off the finite distance determined on the proper axis, and through its extremity drawing a straight line parallel to the other axis. If one of the distances is 0, and the other indeterminate, the asymptote will coincide with one of the axes. If both $A\,T$ and $A\,D$ are 0, the asymptote will pass through the origin; to determine its direction, find the value of $\frac{dy''}{dx''}$ when the point of contact, (x'', y''), is at an infinite distance. This will be the tangent of the angle which the asymptote makes with the axis of X; the asymptote may therefore be constructed.

Amongst the conic sections, the hyperbola is the only one which has asymptotes. It has two, both of which pass through the centre, and coincide with the diagonals of a parallelogram described upon any pair of conjugate diameters.

In general, the equation of every curve which admits of a rectilineal asymptote, will, when referred to rectangular axes, and solved with reference to y, be a particular case of the general form,

$$y = ax + b + cx^{-a} + dx^{-\beta} + ex^{-\gamma} + \&c.$$

or else, when solved with respect to x, it will be of the form

$$x = a'y + b' + c'x^{-a'} + d'x^{-\beta'} + \&c.$$

The equation of the asymptote in the first case is $y = ax + b$, and in the second case, it is $x = a'y + b'$, as may be easily shown.

When the curve is referred to a system of polar co-ordinates, the conditions which indicate an asymptote are, that the subtangent should be finite when the radius-vector becomes infinite.

The formula for the subtangent, in the polar system, is $S = r^2\frac{dv}{dr}$, in which S is the subtangent, r and v being the polar co-ordinates of the point of contact.

To apply this in any particular case, differentiate the polar equation of the curve, and find from the differential equation and the given equation, the value of $\frac{dv}{dr}$ in terms of r, and substitute this value in the formula. Then, make $r = \infty$; if the resulting value of S is finite, there will be a rectilinear asymptote to the curve; if not, there will be none.

2. Curvilinear Asymptotes. In order that one curve line may continually approach another, and become tangent to it at an infi-

nite distance, it is necessary that the general expression for the difference between the corresponding ordinates of points of the curves should be of the form

$$y - y' = kx^{-a'} + k'x^{-\beta'} + k''x^{-\gamma'} + \&c.$$

Or else that the general expression for the difference of the corresponding abscissas of the curves should be of the form

$$x - x' = my^{-a''} + m'y^{-\beta''} + m''y^{-\gamma''} + \&c.$$

It is clear, that in the first instance, the distance between the two curves continually diminishes as x is increased, and that this difference becomes 0 when $x = \infty$. In the second case, the distance between the curves diminishes as the value of y increases, and becomes 0 when $y = \infty$. This could not happen if the difference between y and y' contained any term in which the exponent of x was positive; or if the difference between x and x' contained any term in which the exponent of y was positive.

Every curve, therefore, whose equation when referred to rectangular axes, and solved with reference to y, is a particular case of the form

$$y = ax^{a} + bx^{\beta} + \cdot\cdot + kn^{-a'} + k'n^{-\beta'} + \&c.,$$

admits of an infinite number of curvilinear asymptotes. The equation of the asymptotes, when solved with respect to y, must be of the same form as the equation of the curve up to the first term containing x, with a negative exponent. If the equation of a curve be solved with respect to x, it is of the form

$$x = a'y^{a''} + b'y^{\beta''} + \cdot\cdot\cdot\cdot$$
$$+ S'x^{-a'''} + S'x^{-a^{\text{iv}}} +$$

It will also admit of an infinite number of curvilinear asymptotes whose equations, when solved with respect to x, will be of the same form as the given equation up to the first term containing y, with a negative exponent. This case evidently includes all the cases which admit of rectilinear asymptotes.

3. Circular Asymptotes. It may happen, in the case of a spiral, referred to a system of polar co-ordinates, that the value of r is greater than a for every finite value of v, and that it becomes equal to a when $v = \infty$; in this case, the circle described with a radius equal to a, from the eye of the spiral as a centre, lies entirely within the spiral, and is said to be an asymptote to it, or the spiral is said to be an asymptote to the circle: thus, in the spiral whose equation is

$$r^2 - ar = \frac{1}{v^2}$$

for every finite value of v, r is greater than a but for $v = \infty$, $r = a$.

If, however, the equation of the spiral is of the form

$$ar - r^2 = \frac{1}{v^2},$$

then, for every finite value of v, r will be less than a, and the curve will lie entirely within the circle, in which case, the enveloping circle is called an asymptote to the spiral.

AS-YMP-TOT'IC-AL. Partaking of the nature of an asymptote. Two surfaces are said to be asymptotical with respect to each other, when they continually approach each other, and become tangent to each other at an infinite distance. An idea of this relation may be conceived by the consideration of a single case. If we suppose the hyperbola, with its two asymptotes, to be revolved about either axis, the hyperbola will generate an hyperboloid of one nappe, if we revolve about the conjugate axis, and of two nappes if we revolve about the transverse axis; the asymptotes will generate the surface of a cone which will be asymptotical with respect to the surface. If any plane be passed through the axis of revolution, the elements cut from the cone will be asymptotes to the hyperbolas cut from the hyperboloid. In fine, if any plane cuts an hyperbola from the conic surface, it will cut a second one from the other surface which will be an asymptote to it.

AUG-MENT-Ā'TION. [L. *augmento*, from *augeo*, to increase]. The operation of adding or joining one thing to another, so that the result shall be greater than the original thing. In mathematics, augmentation is nearly equivalent to arithmetical addition.

AUX-IL'IA-RY QUANTITY. [L. *auxiliaris*, from *auxilior*, to aid]. A quantity introduced for the purpose of simplifying some mathematical operation. The practice of employing auxiliary quantities in solving groups of equations, is often of great utility. It is

also advantageous to employ auxiliary quantities in the practical applications of trigometrical formulas.

AV'ER-AGE. A term of commerce and navigation, signifying a damage or loss incurred by any part of a ship or cargo, for the preservation of the rest, as when the goods of a particular merchant are thrown overboard in a storm to save the ship from sinking, or when masts, cables, anchors, or other parts of the ship are cut away or destroyed, for the preservation of the whole. In these, and like cases, when any sacrifices are deliberately made, or any expenses voluntarily incurred, to prevent total loss, such sacrifice or expense becomes justly chargeable upon all the parties concerned, and should be rateably borne by the owners of the ship and cargo.

Average is either *general* or *particular*. *General*, when it is chargeable upon all the interests, viz., the ship, the freight and the cargo; and *particular*, when chargeable only upon some of them. When losses occur from ordinary wear and tear, or from perils incident to the voyage, without being *voluntarily* incurred, or when any particular sacrifice is made for the sake of the *ship only*, or the *cargo only*, these losses must be borne by the parties immediately interested. There are also some small charges called *petty*, or *accustomed* averages, one-third of which is usually charged to the ship, and two-thirds to the cargo.

No *general* average ever takes place, unless it can be shown that the danger was imminent, and that the sacrifice was made indispensable, or supposed to be so by the captain and officers of the ship.

The term *average*, more particularly denotes the quota or proportion which each merchant or proprietor is adjudged, upon reasonable estimation, to contribute towards the common loss. In different countries, different modes are adopted for valuing the articles which are to constitute a general average.

In general, however, the value of the freight is held to be the clear sum which the ship has earned, after seamen's wages, pilotage, and all petty charges are deducted, one-third, and in some cases, one-half being deducted for the wages of the crew. The goods lost, as well as those saved, are valued at the price they would have brought in ready money, at the place of delivery on the ship's arriving there, freight duties and other charges being deducted; indeed, they bear their proportions as well as the goods saved. The ship is valued at the price she would bring on her arrival at the port of delivery. But when the loss of masts, cables, and other furniture of the ship are compensated by a general average, it is usual, as the new articles are of greater value than the old, to deduct one-third, leaving two-thirds only to be charged to the amount to be contributed.

The average value of any number of quantities, is equal to their sum divided by their number.

AV-OIR-DU-POIS'. [Fr. *avoir du pois*, to have weight]. The name given to the system of weights, by which coarser commodities are weighed, such as hay, grain, wool, and all the coarser metals. In this system, the terms *gross* and *net* are used. *Gross* is the weight of the goods, including boxes, bags or casks in which they are contained: *net* is the weight of the goods only. A hundred-weight was formerly 112 pounds, but is now reckoned at 100 pounds.

The standard avoirdupois pound of the United States is equivalent to the weight of 27.7015 cubic inches of distilled water at 62° Fah., the barometer being at 30 inches, and the water weighed with brass weights in the air. The following is the scale of the system of weights, viz:

20   4.   25.   16.   16.

That is, 16 drams make 1 ounce, 16 ounces 1 pound, 25 pounds 1 quarter, 4 quarters 1 hundred-weight, and 20 hundred-weight 1 ton. A pound avoirdupois contains 7000 grains, whilst a pound Troy contains but 5760, hence, one pound avoirdupois is equivalent to $1\frac{31}{144}$ pounds Troy.

AX'IOM. [Gr. αξιωμα, authority]. A self-evident theorem or truth. The expression of an axiom is a self-evident proposition. In order that a truth may be ranked as an axiom, it must not only be self-evident, but it must be a necessary truth, not limited to time or place, but universally true at all times, and at all places. Of such a character is the axiom that "a whole is greater than any of

its parts." The axioms of mathematics are very few in number, and have been universally admitted as truths in all ages. Upon the axioms, and the definitions agreed upon, the whole science is based; from these two sources, every general rule or principle of mathematics is deduced by the strict application of the rules of logic. It is for this reason that the truths of mathematics have stood the test of ages. Carrying with them the most convincing evidence, they have received the assent of every one who has taken the trouble to examine the reasoning on which they are established.

Some of the most useful of the axioms employed in mathematical reasoning are these:

1. A whole is greater than any of its parts.
2. A whole is equal to the sum of all its parts.
3. Things which are equal to equal things are equal to each other.
4. Things which are like parts of equal things are equal to each other.
5. If equals be multiplied or divided by the same quantity, the products or quotients will be equal.
6. If equals be added to equals the sums will be equal.
7. If equals be subtracted from equals the remainders will be equal.
8. The like powers of equals are equal.

An axiom differs from a postulate in the same manner that a theorem differs from a problem, an axiom being a self-evident theorem, that is, one which only needs to be stated to secure the immediate assent of every mind; while a postulate is a self-evident *problem*, that is, a problem whose solution is so obvious as to be at once admitted.

AX-I-O-MAT'IC. Pertaining to an axiom.

AX-I-O-MAT'IC-AL-LY. By the use of axioms, by means of axioms.

AX'IS. [L. *axis*, axletree; Gr. *αξων*, an axle]. A straight line with respect to which the different parts of a magnitude are symmetrically arranged. This appears to be the true meaning of the term axis, but it is often employed in a different sense. We shall point out some of the leading uses to which the term has been applied.

AXIS OF SYMMETRY. In elementary geometry the axis of symmetry of any figure, is a straight line which bisects a system of parallel lines terminating in the boundary of the figure, and the figure is said to be symmetrical with respect to this axis. In a regular polygon, every straight line which bisects a side and is perpendicular to it, is an axis of symmetry. Every straight line which bisects an angle of the polygon, is also an axis of symmetry. Hence, every regular polygon has an axis of symmetry corresponding to each side and each angle.

These principles hold when the number of sides of a polygon becomes infinite; hence, every diameter of a circle is an axis of symmetry.

AXIS OF A PYRAMID OR CONE is a straight line joining the vertex and the centre of the base.

AXIS OF A PRISM OR CYLINDER is a straight line joining the centres of its parallel bases.

AXIS OF REVOLUTION, in descriptive geometry, is a straight line about which some line or plane is revolved; that is, moved in such a manner that all the points of the moving line or plane shall describe the circumferences of circles, whose centres are on the fixed line, and whose planes are perpendicular to it.

In spherical projections, *the axis of a sphere* is the straight line about which it revolves.

In surveying, the *axis of an instrument*, or part of an instrument, is the straight line which remains fixed, whilst the whole or a part of the instrument is revolved. Thus, in the theodolite, *the axis of the instrument* is the line which remains fixed, whilst the whole instrument is revolved upon its support; the axis of the vertical limb is the line which remains fixed, when the vertical limb alone is revolved; and the axis of the telescope is the line which remains fixed, when the telescope is revolved in the Y's.

In analytical geometry, the *axis of a curve* is a straight line which bisects a system of parallel chords of the curve, which are perpendicular to it. This corresponds to the axis of symmetry of polygons. The parabola has but one axis, the ellipse and hyperbola have each two axes, the circle has an infinite number of axes.

AXES OF CO-ORDINATES IN A PLANE, are straight lines intersecting each other, to which points are referred for the purpose of deter-

mining their relative position. They may be rectangular or oblique.

Axes of Co-ordinates in Space, are the straight lines in which the co-ordinate planes intersect each other. The co-ordinates of points are measured on lines parallel to their axes.

When a circle is spoken of with reference to the sphere on which it lies, its axis is that diameter of the sphere which is perpendicular to its plane: thus we speak of the axis of the equator, the axis of the ecliptic, &c.

AZ'I-MUTH ANGLE. In surveying, the angle included between the meridian and a vertical plane, at a point. The azimuth of a line lying in a vertical plane, at any point, is the angle between that plane and the meridian plane passing through the point. If the line does not lie in a meridian plane, its azimuth at different points will be different.

The excess of the azimuth, at one end of the line, over that at the other, is called the *convergence of the meridians.*

In geodesic operations, the azimuths of the principal lines of the survey form very important elements, and are determined with great accuracy, usually by the aid of astronomical observations.

When several lines meet at a station, as is usually the case, the azimuth of some fixed line is determined by astronomical observation, and then the azimuths of each of the lines will be ascertained by measuring the angles between them and the fixed line. It is customary to estimate the azimuth of a line from the south point of the horizon around by the west from 0° to 360°. If we adopt this convention, the *convergence of the meridians* corresponding to two given stations, will be the azimuth of the eastern station, as seen from the western diminished by 180° plus the azimuth of the western, as seen from the eastern station.

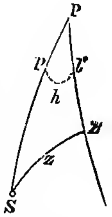
To determine the azimuth of a line by observation upon a star, an altitude and azimuth instrument, or a large theodolite may be used. The instrument is properly leveled over the point at which the observation is to be made, and the middle vertical wire is brought to coincide with the star and the sidereal time of the observation noted. After the reading of the horizontal limb has been taken, the telescope is turned upon a distant signal, lighted by a lamp, and the reading of the horizontal limb again taken.

This operation ought to be repeated a great number of times. Each observation made affords us the means of determining the azimuth of the line joining the station and the signal lighted by the lamp, and by taking a mean of a great number of results, the true azimuth of the line can be found with sufficient accuracy. The difference of the readings on the horizontal limb, gives the difference of azimuth between the signal and the star at the moment of observation. The azimuth of the star, at the instant of observation, may be computed by means of these formulas:

$$\tan \tfrac{1}{2}(z+s) = \cot \tfrac{1}{2}h \frac{\cos \frac{1}{2}(p-l')}{\cos \frac{1}{2}(p+l')}$$

$$\tan \tfrac{1}{2}(z-s) = \cot \tfrac{1}{2}h \frac{\sin \frac{1}{2}(p-l')}{\sin \frac{1}{2}(p+l')}$$

$$A = 180^\circ - z = 180^\circ - [\tfrac{1}{2}(z+s) + \tfrac{1}{2}(z-s)].$$



In which A is the azimuth of the star estimated from the south point of the horizon, h the hour angle, l' the complement of the latitude of the place, and p the star's north polar distance. h is positive when the star is on the west of the meridian, and negative when on the east. Great care should be taken to give the elements, in these formulas, their proper signs. It should be observed that if the star is below the pole, the hour angle is greater than 90°.

It is generally found most convenient to use the pole star (α polaris) in determining the azimuth of a line, because, if the observations are made about the time of its greatest eastern or western elongation, there is less liability to error, and the formulas for determining the azimuth admit of a simplification, which renders the computation more easy.

Having determined the azimuth of one fixed line through the station, the azimuths of all other lines can be determined by simply measuring the angles included between them and the fixed lines, and adding them to or subtracting them from the known azimuth, according as their azimuths are greater or less than that of the fixed line.

If we know the latitudes and longitudes of two stations on the surface of the earth, the azimuth of the second, as seen from the first, may be computed by the following formulas :

$$\frac{\beta}{2} = \frac{\varepsilon^2(L - L')\cos^2\frac{1}{2}(L + L')}{2}$$

$$l = L - \frac{\beta}{2} \qquad x'' = (M' - M)\cos l'$$

$$l' = L' + \frac{\beta}{2} \qquad y'' = (l - l') - \tfrac{1}{2}\sin 1'' x''^2 \tan l$$

$$\cos A = \frac{x''}{y''}.$$

In which L and M represent the latitude and longitude of the first station ; L' and M' of the second station ; ε the eccentricity of the meridian ($\varepsilon^2 = 0.00667435$), and A the azimuth reckoned from the south point of the horizon.

Particular attention must be paid to the sign of $L - L'$, for upon this depends the sign of $\frac{\beta}{2}$, and also to that of $(l - l'')$, in the value of y'', so as to know whether the small quantity, $(-\frac{1}{2}\sin 1'' x''^2 \tan l)$ is to be added to, or to be subtracted from $(l - l')$.

When the latitudes of the stations are known, and the distance between them is also known, the azimuth may be computed by means of the following formulas

$$\frac{\beta}{2} = \frac{\varepsilon^2(L - L')\cos^2\frac{1}{2}(L + L')}{2}$$

$$\left.\begin{array}{l} l = L - \dfrac{\beta}{2} \\ l' = L' + \dfrac{\beta}{2} \end{array}\right\} \cdots \frac{\sin l'}{\sin l} = \cos l''$$

$$\cos A = \frac{2\tan l \sin\frac{1}{2}(u + l'')\sin\frac{1}{2}(u - l'')}{\sin u}.$$

In which L, L', and A, are the same as in the previous formulas, and u denotes the distance between the stations in seconds of arc. If the distance between the stations is in yards, it may be converted into seconds of arc by means of the following formulas,

$$N = \frac{a}{\left\{1 - \varepsilon^2 \sin^2\frac{1}{2}(L + L')\right\}^{\frac{1}{2}}}$$

$$u'' = \frac{K}{N \sin 1''}.$$

In which K denotes the distance in yards, N the normal at the middle latitude, a the equatorial radius, and ε the eccentricity of the meridian ; L, L' are the same as in the previous formulas.

In ordinary surveying, the azimuth of some fixed line is determined for the purpose of finding the variation of the needle. Having the azimuth of a fixed line, the variation of the needle can be determined by taking the bearing of the line. If the azimuth is estimated from the north point of the horizon, as it usually is in plane surveying, the difference between the bearing and azimuth is the variation of the needle.

Magnetic Azimuth. The magnetic azimuth of a line is the angle included between the meridian and a vertical plane through the line. When the magnetic azimuth of a line is known, together with the variation of the needle, the true azimuth may at once be determined.

Azimuth Compass, is a small compass, employed for determining magnetic azimuths of a heavenly body, for the purpose of determining the variation of the needle. It is used in surveying, and in making reconnaissances.

Azimuth Dial—a dial whose style or gnomon is perpendicular to the plane of the horizon.

B. The second letter of the alphabet. It was used by the Hebrews and Greeks as a numeral, and denoted 2. The ancient Romans also employed it to denote 300 : with a dash over it, thus, $\bar{\mathrm{B}}$, it denoted 300,000.

BACK'SIGHT. [*Back* and *sight*]. In leveling, the first reading of the leveling staff taken from any position of the level. All the other readings are called *fore-sights*. The back sight at any station, diminished by the fore-sight at the last station, the leveling-staff being in the same position, is equal to the difference of level between the axis of the instrument at the two stations. If the back-sight is greater than the corresponding fore-sight at the last station, the axis of the level at the new station is *higher* than at the primitive one ; if it is less, it is lower, and the result is shown by the negative sign with which the difference of level is effected.

BACK'STĂFF. [*Back* and *staff*]. An instrument formerly used for measuring the

altitude of the heavenly bodies, but since superseded by the quadrant and sextant. It is named from the fact that the observer stood with his back to the object whose altitude was to be measured.

BAC-U-LOM'E-TRY. [L. *baculus*, a staff, and Gr. $\mu\epsilon\tau\rho o\nu$, measure]. The operation of measuring distances or altitudes by means of a staff or staves.

BAL'ANCE OF TRADE. In mercantile language generally expresses the difference between the amount of exports from, and the imports to a country. It is said to be *favorable* when the amount of exports exceeds that of the imports, and *unfavorable* when the amount of imports exceeds that of the exports.

BAL'ANCE SHEET. A formal statement of the affairs of a business house.

BAL'ANCING THE WORK. In surveying, is the operation of distributing the errors of eastings, westings, northings, and southings, so that the sum of the eastings shall be equal to the sum of the westings, and the sum of the northings equal to the sum of the southings.

The principle on which the distribution is made is, that the amount of error committed is proportional to the length of the course. The rule for distribution is this: Find the sum of the lengths of all the courses; find also the sum of the northings and the sum of the southings, separately; subtract the latter from the former. Find also the sum of the eastings and the sum of the westings, separately. Subtract the latter from the former; the first remainder is the error in latitude, the second in departure. Then, as the sum of the courses is to any course, so is the error in latitude to the correction in latitude for that course. Having found the correction in latitude for each course, it is to be applied to the northing or southing of each course as follows:

If the error in latitude is $+$, and the course makes northing, it is to be subtracted; if southing, it is to be added. If the error in latitude is $-$, and the course makes northing, it is to be added; if it makes southing, it is to be subtracted.

The corrections for departure are found and applied in a similar manner.

BANK DISCOUNT is the charge made by a bank for the payment of money on a note before it becomes due. By the custom of banks, this discount is the interest on the amount named in the note, to be paid in advance, from the time the note is discounted to the time when it falls due, which time always includes the three days of grace. The present value of the note is the difference between the amount named in the note and the discount. The amount named in the note is called the face of it.

There are two kinds of notes discounted at banks: 1st. Notes given by one individual to another, for property actually sold; these are called *business notes* or *business paper*. 2d. Notes made for the purpose of borrowing money, which are called *accommodation notes*, or *accommodation paper*. The first kind is much preferred by banks, as more likely to be paid when they fall due, or in mercantile phrase, "when they come to maturity."

To find the bank discount on a note, *add three days* to the time which the note has to run before it becomes due, and calculate the interest for this time at the given rate per cent.

BAN'KING. Banks are corporations created by law for the purpose of receiving deposits, loaning money, and furnishing a paper circulation based upon a basis of specie. The operation of conducting the business of a bank is called banking. When a private individual is engaged in the business of banking, or any of its principal branches, he is called a *banker*.

Banks are of various kinds. Some confine their business entirely to the care and issue of money deposited with them by their customers, whilst others issue notes or paper money of their own.

Notes issued by a bank circulate as money, being based upon specie contained in the vaults of the bank, and the bank being obligated to redeem them in specie whenever presented for payment. Such notes are called *bank notes* or *bank bills*.

The following account of notes is in a great measure applicable to the operation of banking.

A note is a written agreement to pay a certain sum of money under certain conditions either specified or implied.

The note of an individual, or as it is generally termed, a promissory note, or note of hand, is a positive agreement in writing to pay a given sum at a specified time, and to a person named in the note, or to his order, or sometimes to the bearer at large. The following are some of the forms of notes:

(1). *Negotiable Note.*

$25,50. West Point, N. Y., Dec. 21st, 1854.

For value received I promise to pay on demand to John Smith or order, twenty-five dollars and fifty cents.

AMOS JONES.

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(2). *Note Payable to Bearer.*

$875,39. Cold Spring, N. Y., March 4th, 1855.

For value received I promise to pay, six months after date, to Smith Johnson, or bearer, eight hundred and seventy-five dollars and thirty-nine cents.

PETER PIERCE.

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(3). *Note by two Persons.*

$20,18. Catawissa, Pa., June 3, 1855.

For value received we jointly and severally promise to pay to Richard Survella or order on demand, twenty dollars and eighteen cents.

JOHN BUNDING.
BUNDY ALSTONE.

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(4). *Note Payable at a Bank.*

$187. Newburgh, N. Y., July 1, 1854.

Sixty days after date I promise to pay Henry Bundy or order, at the Highland Bank, in the village of Newburgh, one hundred and eighty-seven dollars, for value received.

SIMEON SIMPSON.

The person who signs a note is called the *drawer* or *maker* of the note: thus, Amos Jones is the drawer of Note No. 1.

The person who has rightful possession of a note is called the *holder* of the note.

A note is said to be *negotiable*, when it is made payable to A. B., or order, who is called the *payee* (see No. 1). Now, if John Smith, to whom this note is made payable, writes his name on the back of it, he is said to indorse the note, and is called the *indorser*; when the note becomes due, the holder must first demand payment of the maker, Amos Jones, and if he declines paying it, the holder may then require payment of John Smith, the indorser.

If the note is made payable to A. B., or bearer, then the drawer alone is responsible, and he must pay to any person who holds the note

The time at which a note is to be paid should always be named, but if no time is specified, the drawer must pay when required to do so, and the note will draw interest after the payment is demanded.

When a note, payable at a future day, becomes due, it will draw interest, though no mention is made of interest.

In each State, there is a *rate* of interest established by law, which is called legal interest, and when no rate is specified, the note will always draw legal interest. If a rate higher than legal interest be taken, the drawer, in most States, is not bound to pay the note.

If two persons jointly and severally give their note (No. 3.), it may be collected of either of them.

The words, *for value received*, should be expressed in every note.

When a note is given, payable on a fixed day, and in a specific article, as in wheat or rye, payment must be offered at the specified time, and if it is not, the holder can demand the value in money.

By mercantile usage, and the custom of banks, a note does not really fall due until the expiration of three days after the time mentioned on its face: for example, Note No. 2 would fall due September 7; the three additional days are called *days of grace.*

When the last day of grace falls on Sunday, or a holiday, as the Fourth of July, New Year's day, &c., the note must be paid on the preceding day, that is, on the second day of grace.

BAR'REL. [Fr. *baril*, a cask]. A unit of liquid measure, which differs in value for the different articles measured. The English beer barrel contains 36 gallons, or 9981.864 cubic inches. The English wine barrel contains $31\frac{1}{2}$ gallons, or 8734.131 cubic inches.

BÄR'TER. [Sp. *baratar*, to traffic]. A rule of arithmetic which treats of the exchange of one commodity for another.

BĀSE. [L. *basis;* Gr. *βασις*]. Of a plane figure, a side upon which it is supposed to stand. In a triangle, the base lies opposite the angular point chosen as the vertex.

The base of a conic or cylindrical surface, is the intersection of the surface by a plane. We see that these surfaces may have an infinite number of bases. The base of a polyhedron, is a plane face on which it is supposed to stand. In the pyramid, the base is opposite the vertex.

Base of a System of Numbers, is the value of a unit of the first order. In all systems of abstract numbers the base is the same, being the abstract number 1. In collections of denominate numbers, the base is 1 thing of the kind numbered.

Base of a System of Logarithms, is that number, the exponents of whose powers constitute the logarithms of the system. In the common system the base is 10, and logarithms in the common system are the exponents of the different powers of 10.

In the Naperian system the base is 2.71828.

Any number may be taken as the base of a system of logarithms, but the two mentioned are the only ones that are much used.

Base Apparatus. A combination of bars, arranged on the compensating principle, used in the measurement of the base line of a trigonometric survey. An outline account of a base apparatus used by Col. Colby, in measuring a base of more than seven miles in extent, in Ireland, will be found under the head of *Geodesy*.

The following account of the base apparatus, in its most approved form, as now used on the coast survey, is extracted from the "proceedings of the American Philosophical Society." It forms a part of a paper read before the society, by Prof Bache, and taken in connection with the article referred to, will give a sufficiently accurate idea of the general arrangement of the base apparatus in its most perfect form.

"The base apparatus presented some novel features in construction, the adaptation of others not hitherto used in field work, and a choice of parts previously used by others. The general plan was devised by me, and the details by Mr. Wm. Wurdeman, mechanician of the coast survey, by whom they were executed, under my direction. The following are the general features of the apparatus: 1. The measuring bars were upon the compensating system first used, I believe, by Colonel Colby in Great Britain, and by Mr. Borden in the United States, but the mode of obtaining the compensation differed entirely from that used by either of these gentlemen. 2. A principle was introduced in reference to the dimensions of the bars which, if at all recognized, has not been hitherto applied. A bar of brass and a bar of iron, of the same dimensions, exposed to the same source of heat, will not heat equally in equal times; this is well known to depend upon the different conducting powers of the two metals, their different specific heats, and the different powers of their surfaces to absorb heat. The bars, then, if of equal sections, when the temperature is rising or falling, have not the same temperature, and the system is not compensating. The surfaces are easily made to absorb equally by the same coating, and the sections must be so proportioned to each other that the bars will have the same temperature when exposed to variable temperatures of the atmosphere and of the case containing them. Having arranged the sections approximately, using numbers taken from the books, the changes, in length, during increase or decrease of temperature, were not perceived when microscopes were used supported upon wooden stands, or even upon stone blocks of small size; the means of measurement were not sufficiently delicate to perceive them, or they were masked by greater changes in the supports. When the level of contact was substituted for the microscopes, or when Mr. Saxton's reflecting pyrometer was employed, these changes became very perceptible, and it was necessary to resort to direct experiment upon the materials of the bars themselves to obtain even approximate results, and then to correct a small residual quantity by applying a covering more absorbent of heat to one bar than to the other. If such changes have not been perceived hitherto, it has been because adequate means were not used to detect them. 3. The lever of contact and level, first used, I believe, in the adjustment of standard measures by Bessel, was applied to indicate the lengths of the bars. The levels were so delicate that several divisions upon them

made up a quantity entirely insignificant in the measurement. The doubt which I had was, whether the sensibility of the apparatus had not been carried too far; this was, however, entirely removed upon finding the rapidity and certainty with which it could be used. The contact between two adjacent measures was between a blunt knife edge and a plane of agate. 4. The trussed support for the bars adapted to bearing the apparatus at two points only, and the tin covering or tube which surrounded the whole, were similar to those used by Mr. Borden, but differed entirely in the adaptation of them; the bars moved freely on the trussed frame upon rollers, and were not attached to the covering tube in which the trussed frame itself was merely supported. The tin covering was conical, and was doubled. 5. The trestles admitted of the various motions required in placing the apparatus, and the length of the whole about 20 feet (six metres), gave a weight which permitted easy and rapid transfer by four men, when covered with several thicknesses of imperfectly conducting material to keep the fluctuations of temperature within moderate limits. The contacts were usually made in much less time than the setting of the forward trestles for the measure."

Base Line. In Geodesic Surveying, a line measured with the greatest possible accuracy, upon which the whole system of triangulation depends. The measurement of a base line in a large survey, is one of the most difficult operations of Geodesy, and one whose successful accomplishment has called for the highest attainments both in science and art. Various methods have been adopted to attain accuracy, such as using rods of platinum or glass, steel rods, and finally a combination of rods of different metals, so adjusted that the combination shall retain a uniform length for all changes of temperature. A most successful application of this principle has been made by Dr. Bache, Superintendent of the United States coast survey. See *Base Apparatus*.

In the survey of the coast of the United States, several base lines have been measured, usually five or six miles in length. With the improved apparatus, about two miles per week can be measured with great accuracy.

In minor surveys, where great accuracy is not so necessary, rods of deal, capped with metal, and thoroughly saturated with oil, so as to resist hygrometric changes, answer very well, and by measuring the base line very carefully two or three times, sufficient accuracy may be attained.

There are some corrections that require to be applied to the measurements made in certain cases.

1st. When the measured line is inclined, or when a rod in any position is inclined, to reduce the oblique distance to a corresponding horizontal distance. Let B denote the length of the oblique line, b the corresponding horizontal distance, and θ the angle of inclination, supposed very small, then is

$$b = B\cos\theta.$$

But as θ is a very small angle, we may compute the excess of B over b, and for this purpose let us suppose that the angle θ is expressed in minutes. We have the formula

$$B - b = B(1-\cos\theta) = 2\,B\sin^2\tfrac{1}{2}\theta$$
$$= \tfrac{1}{2}B\,\theta^2\sin 1' = \frac{\sin^2 1'}{2}\,\theta^2 B;$$

or,

$$B - b = 0.00000004231 \times \theta^2 . B.$$

Whence, by applying logarithms,

$$\log(B-b) = \bar{8}.626422 + 2\log\theta + \log B.$$

2d. When the base line has been measured on elevated ground, to reduce it to the level of the sea. Let r denote the radius of the earth corresponding to the base b at the level of the sea, and $r + a$ the radius of the earth at the level of the measured base B. Then will

$$B - b = B - B\frac{r}{r+a} = B\left(\frac{a}{r} - \frac{a^2}{r^2} + \&c.\right);$$

but since r is very great in comparison with a, all of the terms of this series, except the first, may be neglected; whence

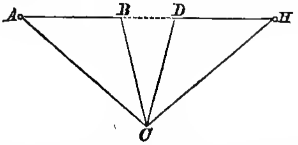
$$B - b = \frac{Ba}{r}.$$

If the measured line is inclined or broken $(r + a)$ may be taken equal to its mean value along the base.

3d. When a portion of the base line cannot be measured directly, on account of some intervening obstacle, as a river or marsh.

Let AH represent the entire base line, and suppose that BD cannot be measured directly on account of an obstacle.

Select a station C from which the points A, B, D and H, can be seen, and from this station measure carefully the angles ACB, which denote by α; ACD, which denote by



β, and the angle ACH, which denote by γ. Denote also the measured portions of the base, AB, by a, and DH by b. The value of BD may be found by means of the following formulas: Let ϕ denote an auxiliary unknown quantity, and we have

$$\tan^2\phi = \frac{4\,a\,b}{(a-b)^2}\cdot\frac{\sin\beta\,\sin(\gamma-\alpha)}{\sin\alpha\,\sin(\gamma-\beta)},$$

$$BD = -\frac{a+b}{2} \pm \frac{a-b}{2\cos\phi}.$$

4. When the two parts of a measured base make an angle with each other, which is nearly equal to 180°. Let θ denote what the angle wants of being equal to 180°, θ being expressed in minutes,

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2,$$

since θ is very small. Let a and b denote the measured parts, including the angle. Then the true length of the base will be equal to

$$a + b - \frac{\sin^2 1'}{2}\cdot\frac{a\,b\,\theta^2}{a+b},$$

$$= a + b - 0.00000004231 \times \frac{a\,b\,\theta^3}{a+b}.$$

The above formulas are taken from a volume of tables and formulas, published by Capt. T. J. Lee, U. S. corps of Topographical Engineers. See *Geodesy*.

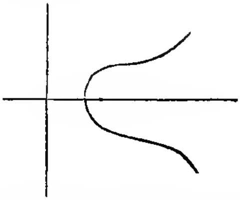
BAT′TER. [Fr. *battre*, to batter]. The slope of a wall, or the inclination of its surface to the horizon.

BEAM COMPASS. See *Compass*.

BEĀR′ING. In Surveying, the angle included between the plane of the meridian and a vertical plane, through a given course. The bearing of one station from another is the angle between the vertical plane through the two stations, and the meridian plane through the instrument. There are two kinds of bearings, the *magnetic* and the *true* bearing. The first is the angle between the plane of the magnetic meridian and the vertical plane through the course. This does not change sensibly for the same course. The second is the angle included between the plane of the true meridian and the vertical plane through the course. This for the same course is invariable. The bearing is estimated from the north or south points of the horizon, around towards the east or west through 90°. The magnetic bearing is estimated from the magnetic north or south points, the true bearing from the true north or south points.

BELL-SHAPED PARABOLA. A parabola having the form of a bell given by the algebraic equation

$$a\,y^2 - x^3 + b\,x^2 = 0.$$



BERNOUILLI'S SERIES. A formula deduced by Bernouilli for developing an integral of a differential of a function of one variable into a series. The formula is as follows:

$$\int X\,dx = Xx - \frac{dX}{dx}\cdot\frac{x^2}{1.2} + \frac{d^2X}{dx^2}\cdot\frac{x^3}{1.2.3} - \frac{d^3X}{dx^3}\cdot\frac{x^4}{1.2.3.4} + \&c.,$$

in which X is any function of x.

BEV′EL-ED. [Fr. *buveau*]. The edge of a ruler, or of the limb of an instrument, is said to be beveled when its cross section is acute angled.

BĪ-AN″GU-LĀR. [L. *bis*, twice, and *angulus*, an angle]. Having two angles. Thus a lune is biangular.

BILL OF EXCHANGE. An order drawn on a distant person, directing him to pay a sum of money to a specified person, or to his order, in consideration of the same sum received by the drawer. See *Exchange*.

BILL′ION. [L. *bis* and *million*]. A thousand millions. In the decimal system, a unit of the tenth order.

BĪ-MĒ'DI-AL. [L. *bis*, twice, *medial*, middle]. In Geometry, when two lines are commensurable only in power (as the side and diagonal of a square, for instance) are added together, and the sum is incommensurable with respect to either, the sum is called by Euclid a bimedial.

BĪ'NA-RY. [L. *binus*, two and two]. A binary number is one expressed by two figures.

BINARY SCALE. In Arithmetic, a uniform scale, whose ratio is 2. See *Arithmetical Scale.*

BINARY ARITHMETIC is that in which numbers are expressed according to the binary scale. Leibnitz perfected such a system, but it is rather curious than useful.

BINARY COMBINATION. A combination by pairs or by twos.

BĪ-NŌ'MI-AL. [L. *bis*, twice, and *nomen*, name]. In Algebra, an expression consisting of two terms connected by the sign $+$ or $-$; a binomial is a polynomial of but two terms. See *Polynomial.*

BINOMIAL DIFFERENTIAL. Any power of a binomial function of one variable, multiplied by the differential of that variable. Every binomial differential may be reduced to the form of

$$x^{m-1}(a + bx^n)^p\,dx;$$

in which m and n are whole numbers, and n positive, p being either positive or negative, entire or fractional.

When a binomial differential has been reduced to the above form, it may be integrated in either of the following cases: 1. When the exponent of the parenthesis is a whole number; 2. when the exponent of the variable, without the parenthesis, plus 1, is exactly divisible by the exponent of the variable within the parenthesis; 3. when this quotient, plus the exponent of the parenthesis, is a whole number.

When neither of these conditions are fulfilled, it may be reduced to such a form as to render it integrable by the aid of formulas *A*, *B*, *C*, *D* and *E*. See *Formula.*

BINOMIAL EQUATION. An equation which can be reduced to the form $x^m - a = 0$, in which m is a positive whole number, and a any known quantity whatever, positive or negative, real or imaginary. The form of a binomial equation may be still farther simplified; for, if we denote an m^{th} root of a by a', and make $a'y = x$, whence $ay^m = x^m$, we shall have, by substitution,

$$ay^m - a = 0, \text{ or } y^m - 1 = 0.$$

In the discussion of binomial equations, we shall, therefore, consider them as reduced to the form $y^m - 1 = 0$.

1. If m is an odd number, the equation will have one real root, equal to 1, since $y = 1$ satisfies the equation; the remaining roots are all imaginary, for if we substitute for y a number greater than 1, the first member will be greater than 0; if we substitute for y any number less than 1, either positive or negative, the first member will be less than 0; and consequently, the equation will not be satisfied in either case; hence, there can be no real root except 1.

2. If m is an even number, both $+1$ and -1 are roots of the equation, since both will satisfy it when substituted for y; all the other roots are imaginary, as may be shown by a course of reasoning similar to that employed above.

3. If r is a root of the equation, then will any power of r be a root of the equation also; for, if r is a root, we have $r^m = 1$; hence, by raising both members to any power denoted by n, we shall have $r^{mn} = 1$; substituting this in the given equation, gives

$$y^m - r^{mn} = 0, \quad \text{whence} \quad y = r^n,$$

which proves the proposition. We see, therefore, that the roots of the equation $y^m - 1 = 0$ may have an infinite variety of forms, $r, r^2, r^3, r^4, r^5, \ldots r^{-1}, r^{-2}, r^{-3} \ldots$ &c.; but amongst all these forms there can only be m essentially distinct expressions, since the equation can only have m different roots.

It may also be shown that of the m roots, no two are equal.

4. If m is an odd number, the algebraic sum of all the imaginary roots is -1; for the algebraic sum of all the roots is equal to the co-efficient of the second term taken with a contrary sign, which in this case is 0; but since the real root is 1, the sum of all the remaining roots, which are imaginary, must be equal to -1.

If m is an even number, the sum of all the imaginary roots is 0, as may be shown in a manner analogous to the above

5. If m is odd, the continued product of all the imaginary roots is equal to $+1$. This follows from the property of an equation, that the product of all its roots is equal to the absolute term taken with a contrary sign, which in this case is $+1$; since the real root is $+1$, the product of all the imaginary roots is $+1$.

In like manner, it may be shown that when m is even, the continued product of all the imaginary roots is equal to $+1$.

For similar reasons, the algebraic sum of all the roots, taken in sets of $2, 3, 4, \cdots n$, (n being less than m) is equal to 0.

6. Finally, all the roots of the equation $y^m - 1 = 0$, can be found from the formula

$$y = \cos\frac{2k\pi}{m} \pm \sin\frac{2k\pi}{m}\sqrt{-1};$$

in which k is either 0, or a positive whole number, and $\pi = 180°$. The roots are found by making, in succession,

$k = 0$, $k = 1$, $k = 2$, $\cdots$ &c., up to $k = \frac{m-1}{2}$ inclusively, if m is an odd number, and up to $\frac{m}{2}$, if m is even. All the expressions obtained for y, up to these limits, are different from each other; but if we continue to substitute for k the consecutive whole numbers beyond these limits, the same expressions continue to recur, and in the same order; a result which might have been expected from what was stated in the third paragraph of the present article.

For example, suppose that we have the equation $y^4 - 1 = 0$: making $k = 0$, $m = 4$, we have $y = +1$; making $k = 1$, $y = \pm\sqrt{-1}$, or, $y = +\sqrt{-1}$, and $y = -\sqrt{-1}$; making $k = 2$, $y = -1$; hence, the four roots are

$$+1, \quad -1, \quad +\sqrt{-1}, \quad -\sqrt{-1}.$$

In the equation $y^3 - 1 = 0$, making $m = 3$ and $k = 1$, we have $y = 1$; making $k = 2$,

$$y = \cos\frac{2\pi}{3} \pm \sin\frac{2\pi}{3}\sqrt{-1}; \qquad \text{but}$$

$\cos\frac{2\pi}{3} = \cos 120° = -\sin 30° = -\frac{1}{2}$, and

$\sin\frac{2\pi}{3} = \sqrt{1-\frac{1}{4}} = \frac{1}{2}\sqrt{-3}$; hence

$y = \frac{-1\pm\sqrt{-3}}{2}$, and the three roots are

$$1, \quad \frac{-1+\sqrt{-3}}{2}, \quad \frac{-1-\sqrt{-3}}{2}.$$

If it is required to solve a binomial equation of the form $x^m - a = 0$, we have simply to solve the equation $y^m - 1 = 0$ by means of the above formula, and then multiply each of the m roots found by the numerical value of the m^{th} root of a; the results will be the required roots.

If, in the above examples, we go on substituting for k consecutive whole numbers, we shall find that the roots already found continually recur in their proper order.

Binomial Formula. A formula which expresses the law of formation of any power of a binomial. It is as follows:

$$(x+a)^m = x^m + max^{m-1} + m\cdot\frac{m-1}{2}a^2x^{m-2}$$
$$+\frac{m(m-1)}{2}\frac{(m-2)}{3}a^3x^{m-3} + \&c.$$

In which x and a represent any two terms, and m any known quantity either positive or negative, entire or fractional, real or imaginary. The form of the development is entirely independent of the value of m.

The second member is a series whose law is evident from simple inspection. First, the exponent of x, in the first term, is equal to the exponent of the power, and goes on *decreasing* by 1 in each term to the right. The exponent of a is 0 in the first term, and goes on *increasing* by 1, in each term to the right. The co-efficient of the first term is 1, that of the second term is the exponent of the power; and in general, the co-efficient of any term is equal to the co-efficient of the preceding term multiplied by the exponent of x in that term, and divided by the number of preceding terms.

From this law of the series it is plain that when m is a positive whole number, the number of terms in the development is finite, and one greater than the exponent of the power. When m is negative or fractional, the number of terms is infinite, and the series will only give an approximate value for the expression when it is convergent.

1. The binomial formula may be used to develop any power of a binomial; thus let it be required to raise the binomial

$$3a^2c - 2bd$$

to the fourth power.

Placing $3a^2c = x$ and $-2bd = a$, we have $(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$; and replacing x and a by their values, we have

$$(3a^2c - 2bd)^4 = 81a^8c^4 - 216a^6c^3bd + 216a^4c^2b^2d^2 - 96a^2cb^3d^3 + 16b^4d^4.$$

2. Any power of a polynomial may be developed by the use of the formula, as in the following example:

Let it be required to find the third power of $2a^2 - 4ab + 3b^2$.

Placing $2a^2 = x$, and $-4ab + 3b^2 = d$, the formula gives

$$(x+d)^3 = x^3 + 3dx^2 + 3d^2x + d^3.$$

finding the values of d^2 and d^3 by means of the formula, and substituting for x, d, d^2, and d^3, their values, we have

$$(2a^2 - 4ab + 3b^2)^3 = 8a^6 - 48a^5b + 132a^4b^2 - 208a^3b^3 + 198a^2b^4 - 108ab^5 + 27b^6.$$

3. A modified form of the binomial formula is used for finding an approximate root of a number.

If in the formula

$$(x+a)^m = x^m\left(1 + m\frac{a}{x} + m\cdot\frac{m-1}{2}\cdot\frac{a^2}{x^2} + m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\frac{a^3}{x^3} + \ldots\right),$$

we make $m = \frac{1}{n}$, and reduce, we find

$$\sqrt[n]{x+a} = \sqrt[n]{x}\left(1 + \frac{1}{n}\cdot\frac{a}{x} - \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2}{x^2} + \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3}{x^3} + \cdots\right).$$

The fifth term within the parenthesis may be found by multiplying the fourth $\frac{3n-1}{4n}$ and by $\frac{a}{x}$, and changing the sign of the result, and so on.

To apply this formula to find the cube root of 31. Let $x = 27$ and $a = 4$; substituting these in the formula, and writing 3 in the place of n, it becomes

$$\sqrt[3]{31} = 3\left(1 + \frac{1}{3}\cdot\frac{4}{27} - \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{16}{729} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{5}{9}\cdot\frac{64}{19683} - \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{5}{9}\cdot\frac{2}{3}\cdot\frac{256}{531441} + \&c.\right);$$

or, by reducing,

$$\sqrt[3]{31} = 3 + \frac{4}{27} - \frac{16}{2187} + \frac{320}{531441} - \frac{2560}{43046721} + \&c.\Big) = 3.14138+$$

which is exact to within less than .00001: all similar cases may be treated in like manner. Hence, we can approximate to the value of any root of a number by means of the binomial formula, by the following rule:

Find the perfect power of the degree indicated, which is nearest to the given number, and place this in the formula for x. *Subtract this power from the given number, and substitute the remainder, which will often be negative in the formula for* a. *Perform the operations indicated, and the result will be the approximate root.*

Thus, since $\sqrt[7]{128} = 2$, we find

$$\sqrt[7]{108} = \sqrt[7]{128 - 20} = \sqrt[7]{128}\left(1 - \frac{5}{32}\right)^{\frac{1}{7}} = 1.95204.$$

4. The binomial formula is employed to develop algebraic expressions into series. To develop $\frac{1}{1-z}$ into a series, according to the ascending powers of z, we observe that the expression is equivalent to $(1-z)^{-1}$. If in the binomial formula we make $x=1$, $a=-z$, and $n = -1$, it gives

$$\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \cdots + z^n + \cdots$$

Again, to develop $\sqrt[3]{2z - z^2}$, we place it under the form $\sqrt[3]{2z}\left(1 - \frac{z}{2}\right)^{\frac{1}{3}}$; then by the application of the formula, we find

$$\left(1 - \frac{z}{2}\right)^{\frac{1}{3}} = 1 - \frac{1}{6}z - \frac{1}{36}z^2 - \frac{5}{648}z^3 - \cdots$$

and by substitution we have, finally,

$$\sqrt[3]{2z - z^2} = \sqrt[3]{2z}\left(1 - \frac{1}{6}z - \frac{1}{36}z^2 - \frac{5}{648}z^3 - \cdots\right).$$

In like manner, a variety of algebraic expressions may be developed into series.

Imaginary Binomial. A binomial in which one term is imaginary, as $a \pm \sqrt{-b^2}$, or $a \pm b\sqrt{-1}$. Such is the form of the imaginary roots of equations.

The product of each pair of imaginary roots of an equation is real, and of the form a^2+b^2. See *Imaginary Quantity*.

Binomial Line or Curve. A line whose ordinate may be expressed by some power of a binomial function of the abscissa: thus, the line whose equation is $y = x^m(a+bx^n)^p$, is a binomial curve.

Binomial Theorem. The theorem which has for its object to demonstrate the law of formation of any power of a binomial. The algebraic expression of this law constitutes the binomial formula already considered. The simplest complete demonstration of this theorem for any exponent whatever, is that by means of the principle of indeterminate coefficients. See *Davies' Bourdon*, p. 257.

Binomial Surd. A binomial in which one or both terms are surds or radicals. Thus, $a \pm \sqrt{b}$, or $\sqrt{a} \pm \sqrt{b}$, are binomial surds.

BI-PÄR′TIENT. [L. *bis*, twice, and *partio*, *partiens*, to divide]. Dividing into two parts. A bipartient number is one which is contained twice in a given number; thus, 2 is a bipartient of 4.

BĪ-QUAD′RATE. [L. *bis*, twice, and *quadratus*, squared]. A fourth power, or the square of the square; thus, 16 is the biquadrate of 2.

BĪ-QUAD-RAT′IC. Pertaining to a biquadrate.

Biquadratic Equation. An equation of the fourth degree, containing but one unknown quantity. Thus,

$$ax^4 + bx^3 + cx^2 + dx + f = 0,$$

is a complete biquadratic equation, a, b, c, d and f, denoting any known quantities whatever.

The solution of every biquadratic equation may be made to depend upon that of a cubic equation. There are various methods of making this reduction. The following is Euler's: In the first place, the equation, by a well known transformation, is deprived of its second term, and reduced to the form

$$y^4 + py^2 + qy + r = 0 \dots . (1).$$

Let us assume

$$y = \sqrt{a} + \sqrt{b} + \sqrt{c} \dots . (2),$$

in which a, b and c, are the roots of a cubic equation,

$$z^3 + Pz^2 + Qz - R = 0 \dots . (3).$$

From the general properties of the roots of an equation, we have the relations

$$a+b+c=-P, \quad ab+ac+bc=Q \text{ and } abc=R.$$

If we square both members of equation (2), and substitute for $a+b+c$ its value $-P$, we shall find, after transposing P,

$$y^2 + P = 2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}) \dots . (4).$$

If we square both members of (4), and substitute for $ab+ac+bc$ its value Q, and for $\sqrt{a}+\sqrt{b}+\sqrt{c}$ its value y, we have, after transposition,

$$y^4 + 2Py^2 - 8\sqrt{R}.\ y + P^2 - 4Q = 0 \dots (5).$$

Now, we know that one of the roots of equation (5) is equal to $\sqrt{a}+\sqrt{b}+\sqrt{c}$, and since a, b, and c, are the roots of equation (3), it follows that one of the roots of equation (5) may be found by solving equation (3). But equation (5) will be the same as equation (1), if we impose the following conditions, viz.:

$$2P = p, \quad -8\sqrt{R} = q \text{ and } P^2 - 4Q = r,$$

$$\text{or } P = \frac{p}{2}, R = \frac{q^2}{64} \text{ and } Q = \frac{p^2-4r}{16}.$$

Substituting these values in equation (3), it becomes

$$z^3 + \frac{p}{2}z^2 + \frac{p^2-4r}{16}z - \frac{q^2}{64} = 0 \dots . (6).$$

It follows, therefore, that one root of equation (1) is equal to $\sqrt{a}+\sqrt{b}+\sqrt{c}$, in which a, b, and c, are roots of equation (6). Furthermore, all of the roots of equation (1) are involved in the general expression,

$$y = \pm\sqrt{a} \pm \sqrt{b} \pm \sqrt{c}.$$

In order to discover the form of each root, we have to determine the respective signs with which each of the three terms is to be effected.

It would appear, from inspecting the formula, that there might be eight combinations of signs, and consequently eight roots; but the condition that

$$\sqrt{a} \times \sqrt{b} \times \sqrt{c} = \sqrt{R} = -\frac{q}{8}$$

limits the number to four. It is obvious that when q is essentially positive, the product must be positive, which can only be the case when they are all positive, or when one is positive, and both of the other two negative. If q is essentially negative, the product is

negative, which requires that the three should be negative, or that one should be negative, whilst the other two are positive. There can, therefore, be only four different expressions for y when q is positive, and four when q is negative, which in either case, are the roots of the equation.

Since equation (6) involves fractional co-efficients, it will be found more convenient to transform it to one in which they are entire. This may be effected by making $z = \frac{v}{4}$, which gives, after reduction,

$$v^3 + 2pv^2 + (p^2 - 4r)v - q^2 = 0 \ldots . (7).$$

The roots of this new equation are respectively $\frac{1}{4}a$, $\frac{1}{4}b$ and $\frac{1}{4}c$.

In order to solve a biquadratic equation, we have therefore the following rule:

Reduce it to the form

$$y^4 + py^2 + qy + r = 0,$$

and form the cubic equation

$$v^3 + 2pv^2 + (p^2 - 4r)v - q^2 = 0.$$

Solve the cubic equation, and denote its roots by $4a$, $4b$, and $4c$, then the roots of the given equation will be

When q *is positive.*	*When* q *is negative.*
$y=-\sqrt{a}-\sqrt{b}-\sqrt{c}$.	$y=+\sqrt{a}+\sqrt{b}+\sqrt{c}$.
$y=+\sqrt{a}+\sqrt{b}-\sqrt{c}$.	$y=+\sqrt{a}-\sqrt{b}-\sqrt{c}$.
$y=+\sqrt{a}-\sqrt{b}+\sqrt{c}$.	$y=-\sqrt{a}-\sqrt{b}+\sqrt{c}$.
$y=-\sqrt{a}+\sqrt{b}+\sqrt{c}$.	$y=-\sqrt{a}+\sqrt{b}-\sqrt{c}$.

1. Let it be required to solve the equation

$$y^4 - 25x^2 + 60x - 36 = 0.$$

Here $p = -25$, $q = 60$, and $r = -36$, and the auxiliary cubic equation is

$$v^3 - 50v^2 + 769v - 3600 = 0.$$

The roots of this equation are, $4a = 9$, $4b = 16$, and $4c = 25$, which give

$$a = \frac{9}{4},\ b = 4,\ c = \frac{25}{4}, \text{ and}$$

$$\sqrt{a} = \frac{3}{2},\ \sqrt{b} = 2, \text{ and } \sqrt{c} = \frac{5}{2}.$$

And because q is positive, we find

$$y = -\frac{3}{2} - 2 - \frac{5}{2} = -6,\ y = \frac{3}{2} - 2 + \frac{5}{2} = 2,$$

$$y = +\frac{3}{2} + 2 - \frac{5}{2} = 1,\ y = -\frac{3}{2} + 2 + \frac{5}{2} = 3.$$

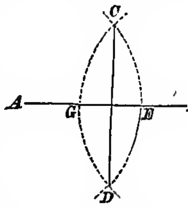
The following rule is useful in determining the nature of the roots of a biquadratic equation when it has all its terms: If $\frac{3}{8}$ of the square of the co-efficient of the second term is greater than the product of the co-efficients of the first and third terms, and if $\frac{3}{8}$ of the square of the co-efficient of the fourth term is greater than the product of the co-efficients of the third and fifth terms, and if $\frac{1}{3}$ of the square of the co-efficient of the third term is greater than the product of the co-efficients of the second and fourth terms; then, all the roots of the equation are real and unequal; but if either of these conditions is not fulfilled, the equation will have imaginary roots. We have already explained the method of constructing the roots of a biquadratic equation under the head of *Application* of Geometry to Algebra. See *Application.*

BĪ-RECT-AN″GU-LAR. [L. *bis*, twice, *rectus*, eight, and *angulus*, angle]. Having two right angles. A spherical triangle is birectangular when two of its angles are right angles.

BĪ-RHOM-BOID′AL. [L. *bis*, twice, and *rhomboid*]. Having a surface composed of twelve rhombic faces, such that, taken six and six, and produced, they will form two rhombohedrons.

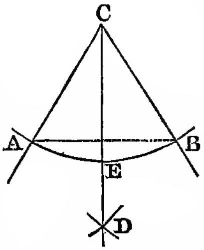
BĪ-SECT′. [L. *bis*, twice, and *seco*, to cut]. To divide a magnitude into two equal parts.

Bisect a Straight Line, AB. With A as a centre, and a radius greater than half of the line AB, construct an arc of a circle, CGD; then with B as a centre, and the same radius, construct an arc of a circle, cutting the former arc in the two points C and D; join these points by a straight line, and the point in which it cuts the given line will be the middle of AB. The radii with which the auxiliary circles are constructed, ought to be as nearly equal to the line AB as possible, since the arcs intersect under more favorable circumstances.



Bisect an Angle or an Arc of a Circle. Let BCA represent any plane angle which it

is required to bisect. With C as a centre, and any radius, as CB, describe an arc of a circle limited by the sides of the angle in the points B and A. Then with any radius greater than one half of AB, and from B and A



as centres, construct arcs of circles intersecting each other in the point D. Join the points D and C by a straight line, it will bisect both the angle BCA, and the arc BEA.

Bisect a Diedral Angle. First, pass a plane perpendicular to the edge of the angle, cutting out two straight lines, one from each face. Bisect the angle between these lines by the rule already given, and pass a plane through the bisecting line and the edge of the angle, and it will bisect the diedral angle.

Bisect a Spherical Angle. Pass planes through the sides of the angle forming a diedral angle; bisect this by the rule just given, the intersection of the bisecting plane with the surface of the sphere will bisect the spherical angle.

In the use of surveying instruments, a hair or spider line is said to *bisect an object*, when the instrument is so directed that, to the observer, the hair or line appears to pass over the middle of the object.

BI-SEG'MENT. [*bis*, twice, and *segmentum*, a segment]. Half of a segment. A straight line, which bisects both the arc and chord of a segment of a circle, also divides the segment into two equal and symmetrical parts, which are called bisegments.

BLACK'BOARD. A board or other surface painted black and employed almost universally in giving instruction in mathematics. Upon it figures or symbols are traced in chalk, which may be used for the purpose of demonstration and illustration.

BLAZE. [Fr. *blazer*]. A white spot made on the side of a tree, by removing the bark with an axe. It is used to mark prominent points in surveying, and for the purpose of distinguishing different points. Each blaze is generally marked with a sharp instrument, in accordance with some conventional plan of marking. The method of blazing trees, as marks of survey, is extensively employed in the survey of the public lands.

BOD'Y. A term sometimes used in geometry to denote the limited portion of space occupied by any material object. In geometry we often speak of the five regular bodies or solids. This use of the term is evidently improper, and its place would be better supplied by the term *volume*, which, in this acceptation, would denote a quantity of space having three dimensions, and limited in every direction. The term body implies the existence of matter, whereas the reasonings of geometry carefully exclude every such idea. For this reason, volume, which implies quantity of space, should in all cases be used in preference to either of the terms *body* or *solid*.

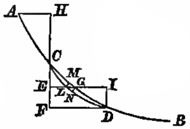
BRA-CHYST'O-CHRONE. [Gr. *βραχιστος*, shortest and *χρονος*, time]. The name given by John Bernouilli to a curve which possesses this property, viz: "That a body setting out from any point of it, as A, and impelled solely by the force of gravity, will reach another point of it, as B, in a shorter space of time than it could reach the same point by following any other path." This curve possesses an historical interest entirely independent of the particular nature of the curve, for the determination of the nature of this line suggested to Lagrange the idea of an entirely new branch of mathematics, that of the *Calculus of Variations*. On this account, we subjoin the solution of the problem as given by James Bernouilli, and also the solution of the same problem by means of the Calculus of Variations.

A comparison of the two methods affords a very fair illustration of the superiority of the latter method, besides which, Bernouilli's method affords a good example of the kind of reasoning employed by the ancient geometers.

Bernouilli's solution:

Let A represent the point from which the body is to move, and B the point to which it is to go, and ACDB the path it must follow, which is evidently in a vertical plane through the two points, A and B. Take any small portion of the curve CD; then it is obvious that if ACDB be the path by which the body will descend from A to B in the shortest possible time, it must also pass from C to D

along this path in a shorter time than if it followed any other path. For, suppose it to pass from C to D by another path, CLND, in a shorter time than by the path CMGD; then must it pass from A to B through CND in a shorter time than it will through CMG, which is contrary to the hypothesis. This being premised, let AH be drawn horizontally through A, CH perpendicular, and DF parallel to AH, through the points C and D. Also let CF be bisected in E, and complete the rectangle EFDI. The object is to find the point G in the straight line EI, at which the required curve crosses it; that is, so that the time of descent through CG, plus the time of descent through GD, may be a minimum.



Let us denote the time of descent through CG by $t(CG)$, and the time of descent through GD by $t(GD)$. Assume another point L, on EI, so that LG may be regarded as infinitely small in comparison with EG; and having drawn CL and DL, let LM and GN be respectively perpendicular to CG and DL. Now variable quantities, when they are infinitely near their maxima or minima, may be regarded as constant; therefore,

$$t(CL) + t(LD) = t(CG) + t(GD),$$

and consequently,

$$t(CG) - t(CL) = t(LD) - t(GD).$$

From the principles of the descent of heavy bodies,

$$CE : CG :: t(CE) : t(CG), \text{ and}$$
$$CE : CL :: t(CE) : t(CL) \cdots\cdots (1).$$

Therefore,

$$CE : CG - CL :: t(CE) : t(CG) - t(CL) \quad (2).$$

But from similar triangles,

$$MG : GL :: EG : CG \cdots\cdots (3).$$

Combining proportions (2) and (3), since

$$CG - CL = MG;$$

$$CE : GL :: EG \times t(CE) : CG \times [t(CG) - t(CL)] \cdots\cdots (4).$$

In like manner,

$$EF : GD :: t(EF) : t(GD), \text{ and}$$
$$EF : LD :: t(EF) : t(LD) \cdots\cdots (5).$$

Therefore,

$$EF : LD - GD :: t(EF) : t(LD) - t(GD) \; (6).$$

But $LN : GL :: GI : GD$ (7).

Therefore, since

$$LN = LD - GD;$$

$$EF : GL :: GI \times t(EF) : GD \times [t(LD) - t(GD)] \cdots\cdots (8).$$

Recollecting that $EF = CE$, and combining proportions (4) and (8),

$$EG \times t(CE) : CG \times [t(CG) - t(CL)] :: GI \times t(EF) : GD \times [t(LD) - t(GD)] \quad (9)$$

or,

$$EG \times t(CE) : GI \times t(EF) :: CG \times [t(CG) - t(CL)] : GD \times [t(LD) - t(GD)] \; (10),$$

but it was proved that

$$t(CG) - t(CL) = t(LD) - t(GD),$$

and by the laws of falling bodies,

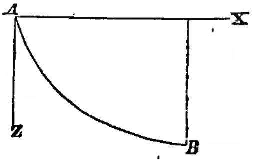
$$t(CE) : t(EF) :: \frac{1}{\sqrt{HC}} : \frac{1}{\sqrt{HF}},$$

we have, therefore, by substitution,

$$\frac{EG}{\sqrt{HC}} : \frac{GI}{\sqrt{HE}} :: CG : GD,$$

a known property of the common cycloid. Hence, the brachystochrone is the common cycloid.

The following is the method of investigating the nature of the curve of quickest descent, by the aid of the *Calculus of Variations.*



Let the axis of Z be vertical through the point A, and let the axis of X be horizontal through the same point. From the principle of falling bodies when constrained to move upon a curve, we have the time of descent from A to B, if we place

$$\sqrt{dx^2 + dz^2} = ds, \qquad t = \int_0^a \frac{ds}{\sqrt{2gz}};$$

or placing

$$\sqrt{2gz} = u, \qquad t = \int_0^a \frac{ds}{u};$$

taking the variation of $\frac{ds}{u}$, we have

$$\delta\frac{ds}{u} = -\frac{ds\delta z}{u^3} + \frac{dx\delta dx}{uds} + \frac{dz\delta dz}{uds}.$$

In order that t may be a minimum, we must have

$$d\left(\frac{dx}{uds}\right) = 0, \quad \text{or} \quad \frac{dx}{uds} = c\,;$$

whence, by including g in the constant of integration,

$$dx = \frac{cdz\sqrt{2z}}{\sqrt{1-2c^2z}}\,;$$

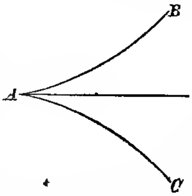
which is characteristic of the cycloid. We must have also

$$\frac{ds}{u^3} + d\left(\frac{dy}{uds}\right) = 0,$$

which leads to the same result; hence, the curve of quickest descent is a cycloid.

BRANCH OF A CURVE. [Fr. *branche*, branch]. A portion of a curve extending continuously from one singular point, or point of discontinuity, to another; or which extends continuously from any point till it returns upon itself at that point. The latter kind of branch is called an *oval*. Thus, an ellipse is an oval branch.

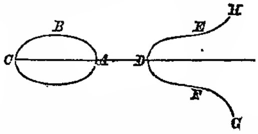
The branches of a curve may *meet* at singular points, or they may be entirely *disconnected*. Thus, the cubic parabola BAC has two branches BA and CA meeting at the cusp point A. The ordinary hyperbola affords an example of a curve having two branches, which are entirely disconnected except by their common equation.



The branches of a curve may be *closed*, *finite and not closed*, or *infinite*. Thus, the curve whose equation is

$$ay^2 - x^3 + (b-c)x^2 + bcx = 0 \quad (1),$$

has four branches:



1st. A closed branch CBA of an oval form, which returns upon itself.

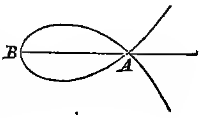
2d. A finite branch EDF extending from the point of inflexion E, to the corresponding point of inflexion F.

3d. Two infinite branches, EH, FG, having their convexities turned towards the axis CD.

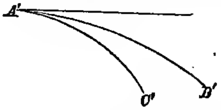
If, in equation (1) $c = 0$, the oval or closed branch becomes the point A; whence we see that a conjugate point is, analytically speaking, a *branch* of a curve.

Again, if we make, in (1), both $c = 0$, and $b = 0$, the branch EDF disappears, and the curve becomes a cubic parabola; the two infinite branches EH and FG uniting on the axis at a cusp point, as shown in the preceding figure.

Had we made b alone equal to 0, the curve would have taken the form shown in the annexed figure, in which the branch cuts itself at A, forming a multiple point.



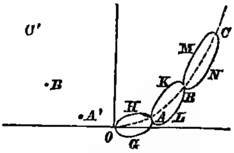
Branches which meet in a cusp point, may have their convexities turned in the same direction at the cusp point or in different directions.



Thus, BAC in the cubic parabola is an example of the first relation, and B′ A′ C′ one of the second.

Finally, there is a kind of branch called the pointed branch, which is, strictly speaking, an infinite number of points whose positions are given by the equation of a curve, but which are entirely disconnected. They lie scattered along a curve, which has caused the group to be termed a pointed branch. To illustrate the nature of a pointed branch, let us take the equation

$$y = ax^2 \pm \sqrt{x}\cdot\sin bx.$$



For every positive value of x there are two real values of y, which become absolutely equal as often as $\sin bx$ becomes 0. These

values give a curve O G A K B N, &c., formed of a system of branches uniting like the links of a chain at the points O, A, B, C, &c., situated upon a parabola whose equation is $y = ax^2$. This parabola bisects all chords of the curve which are parallel to the axis of Y, and is called, for this reason, a diametral curve. For all negative values of x, except such as make $\sin bx$ equal to 0, that is, which do not make bx some multiple of π, render y imaginary, and do not therefore correspond to points of the curve; but these negative values of x which reduce the radical to 0, give with the corresponding values of y, a system of points which lie upon the diametral parabola, as A′, B′, C′, &c. These, taken together, constitute a pointed branch.

Of the *infinite branches*, two kinds are considered, *hyperbolic* and *parabolic*.

In general, when a branch admits of an hyperbolic or rectilinear asymptote, it is called *hyperbolic:* thus, in the curve whose equation is

$$y = \frac{x^3}{a} - \frac{b^3}{x^2},$$

the axis of Y is an asymptote, and it has also a cubic hyperbola for an asymptote.

Parabolic Branches can never have either a right line, or an hyperbola for an asymptote, but they admit of parabolic asymptotes.

Thus, the curve whose equation is

$$y = \frac{x^2}{a} + \frac{b^2}{x},$$

has a parabolic branch whose asymptote is the common parabola.

Finally, all infinite branches of curves belong to either one or the other of these two classes. All curves whose equations, or the equations of whose branches can be reduced to the form $x^m y^n = a$, in which m and n are both positive, have hyperbolic branches, and are called hyperbolas. All curves whose equations, or the equations of whose branches can be reduced to the form $y^m = ax^n$, have parabolic branches, and are called parabolas.

A Branch of Science, is one of its subdivisions: thus, arithmetic, algebra, geometry, &c., are called branches of mathematics.

BRILLIANT POINT. [Fr. *brilliant*, shining, bright]. Of a surface, is that point of a surface from which the light is reflected directly to the eye. According to the laws of Optics, the reflected ray makes an angle with the normal to the surface at the point of incidence equal to that which the incident ray makes with the normal at the same point. From this principle, we deduce the following construction for the brilliant point of a surface, under the supposition, that the eye is at an infinite distance.

Assume any point in space, and draw through it a line parallel to a ray of light, and also a line to the position of the eye; bisect the angle formed by these two lines, and pass a plane which shall be tangent to the surface, and perpendicular to this bisecting line, the point of contact will be the *brilliant point*. The reason for this construction is obvious; for, if at the point of contact lines be drawn respectively parallel to the three lines constructed through the point in space, the first will be the direction of the incident ray, the second that of the reflected ray, and the third the normal to the surface at the point. Whence, this point satisfies the conditions of a brilliant point.

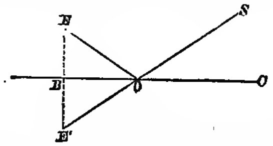
If the surface is a single curved surface, the plane will be tangent all along an element of the surface. This element is called a *brilliant element*.

This construction is only correct when the eye is at an infinite distance, and the rays of light parallel. If, however, the eye is at a considerable distance compared with the magnitude of the object viewed, and if the light considered comes from the sun, the construction gives an approximate result which is abundantly accurate.

When the eye is at a finite distance, and the surface single curved, it may be desirable to determine the position of the most brilliant point of the brilliant element. This may be effected by passing a plane through the point of sight, and perpendicular to the brilliant element; the point in which it cuts the element is the most brilliant point.

To find the brilliant point upon a plane, as for instance, the brilliant point upon still water, as viewed by the eye above the surface, the eye being at a finite distance: Pass a plane through the point of sight and the source of light, and perpendicular to the plane surface; this will cut the surface of the water in a straight line BC. From the point of sight E, draw a perpendicular to BC, and

prolong it till BE′ equals BE: through the point E′ draw a ray of light E′S, and the



point O in which it cuts the line BC will be the brilliant point. This construction is apparent from the figure.

BRŌ′KEN LINE. In Geometry, a line composed of limited straight lines meeting each other at their extremities: thus, the sides of a plane polygon, or any number of them, taken together, constitute a broken line.

BUŎY. [Fr. *bouée*, buoy]. In Surveying, a floating body moored or anchored over a point, generally for the purpose of directing a line of soundings in the hydrographic portion of a survey.

BUSH′EL. [Fr. *boisseau*, bushel]. A measure of capacity. According to the ordinance manual, the bushel of the United States contains 2150.4 cubic inches, and holds 77.627413 lbs. of distilled water, at its maximum density (39°.8 Fahr.), the barometer being 30 inches.

A cubic yard contains 21.69 bushels.

A cylinder 14 inches in diameter, and 14 inches high, contains 1 bushel. A box 16.8 inches long, 16 inches wide, and 8 inches deep, holds 1 bushel.

The standard bushel of the United States described above, is the same as the Winchester bushel or English bushel, which is a right cylinder 18½ inches in diameter and 8 inches deep.

BUTT. A measure for liquids. In the English beer measure, a butt contains 108 imperial gallons.

C. The third letter of the English alphabet. In the Roman system of arithmetical notation it stands for 100.

CAL′CU-LĀTE. [L. *calculo*, to calculate, from *calculus*, a pebble]. To compute, to reckon.

CAL-CU-LĀ′TION. The name given to any operation requiring the application of any of the ordinary rules of arithmetic. All applications of mathematical rules, except those of pure geometry, may be styled calculations.

CAL′CU-LĀTOR. One who calculates.

CAL′CU-LUS. A term used to denote the *manner* of performing certain mathematical operations, or of making mathematical investigations by the aid of symbols: thus, we find arithmetic called the arithmetical or numeral calculus, algebra, the algebraic calculus. We also find the terms exponential calculus, fluxional calculus, literal or symbolical calculus, &c.

The term has also been applied in a sense nearly synonymous with *calculation* and *operation*: thus, the method of transforming and operating upon radicals, is called the calculus of radicals.

By general consent, however, the term *calculus* is now applied to the highest branch of mathematics, that which treats of the *forms of functions*. We shall, therefore, consider the term under this view, rejecting all other senses in which the term may have been applied. With this understanding, we may define calculus to be that branch of mathematics which treats of the nature and the forms of functions.

It also has for its object the laws of derivation of one *form* from another, and the application of these laws to other branches of mathematics, as algebra, analytical geometry, trigonometry, &c.

Functions are of two kinds, *determinate*, and *indeterminate*; *determinate functions* being those whose forms are given, and *indeterminate functions* those whose forms are not given, being only required to fulfill certain conditions. Hence, there are two branches of calculus: the *ordinary calculus*, which treats of determinate functions, and the *calculus of variations*, which treats of the nature and relations of indeterminate functions.

Again, in the ordinary calculus, there are two different methods of deriving determinate functions from other determinate functions. These give rise to the division of the ordinary calculus into two parts, *differential* and *integral calculus*. We shall consider, in succession, *differential calculus*, *integral calculus*, and the *calculus of variations*.

Differential Calculus. Differential calculus is that branch of mathematics which

has for its object to explain the method of deriving one determinate function from another, by the process of differentiation. If, in any determinate function of one variable, we give to the variable a constant increment and find the corresponding increment of the function, and then divide the increment of the function by the increment of the variable, we shall find a ratio which will in general be dependent upon the increment of the variable. If now we pass to the limit of this ratio, by making the increment of the variable equal to 0, we shall in general obtain a function of the original variable, which is called the *differential co-efficient of the function.* If the differential co-efficient of the function be multiplied by the differential of the variable, the result is called the *differential of the function.* Any function of a single variable will have one, and only one differential co-efficient, and consequently it will have but one differential of the same order.

If the original function be one of several variables, we may find its differential with respect to each one of the variables, that is, as though all of the others were constant. These differentials are called *partial differentials*, and the sum of all the partial differentials of a function of several variables is called the *differential* or *total differential of the function.* The differential co-efficient taken with respect to any variable, is called a *partial differential co-efficient of the function taken with respect to that variable.* There is no such thing as a total differential co-efficient of a function of several variables.

The operation of finding the differential of any function is called *differentiation.*

If any differential be differentiated, the result is called a differential of the *second order*; if a differential of the *second order* be differentiated, the result is called a differential of the *third order*, and so on; generally, if a differential of the n^{th} order be differentiated, the result is a differential of the $(n+1)^{\text{th}}$ order, and the continued operation of differentiating succeeding differentials, is called *successive differentiation*, and the differentials obtained, taken in their order, are *successive differentials.*

The number of successive differentials of any given function may be finite or it may be infinite. If the function does not involve the variable or variables (either directly or indirectly) with negative or fractional exponents, the number of successive differentials will be finite, and if all the operations indicated in the function be performed, the highest exponent of either variable, in any term, will denote the number of successive differentials. If the function involve the variables with either negative or fractional exponents, the number of successive differentials will be infinite.

When the given function depends upon more than one variable, the process of successive differentiation requires that a partial differential taken with respect to one variable, be differentiated with respect to some other variable: this result is called a partial differential, and its nature is expressed by describing the successive operations. Thus, if a partial differential of the first order, taken with respect to x, be differentiated with respect to y, the result is called a partial differential of the second order, taken once with respect to x, and once with respect to y, and so on, for partial differentials of the higher orders.

The Differential Calculus consists of two parts. The *first* embraces the *science* of the differential calculus, and explains the methods of finding the differentials and successive differentials of all determinate functions. The *second* treats of the applications of the differential calculus to the other branches of mathematics, as Algebra, Analytical Geometry, &c.

We shall give the formulas for differentiating every kind of function of one variable, observing that these will, when properly applied, in connection with the principles already laid down, give the differentials of all functions of several variables, as well as all successive differentials of functions of any number of variables. The only additional remark necessary to premise, is, that the differential of every independent variable is to be regarded as *constant*, whilst the differential of the function will in general be *variable.*

The various applications of the principles of the differential calculus will be considered separately under their appropriate headings.

In the following formulas, a, b, c, in short, all of the leading letters of the alphabet, will denote known or constant quantities; u and v,

will be employed to denote functions of one variable, and x will designate the independent variable.

Formulas for Differentiating any Function of one Variable.

(1). $d(a)=0.$ (2). $d(u+v)=du+dv.$

(3). $d(u-v)=du-dv.$ (4). $d(uv)=udv+vdu.$

(5). $d\left(\frac{u}{v}\right)=\frac{vdu-udv.}{v^2}$ (6). $d\left(\frac{u}{a}\right)=\frac{du}{a}.$

(7). $d\left(\frac{a}{u}\right)=-\frac{adu}{u^2}.$ (8). $d(u^m)=mu^{m-1}du.$

(9). $d(\sqrt[n]{u})=\frac{du}{n\sqrt[n]{u^{n-1}}}.$ (10). $d(a^u)=a^u ladu.$

(11). $d(\log u)=M\frac{du}{u}.$ (12). $d(lu)=\frac{du}{u}.$

(13). $d(u^v)=u^v ludv+vu^{v-1}du.$

(14). $d(l(lu))=d(l^2u)=\frac{du}{ulu}.$

(15). $d(\sin u)=\cos udu.$

(16). $d(\cos u)=-\sin udu.$

(17). $d(\text{ver-sin}\, u)=\sin udu.$

(18). $d(\tan u)=\frac{du}{\cos^2 u}.$

(19). $d(\cot u)=-\frac{du}{\sin^2 u}.$

(20). $d(\sec u)=\tan u \sec udu.$

(21). $d(\operatorname{cosec} u)=-\cot u \operatorname{cosec} udu.$

(22). $d(\sin^{-1}u)=\frac{du}{\sqrt{1-u^2}}.$

(23). $d(\cos^{-1}u)=\frac{-du}{\sqrt{1-u^2}}.$

(24). $d(\operatorname{versin}^{-1}u)=\frac{du}{\sqrt{2u-u}}.$

(25). $d(\tan^{-1}u)=\frac{du}{1+u^2}.$

(26). $d(\cot^{-1}u)=-\frac{du}{1+u^2}.$

(27). $d(\sec^{-1}u)=\frac{du}{u\sqrt{u^2-1}}.$

(28). $d(\operatorname{cosec}^{-1}u)=-\frac{1}{u\sqrt{u^2-1}}.$

If in any curve whose equation is $y=f(x)$, we denote the length of any portion by z, the area included between the curve, the axis of x, and any two ordinates, by s, the area of the surface generated by revolving a portion of the curve between any two ordinates, around the axis of x, by u, and the solid generated by v, we have

(29). $dz=\sqrt{dx^2+dy^2}.$ (30). $ds=ydx.$

(31). $du=2\pi y\sqrt{dx^2+dy^2}.$ (32). $dv=\pi y^2dx.$

If w denote the length of any portion of a curve in space, the co-ordinates of whose points are x, y, and z.

(33). $dw=\sqrt{dx^2+dy^2+dz^2}$

If a plane curve be referred to a system of polar co-ordinates, its equation will be of the form $r=f(v)$, in which r is the radius vector, and v the variable angle. If we denote the length of any portion of this curve by z, and the area included between any two positions of the radius vector, and the curve by s, we shall have

(34). $dz=\sqrt{dr^2+r^2dv^2}.$ (35). $ds=\frac{r^2dv}{2}.$

By means of the first 28 formulas, and a proper application of the principles laid down for differentiating functions of two or more variables, every possible function may be differentiated. The remaining seven formulas are useful in the practical applications of the calculus.

Integral Calculus. The object of the integral calculus is the converse of that of the differential calculus. Having a given or known differential, the integral calculus has for its object to find a function, such that being differentiated, it will produce the given differential: such expression is called the *integral* of the differential.

The operation of finding the primitive function or integral is called *Integration.* Besides the method of finding the integrals of given differentials, the integral calculus is also applied to various branches of mathematics, as well as to almost every branch of natural philosophy and engineering.

The method of integrating a given differential is by no means obvious, nor is it capable of being reduced to definite and fixed rules. We have seen, in finding the differentials of functions, that the rules are certain, and their number definite, being determined by the particular form of the function. In returning from the differential to the integral, from which it may have been derived, we can only compare the differential expression with other expressions which we know to be differentials of given functions, and thus arrive at the form of the integral.

The great object of the integral calculus is

to transform the given expressions into others which are differentials of known functions, and thus deduce formulas which may be applied to all similar forms. The number of formulas is unlimited, and of these we shall only attempt to give some of the most useful, referring the reader for a more complete system of formulas to special treatises on the subject of Integral Calculus. Of these PEACOCK and HERSCHEL's examples will probably be found the most accessible, and is altogether a very complete compendium of formulas on the subject. It is from that work that most of those subjoined are taken. The most complete collection of integral formulas that have been published is probably that of MEYER HIRSCH of Berlin, who has filled a large volume, so arranged that they can be readily referred to.

The symbol of integration is this, $\int$, which is only a particular form of the letter *s*, which originally stood for the word *summa*, or sum. In fact, the integral is the sum of all the differentials, these being infinitely small. For integrating between limits, the symbol $\int_a^b$ is used, and is read, the integral between the limits a and b, the subtractive limit being written at the bottom of the symbol.

Were the notation of mathematics to be revised, the proper symbol for integration would appear to be d^{-1}, which, in accordance with the conventional principles of notation, would be read, *the function whose differential is* ...; thus, $d^{-1}(xdx)$, would mean *the function whose differential is xdx*, which is the same thing as *the integral of xdx*.

It was shown in the differential calculus that the differential of a constant is 0, hence to every integral found by applying a formula, a constant must be added. This constant is arbitrary, and serves to make the integral fulfill any one reasonable condition.

We shall first give the fundamental formulas, then indicate the most useful transformation for bringing particular cases under them, and afterwards give some of the formulas most commonly needed in the practical application of the Integral Calculus.

Fundamental Formulas.

(1). $\int a x^n \, dx = \frac{x^{n+1}}{n+1} + C.$

(2). $\int a x^{-1} \, dx = \int a \frac{dx}{x} = alx + C.$

(3). $\int a^x \, dx = \frac{a^x}{la} + C.$

(4). $\int e^{ax} \, dx = \frac{e^{ax}}{a} + C.$

(5). $\int \sin x dx = -\cos x + C.$

(6). $\int \cos x dx = \sin x + C.$

(7). $\int \frac{dx}{\cos^2 x} = \tan x + C.$

(8). $\int \frac{dx}{\sin^2 x} = -\cot x + C.$

(9). $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$

(10). $\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C.$

(11). $\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$

(12). $\int \frac{+dx}{\sqrt{2x-x^2}} = \text{ver-sin}^{-1} x + C.$

To which may be added the following, deduced from the preceding, but which should be memorized on account of their frequent occurrence in practice:

(13). $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C.$

(14). $\int -\frac{dx}{\sqrt{a^2-x^2}} = \cos^{-1} \frac{x}{a} + C.$

(15). $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$

(16). $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = l \left\{ x \pm \sqrt{x^2 - a^2} \right\} + C.$

(17). $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} l \left(\frac{a+x}{a-x} \right) + C.$

(18). $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} l \left(\frac{x-a}{x+a} \right) + C.$

(19). $\int \frac{dx}{(a^2 \pm x^2)^{\frac{3}{2}}} = \frac{1}{a^2} \cdot \frac{x}{\sqrt{a^2 \pm x^2}} + C.$

(20). $\int \frac{dx}{\sin x} = l (\tan \tfrac{1}{2} x) + C.$

(21). $\int \frac{dx}{\cos x} = l \left(\cot \left(\frac{\pi}{4} - \frac{x}{2} \right) \right) + C.$

(22). $\int \tan x dx = -l (\cos x) + C.$

(23). $\int \cot x dx = l (\sin x) + C.$

(24). $\int lx dx = x (lx - 1) + C.$

Principal Transformations.

I. The transformation of an expression of the form $\int X dx$ to one of the form of $\int V dv$, (X and V being respectively functions of x and v) so that the latter expression shall come under one of the fundamental formulas. This is the most common transformation of the Calculus, and the cases are so numerous and so varied that no general rule can be given. The most common and useful ones are given in every work on the Differential and Integral Calculus. The following is an example of this transformation: The expression

$$\int (a^2 + x^2)^n x dx,$$

can be immediately placed under the form,

$$\tfrac{1}{2}\int (a^2 + x^2)^n d(a^2 + x^2),$$

and by placing v for $x^2 + a^2$, it takes the form $\frac{1}{2}\int v^n dv$, an expression to which formula (1) is immediately applicable. To find the integral in terms of x, we have to substitute in the integral found in terms of v for v, its value $x^2 + a^2$.

As a second example, take the expression

$$\int (1 + a e^x)^{-1} dx,$$

which can be written

$$\int (e^{-x} + a)^{-1} e^{-x} dx.$$

and by making $e^{-x} + a = v$, it reduces to $-\int v^{-1} dv$, which comes under formula (2).

II. The transformation of algebraical to trigonometrical functions, or the reverse. Thus, if in the expression

$$\int (a^2 - x^2) x^n dx$$

we make $x = a \sin\theta$, it reduces to

$$a^{2n+n+1} \cos^{n+1}\theta \sin^n\theta\, d\theta.$$

Also, if in the expression

$$\int (\sin\theta, \cos\theta)\, d\theta,$$

we make $\sin\theta = x$, it becomes

$$\int (x, \sqrt{1 - x^2})(1 - x^2)^{-\frac{1}{2}} dx.$$

III. When there are rational factors in the denominator of a fraction, they may be changed into the numerator by making $x = \frac{1}{z}$. Thus, in the expression

$$\frac{dx}{x^m \sqrt{a + bx + cx^2}},$$

if we make $x = \frac{1}{z}$, it reduces to

$$\frac{-z^{m-1} dz}{\sqrt{c + bz + az^2}},$$

a form more readily integrated.

IV. When the numerator of a fractional differential is irrational, it may be rendered rational by multiplying both terms of the fraction by some power of the numerator, and the transformation will generally render the fraction more easily integrable. Thus the expression $\int \sqrt{X dx}$ may be transformed to $\int \frac{X dx}{\sqrt{X}}$; also, the expression

$$\int \sqrt{a^2 + x^2}\, dx \quad \text{to}$$

$$\int \frac{a^2 dx}{\sqrt{a^2 + x^2}} + \int \frac{x^2 dx}{\sqrt{a^2 + x^2}},$$

both of which are simpler forms.

V. When by addition of simple terms to the numerator, it can be made the differential of a function of the denominator, such additions, with compensating subtractions, will often reduce integration to a simple form.

Thus, the expression

$$\int \frac{x dx}{\sqrt{a + bx + cx^2}}$$

may be transformed to

$$\frac{1}{2c}\int \frac{2cx + b - b}{\sqrt{a + bx + cx^2}} \times dx,$$

or to

$$\frac{1}{2c}\int \frac{d(a + bx + cx^2)}{\sqrt{a + bx + cx^2}} - \frac{b}{2c}\int \frac{dx}{\sqrt{a + bx + cx^2}},$$

the first term of which can be integrated by preceding methods, and the second term may also be integrated by a simple operation.

VI. The method of integration by parts, consists in resolving the differential into two factors, and then applying the formula

$$\int u dv = uv - \int v du.$$

In this case, when the expression $\int v du$ is simpler than $u dv$, a saving is effected. Thus, having $\int x^m lx\, dx$, we may place

$$u = lx \quad \text{and} \quad dv = x^m dx;$$

whence,

$$du = \frac{dx}{x} \quad \text{and} \quad v = \frac{x^{m+1}}{m+1},$$

which in the formula gives

$$\int x^m lx dx = \frac{lx \,.\, x^{m+1}}{m+1} - \int \frac{x^{m+1} dx}{(m+1)x}.$$

The last term is easily integrated.

By continued application of the method by parts, integrals may oftentimes be much simplified: thus,

$\int e^x x^6\,dx = x^6 e^x - 6\int e^x x^5\,dx.$
$\int e^x x^5\,dx = x^5 e^x - 5\int e^x x^4\,dx.$
$\int e^x x^4\,dx = x^4 e^x - 4\int e^x x^3\,dx.$
&c. &c. &c.

In cases like the preceding, it is customary to deduce general formulas, called equations of reduction.

In the case last considered, the general equation of reduction is

$$\int e^{ax} x^n\,dx = a^{-1} e^{ax} x^n - \frac{n}{a}\int e^{ax} x^{n-1}\,dx.$$

The utility of an equation of reduction depends upon our being able to arrive, by continued application, at an expression that will come under one of the fundamental formulas.

In the example just considered, if n is a whole number, we must, after n applications of it, reach an integral expression $\int e^{ax}\,dx$, which comes under formula (4); but if n is a fraction, no integrable result can be arrived at by using the formula.

2. *Particular Cases. Binomial Differentials.*

(25). $$\int (a+bx^n)^m x^{n-1}\,dx = \frac{(a+bx^n)^{m+1}}{bn(m+1)}.$$

(26). $$\int \frac{x\,dx}{a+bx} = \frac{x}{b} - \frac{a}{b^2}\,l\,(a+bx).$$

(27). $$\int \frac{x\,dx}{(a+bx)^3} = -\left(\frac{x}{b}+\frac{a}{2b^2}\right)\left(\frac{1}{a+bx}\right)^2.$$

(28). $$\int \frac{x^2\,dx}{(a+bx)^2} = \left(\frac{x^2}{b}-\frac{2a^2}{b^2}\right)\left(\frac{1}{a+bx}\right) - \frac{2a}{b^3}\,l\,(a+bx).$$

(29). $$\int \frac{x^3\,dx}{a+bx} = \frac{x^3}{3b} - \frac{ax^2}{2b^2} + \frac{a^2x}{b^3} - \frac{a^3}{b^4}\,l\,(a+bx).$$

(30). $$\int \frac{x^3\,dx}{(a+bx)^3} = \left(\frac{x^2}{b} - \frac{6a^2x}{b^3} - \frac{9a^3}{2b^4}\right) \times \left(\frac{1}{a+bx}\right)^2 - \frac{3a}{b^4}\,l\,(a+bx).$$

(31). $$\int \frac{x^{-1}\,dx}{a+bx} = \frac{1}{a}\,l\left(\frac{x}{a+bx}\right).$$

(32). $$\int \frac{x^{-1}\,dx}{(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a}\,l\left(\frac{a+bx}{x}\right).$$

(33). $$\int \frac{x^{-1}\,dx}{(a+bx)^3} = \left(\frac{3}{2a}+\frac{bx}{a^3}\right)\left(\frac{1}{a+bx}\right)^2 - \frac{1}{a^3}\,l\left(\frac{a+bx}{x}\right).$$

(34). $$\frac{x^{-1}\,dx}{(a+bx)^4} = \left(\frac{11}{6a}+\frac{5bx}{2a^2}+\frac{b^2x^2}{a^3}\right) \times \left(\frac{1}{a+bx}\right)^3 - \frac{1}{a^4}\,l\left(\frac{a+bx}{x}\right).$$

(35). $$\int \frac{x^{-2}\,dx}{(a+bx)^2} = -\left(\frac{1}{ax}+\frac{2b}{a^2}\right)\left(\frac{1}{a+bx}\right) + \frac{2b}{a^3}\,l\left(\frac{a+bx}{x}\right).$$

(36). $$\int \frac{x^{-2}\,dx}{(a+bx)^3} = -\left(\frac{1}{ax}+\frac{9b}{2a^2}+\frac{3b^2x}{a^3}\right) \times \left(\frac{1}{1+bx}\right)^2 + \frac{3b}{a^4}\,l\left(\frac{a+bx}{x}\right).$$

(37). $$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{b}{a}}\right) = \frac{1}{\sqrt{ab}}\sin^{-1}\left(\sqrt{\frac{bx^2}{a+bx^2}}\right).$$

(38). $$\int \frac{dx}{a-bx^2} = \frac{1}{2\sqrt{ab}}\,l\left(\frac{\sqrt{a}+x\sqrt{b}}{\sqrt{a}-x\sqrt{b}}\right) = \frac{1}{\sqrt{ab}}\,l\left(\frac{\sqrt{a}+x\sqrt{b}}{\sqrt{a-bx^2}}\right).$$

(39). $$\int \frac{x^3\,dx}{a+bx^2} = \frac{x^2}{2b} - \frac{a}{2b^2}\,l(a+bx^2).$$

(40). $$\int \frac{dx}{x^3(a+bx^2)} = -\frac{1}{2ax^2} - \frac{b}{2a^2}\,l\left(\frac{x^2}{a+bx^2}\right).$$

(41). $$\int \frac{dx}{x(a+bx^3)} = \frac{1}{3a}\,l\left(\frac{x^3}{a+bx^3}\right).$$

(42). $$\int \frac{dx}{x(a+bx^3)^2} = \frac{1}{3a(a+bx^3)} - \frac{1}{3a^2}\,l\left(\frac{a+bx^3}{x^3}\right)$$

3. *Particular Cases. Trinomial Differentials.*

(43). $$\int \frac{dx}{a+bx+cx^2} = \frac{1}{\sqrt{b^2-4ac}}\,l\left(\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}\right).$$

(44). $$\int \frac{dx}{(a+bx+cx^2)^2} = \frac{2cx+b}{(4ac-b^2)(a+bx+cx^2)} + \frac{2c}{4ac-b^2}\int \frac{dx}{a+bx+cx^2}.$$

(45). $$\int \frac{xdx}{a+bx+cx^2} = \frac{1}{2c}\, l\,(a+bx+cx^2) - \frac{b}{2c}\int \frac{dx}{a+bx+cx^2}.$$

(46). $$\int \frac{dx}{x(a+bx+cx^2)} = \frac{1}{2a}\, l\left(\frac{x^2}{a+bx+cx^2}\right) - \frac{b}{2a}\int \frac{dx}{a+bx+cx^2}.$$

4. *Particular Cases. Differentials of Circular and Logarithmic Functions.*

(47). $$\int \frac{dx}{1+x^3} = \frac{1}{3}\cdot l\left(\frac{1+x}{1-x+x^2}\right) + \frac{1}{\sqrt{3}}\, tan^{-1}\, \frac{x\sqrt{3}}{2-x}.$$

(48). $$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\, l\left(\frac{1+x\sqrt{2+x^2}}{1-x\sqrt{2+x^2}}\right) + \frac{1}{2\sqrt{2}}\, tan^{-1}\, \frac{x\sqrt{2}}{1-x^2}.$$

(49). $$\int \frac{dx}{1-x^3} = \frac{1}{3}\, l\left(\frac{1-x}{\sqrt{1+x+x^2}}\right) + \frac{1}{3\sqrt{3}}\, tan^{-1}\, \frac{x\sqrt{3}}{2+x}.$$

(50). $$\int \frac{dx}{1-x^4} = \frac{1}{4}\, l\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\, tan^{-1}\, x.$$

5. *Particular Cases. Rational Fractions.*

(51). $$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\, l\left(\frac{x+a}{x+b}\right).$$

(52). $$\int \frac{dx}{(x+a)(x+b)^2} = \frac{1}{(b-a)(x+b)} + \frac{1}{(b-a)^2}\, l\left(\frac{x+a}{x+b}\right).$$

(53). $$\int \frac{xdx}{(x^2+a)(x+b)} = \frac{1}{b^2+a} \times \left\{ b\cdot l\,\frac{\sqrt{x^2+a}}{x+b} + \sqrt{a}\, tan^{-1}\, \frac{x}{\sqrt{a}} \right\}.$$

(54). $$\int \frac{xdx}{(x^2+a)(x^2+b)} = \frac{1}{2(b-a)}\, l\left(\frac{x^2+a}{x^2+b}\right).$$

(55). $$\int \frac{dx}{x^3-x^2+x-1} = -\frac{1}{2}\, l\,\frac{\sqrt{x^2+1}}{x-1} - \frac{1}{2}\, tan^{-1}\, x.$$

(56). $$\int \frac{x^2dx}{x^3+x^2+x+1} = \frac{1}{2}\, l\,(x+1) + \frac{1}{2}\, l\,\sqrt{x^2+1} - \frac{1}{2}\, tan^{-1} x.$$

6. *Particular Cases. Irrational Differentials.*

(57). $$\int \frac{dx}{\sqrt{a+bx}} = \frac{2}{b}\sqrt{a+bx}.$$

(58). $$\int \frac{dx}{\sqrt{1+x^2}} = l\,(x+\sqrt{1+x^2}).$$

(59). $$\int \frac{dx}{\sqrt{x^2-1}} = l\,(x+\sqrt{x^2-1}).$$

(60). $$\int \frac{dx}{x\sqrt{1+x^2}} = l\left(\frac{\sqrt{1+x^2}-1}{x}\right).$$

(61). $$\int \frac{dx}{x\sqrt{1-x^2}} = l\left(\frac{\sqrt{1-x^2}-1}{x}\right).$$

(62). $$\int \frac{dx}{x\sqrt{x^2-1}} = \tan^{-1}\sqrt{x^2-1}.$$

(63). $$\int \frac{dx}{\sqrt{x-x^2}} = 2\sin^{-1}\sqrt{x}.$$

(64). $$\int \frac{dx}{\sqrt{x+x^2}} = l\,(2x+1+2\sqrt{x^2+x}).$$

(65). $$\int \frac{dx}{\sqrt{x^2-x}} = l\,(1-2x-\sqrt{x^2-x}.$$

(66). $$\int \frac{dx}{\sqrt{1+x-x^2}} = \sin^{-1}\frac{2x-1}{\sqrt{5}}.$$

(67). $$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\, l\left(2cx+b+2\sqrt{c}\times\sqrt{a+bx+cx^2}\right).$$

(68). $$\int \frac{dx}{\sqrt{1+x+x^2}} = l\left(2x+1+2\sqrt{1+x+x^2}\right).$$

(69). $$\int \frac{dx}{(1+x)\sqrt{1-x}} = \frac{1}{\sqrt{2}}\, l\left(\frac{x-3+2\sqrt{2}\times\sqrt{1-x}}{1+x}\right)$$

(70). $$\int \frac{dx}{x^2\sqrt{a+bx^2}} = -\frac{\sqrt{a+bx}}{ax}.$$

(71). $$\int \frac{xdx}{(a+bx^2)^{\frac{3}{2}}} = -\frac{1}{b\sqrt{a+bx^2}}.$$

(72). $\int \frac{dx}{(ax+bx^2)^{\frac{3}{2}}} = -\frac{2(2bx+a)}{a^2\sqrt{ax+bx^2}}$.

(73). $\int \frac{xdx}{(ax+bx^2)^{\frac{3}{2}}} = \frac{2x}{a\sqrt{ax+bx^2}}$.

(74). $\int \frac{dx}{(a+bx+cx^2)^{\frac{3}{2}}} =$

$$\frac{2(2cx+b)}{(4ac-b^2)\sqrt{a+bx+cx^2}}.$$

(75). $\int \frac{xdx}{\sqrt{a+bx+cx^2}} = \frac{\sqrt{a+bx+cx^2}}{c}$

$$-\frac{b}{2c}\int \frac{dx}{\sqrt{a+bx+cx^2}} \text{ (Ex. 51).}$$

7. *Particular Cases. Transcendental Differentials.*

(76). $\int x^n lxdx = \frac{x^{n+1}}{n+1}\left(lx - \frac{1}{n+1}\right)$.

(77). $\int \frac{dx}{x} lx = \frac{1}{2}(lx)^2$.

(78). $\int x^m (lx)^2 dx = \frac{x^{m+1}}{m+1}\left((lx)^2\right.$

$$\left.-\frac{2}{m+1} lx + \frac{2}{(m+1)^2}\right).$$

(79). $\int \frac{lx}{(1-x)^2} dx = \frac{xlx}{1-x} + l(1-x)$.

(80). $\int \frac{dx}{xlx} = l(lx) = l^2 x$.

(81). $\int \frac{dx}{x(lx)^2} = -\frac{1}{lx}$.

(82). $\int \frac{dx}{x\sqrt{x}} \log\frac{1}{1-x} = 2l\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$

$$-\frac{2}{\sqrt{x}} l\left(\frac{1}{1-x}\right).$$

(83). $\int a^x xdx = \frac{1}{la} a^x x - \frac{1}{(la)^2} a^x$.

(84). $\int dx \sin^{-1} x = x\sin^{-1} x + \sqrt{1-x^2}$.

(85). $\int xdx \sin^{-1} x = \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x$

$$+\frac{1}{4} x\sqrt{1-x^2}.$$

(86) $\int \frac{d\theta}{\tan\theta} = l(\sin\theta)$.

(87). $\int \frac{d\theta}{\cos\theta\sin\theta} = l(\tan\theta)$.

Many other formulas might be added, but the limits of the present work exclude any further selections. The various applications of integral calculus will be mentioned under their appropriate headings.

Calculus of Variations. This is the highest branch of mathematics, and in its proper acceptation, treats only of the laws of variation, the forms of indeterminate functions, and the application of these laws to other branches of mathematics, mechanics, &c. The subject is, however, so intimately connected with the differential and integral calculus, that many operations which strictly belong to these branches, are often referred to the calculus of variations.

We can only give an imperfect outline of the nature of this branch of the calculus, and in doing so, shall endeavor to confine our remarks to what strictly belongs to the Calculus of Variations.

It has already been stated, that functions are either determinate or indeterminate. An indeterminate function, is one in which a relation between the function and variables is expressed, but in which the nature or form of the relation is entirely arbitrary, or only subject to certain general conditions.

Thus, in the expression $u = \phi(x, y, z, \&c.)$ u is an indeterminate function of the variables x, y, z, &c., and, so far as the expression indicates, the form of the function or the relation between u and these variables is entirely arbitrary.

It is evident that the form of one function may be so related to that of another, that if the form of the latter be determined or given, that of the former may also be determined. Thus, the *differential coefficient* of a function depends upon and may be deduced from the form of the function itself: we may conceive many other relations between the forms of functions which make them dependent upon each other. A function whose form depends upon that of another, is called a *derived function* of the former, which, with respect to the latter, is called the *primitive function.*

If we attribute an arbitrary change of form to the primitive function, the derived function will experience a change of form, which will not be arbitrary, but will be connected by a fixed law of relation with the change of form attributed to the primitive function.

It is the investigation of this law of relation in every possible case, which constitutes the principal object of the calculus of variations.

To acquire an idea of what is meant by a *variation*, let us first consider a primitive function. Let $u = \phi(x, y, z, \&c.)$ be any indeterminate function of x, y, z, &c. If we add to this function $c\phi'(x, y, z, \&c.)$, and denote the resulting form by u', we shall have

$$u' - u = c\phi'(x, y, z, \&c.)$$

If now an infinitely small value be attributed to c, which is supposed constant, the difference between u' and u will be infinitely small for all values of the variables x, y, z, &c., whatever may be the form indicated by ϕ'. The increment $c\phi'(x, y, z, \&c.)$, is called the variation of the function u, and since the form indicated by ϕ' is perfectly indeterminate, we conclude that the variation of the primitive function is *arbitrary*. The only restriction laid upon the variation is, that it must not be of such a form that it, or any of its successive differentials, will become infinite for any values of the variables except those which render the primitive function and its corresponding differential infinite also. This restriction does not impair the arbitrary character of the variation.

We are now prepared to explain what is meant by a variation of a derived function. Let F be a symbol of derivation, indicating simply that the function is derived by some fixed law from the primitive function; thus, in the expression $u = F[\phi(x, y, z, \&c.)]$, u is derived from the function $\phi(x, y, z, \&c.)$ by a determinate law. If now we add to $\phi(x, y, z, \&c.)$ the variation $c\phi'(x, y, z, \&c.)$, and then perform upon the resulting function the operation indicated by F, continuing the operation sufficiently far to obtain that term which is of the first degree with respect to c, then will this term of the first degree, with reference to c, be the variation of the derived function. The fact is rendered obvious by the same train of reasoning as is used in explaining the nature of a differential of any function.

Hence, to find the variation of any derived function, add to the primitive function the product of an infinitely small and constant quantity by any arbitrary function of the same variables, perform the operations indicated, and that term of the result which is of the first degree with respect to the constant, is the required variation.

This is the general problem of the calculus of variations, but in the present state of mathematical science, attention is only directed to two laws of derivation, viz.: that by differentiation and integration, and these have been found sufficient, thus far, for all practical purposes.

We may, therefore, consider the symbol F as replaced by either the symbol d or $\int$, and when so replaced we know from the laws of differentiation and integration that the operation denoted heretofore by F becomes distributive, that is, letting ϕ and ϕ' stand for the functions whose form they indicate,

$$F(\phi + c\phi') = F\phi + Fc\phi';$$

and further, from the same principles,

$$Fc\phi' = cF\phi',$$

whether F denotes differentiation or integration. Upon this basis, the entire science of the calculus of variations is founded. For further information as to the principles and their applications, the reader is necessarily referred to special treatises on the subject. One of the most complete works on the subject in the English language is that of Prof. Jellet of the University of Dublin.

Calculus of Finite Differences. That branch of analysis which treats of the *finite differences* of functions. This branch is usually regarded as being very closely allied to the differential and integral calculus, but, with the exception of a general similarity in the notation employed, and in the terms made use of, they have little in common.

In order to explain what is meant by finite differences, let u be any function of x, expressed thus, $u = f(x)$. If in this function we substitute $x + h$, for x, h being a finite increment of x, and denote the new state of the function by u', then is the difference between the new and primitive states of the function called a finite difference of the unction, h being the finite difference of the variable x. The symbol Δ, is employed to designate a finite difference, so that we should havs

$$\Delta u = u' - u = f(x + \Delta x) - f(x).$$

If now we take the finite difference of a finite

difference of the first order, it is called a finite difference of the second order. The finite difference of a finite difference of the second order is a finite difference of the third order, and so on. The successive finite differences are represented by the symbols

$$\Delta u,\ \Delta^2 u,\ \Delta^3 u, \ldots \Delta^n u.$$

Any function being given, its finite differences of the different orders may be found by means of the principles of analysis, or by means of certain rules or formulas, deduced for the purpose, in this branch of mathematics.

The calculus of finite differences consists of two parts: 1st. Having given a function, to determine its successive orders of differences. The successive finite difference of the *independent* variable is generally taken constant. 2d. Having given any one of the successive orders of differences, to find a function for which it might have been derived.

These divisions, similar to those of the differential and integral calculus, are called the *direct* and the *inverse calculus of finite differences*.

It has already been remarked that the calculus of finite differences is logically unconnected with the differential and integral calculus. What constitutes the peculiar character of the latter branches, and gives them their great power as instruments of scientific investigation, is the fact that the derived functions are of an entirely different nature from the primitive ones, giving rise to relations not only more general but also more simple and more easily deduced: in the calculus of finite differences the derived functions are essentially similar in their nature to the primitive functions from which they are derived. This circumstance prevents their being used in deducing more general relations than those existing between the primitive functions and their variables. The calculus of finite differences, aside from its peculiar notation, is nothing else than an extended branch of ordinary analysis, and all the truths deduced by means of this calculus, may be established by the ordinary operations of analysis, without any reference whatever to the notation explained. As a branch of analysis, it finds its proper application in the investigation of the nature and properties of series.

The *direct calculus of finite differences*, enables us to find the general term of a series, knowing the law of the series, and thus enables us to find any term by a simple substitution. It also enables us to deduce formulas for the sum of any number of terms of a series from a knowledge of the law of the series. From these the particular sum in any given instance, may be found without the trouble of continually adding the terms together.

The *inverse calculus of finite differences* enables us to determine the law of the series from the sum of any number of terms, or from the expression of the general term.

Since there may be conceived an infinite number of laws of series, there are an infinite number of different series. Besides these applications, another very important one is its application to interpolation, that is, finding from a series of terms corresponding to equal finite differences of the variable any intermediate term which shall conform to the law of the series. See *Interpolation*.

Another important application is to the approximate rectification and quadrature of curves. By this method we may sometimes arrive at good practical results which might not otherwise have been obtained. It is to be observed, however, that all these applications are in no wise peculiar to this kind of calculus, for the results may be reached by the principles of ordinary algebraic analysis. See *Series*, *Summation*, &c.

CAN′CEL [L. *cancello*, to deface, to make cross-bars or lattice work]. To cross out. In Arithmetic, the operation of striking out the common factors in both dividend and divisor, before performing the operation of division, is called *canceling*.

When several factors are found in both dividend and divisor, the operation of division is often much simplified by canceling such factors as are common to both.

CAN-CEL-LĀ′TION. The operation of canceling.

CAP′I-TAL. [L. *capitellum*, from *caput*, the head]. The uppermost part of a column or pylaster. The sum of money which a merchant, banker or manufacturer employs in his business.

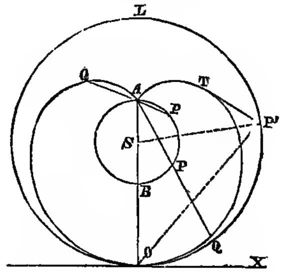
CAR′AT. [Fr. *carat*, weight for diamonds]. A weight of four grains employed in weighing diamonds. The term is also used in

estimating the fineness of gold. The whole mass of the alloy is supposed to be divided into 24 equal parts; then the number of these parts which are pure gold will express the number of carats of fineness of the alloy: thus, if in a certain alloy there is contained $\frac{22}{24}$ths of pure gold, the alloy is said to be 22 carats fine.

CĂR′DI-NAL POINTS. [L. *cardinalis*, from *cardo*, a hinge]. In Navigation, the four principal points of the compass: *North*, *South*, *East*, and *West*.

CĂR′DI-OID. [Gr. χαρδια, a heart, and ειδος, shape or form]. The name of a heart-shaped curve, which may be generated as follows:

Let APB represent any circle, and AB one of its diameters: if, through one extremity A, a straight line APQ be drawn, and the



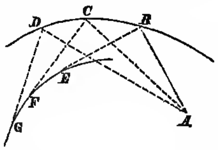
distance PQ be made equal to AB, then will the locus of the point be a cardioid. The line is algebraic, and if the origin be taken at O, and the axes of X and Y coinciding with OX and OA respectively, its equation is

$$y^4 - 6ay^3 + (2x^2 + 12a^2)y^2 - (6ax^2 + 8a^3)y + x^4 + 3a^2x^2 = 0,$$

in which a is the diameter of the directing circle.

CAT-A-CAUS′TIC CURVE. [Gr. κατακαυσις, a burning]. A curve of the higher geometry, which may be generated as follows:

Let BCD be any plane curve, and A, a point in its plane. From A draw any line AB to any point of the curve, as B, and from the point of intersection B draw a second line BE, making with the normal at the point B, an angle equal to that made by the line AB; conceive the same construction to be made at each point of the curve; then will the curve drawn tangent to all of the lines BE, CF, DG, &c., be a catacaustic curve. If we conceive the line BCD to be a reflector, and A a radiant point, then will the lines BE, BC, &c., be reflected rays; the catacaustic is, therefore, a curve tangent to all of the reflected rays.



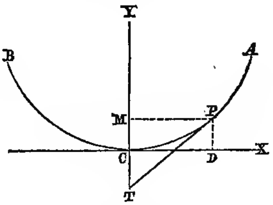
If the curve BCD is a circle, and the rays incident upon it are parallel, the catacaustic is an epicycloid, a curve generated by a point of the circumference of a circle when it is rolled upon the circumference of another circle.

In like manner, if the incident rays are parallel, and the reflecting curve is a cycloid whose axis is parallel to the rays, then will the catacaustic be a cycloid.

The catacaustic of the logarithmic spiral, is a logarithmic spiral. See *Caustic*.

CAT′E-NA-RY. [L. *catenarius*, from *catena*, a chain]. The curve which a heavy cord or flexible chain of uniform thickness and density forms, by reason of its own weight, when freely suspended by two of its points. It is chiefly interesting on account of the light which its investigation has thrown upon the theory of arches, and also by reason of its application in the construction of suspension bridges.

Let ACB represent the curve assumed by a chain or rope of uniform thickness and density, when freely suspended by its two extremities A and B.



Let C be the lowest point of the curve, and let CX, a horizontal tangent at C, be taken as the axis of X, and let CY, perpendicular to CX, and lying in the plane of the curve, be taken for the axis of Y, and let P be any point of the curve.

Let a denote the length of a portion of the chain whose weight is equal to the tension at C.

Let t denote the length of a portion of the chain whose weight is equal to the tension at P.

Let s denote the length of the portion of the chain between C and P.

Denote the co-ordinates of P by x and y, and by θ, the angle included between the tangent PT, at P, and the axis of y; that is, the complement of the inclination of the curve to the horizon at that point.

Now, the portion CP is held in equilibrium by the tension at P, the tension at C, and its own weight. These forces are proportional to t, a and s respectively, and their directions are respectively parallel to the three sides of the triangle PMT.

From these principles, we have

$$\frac{s}{a} = \frac{MT}{PM} = \frac{dy}{dx}\,;\ \frac{t}{a} = \frac{TP}{PM} = \frac{ds}{dx} \cdots\cdots (1).$$

From (1), we have

$$\frac{dy^2}{dx^2} + 1 = \frac{dy^2 + dx^2}{dx^2} = \frac{ds^2}{dx^2} = \frac{a^2 + s^2}{a^2} \cdot\cdot (2);$$

whence,

$$\frac{dx}{ds} = \frac{a}{\sqrt{a^2 + s^2}} \cdots\cdots (3).$$

By integration, observing that $s = 0$ when $x = 0$, we have,

$$x = al\left(\frac{s + \sqrt{s^2 + a^2}}{a}\right) \cdots\cdots\cdot (4).$$

This gives

$$e^{\frac{x}{a}} = \frac{\sqrt{s^2+a^2}+s}{a} \text{ and } e^{-\frac{x}{a}} \frac{\sqrt{s^2+a^2}-s}{a} \quad (5).$$

Subtracting and reducing,

$$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \cdots\cdots (6).$$

Substituting in (1),

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) = \cot\theta \cdot \ (7).$$

Integrating and remembering that $x = 0$ when $y = 0$, we have

$$y + a = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \cdots\cdots (8),$$

which is the equation of the catenary.

Differentiating equation (6),

$$\frac{ds}{dx} = \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \cdots\cdots\cdot (9).$$

Substituting in the second of equations (1), we find

$$t = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \cdots\cdots (10).$$

Now, if we denote by N the number whose Naperian logarithm is $\frac{x}{a}$, we have, from the above equations, the following group of formulas:

$$\left.\begin{aligned} s &= \frac{a}{2}\left(N - \frac{1}{N}\right) = a \cot\theta \\ \cot\theta &= \frac{1}{2}\left(N - \frac{1}{N}\right) \\ y &= \frac{a}{2}\left(N + \frac{1}{N}\right) - a = t - a \\ t &= \frac{a}{2}\left(N + \frac{1}{N}\right) \end{aligned}\right\} (A)$$

By assuming different values for $\frac{x}{a}$, the corresponding values of y, s, t, and θ, can be found from the formulas and tabulator for practical purposes.

The following are taken from a set of tables published in the Philosophical Transactions of 1826.

TABLE I.—ORDINARY CATENARY.

$x = 100.$

a	N	y	s	t	θ ° ′ ″
1000	1.105170	5.004084	100.165906	1005.004840	94 16 48
900	1.117519	5.561266	100.205825	905.561266	83 38 48
800	1.133148	6.258102	100.260296	806.258102	82 51 23
700	1.153564	7.154926	100.339869	707.154926	81 50 33
600	1.181360	8.352608	100.463404	608.352608	80 29 40
500	1.221402	10.033315	100.667683	510.033315	78 30 59
400	1.284025	12 565207	101.044792	412.565207	75 40 22
300	1.395612	16.821529	101.862069	316 821529	71 14 44
200	1.648721	25 525175	104.219022	225.525175	62 28 34
100	2.718281	54.308027	117.520071	154.308027	40 23 42
90	3.037731	61.511583	121.884206	151.511583	36 20 34
80	3.490342	71.073875	128.153485	151.073875	31 58 28
70	4.172733	84.433443	137.657866	154.433443	26 57 10

TABLE II.—ORDINARY CATENARY.

$a = 100.$

x	N	y	s	t	θ ° ′ ″
1	1.010050	.004999	1.000000	100.004999	89 25 39
2	1.020201	.020000	2.000100	100.020000	88 51 15
3	1.030454	.045001	3·000398	100 045001	88 16 53
4	1.040810	.080007	4.000992	100.080007	87 42 31
..	......	......	....	....	...
20	1.221402	2.006663	20 133536	102.006663	78 36 59
21	1.233678	2.213114	21.154685	102.213114	78 3 19
22	1.246076	2.429763	22.177836	102 429763	77 29 43
23	1.258600	2.656680	23 203319	102.656680	70 50 11
24	1.271249	2.893847	24.231042	102.893847	76 22 45
25	1.284025	3.141302	25.261197	103.141302	75 49 22

To show the use of these tables, let the span proposed for a suspension bridge be 800 feet, and let the weight of suspension rods, roadway, &c., be taken at one-half of the weight of the chains; and let it be determined to load the chains at the points of greatest strain, that is, at the points of suspension, with one-sixth of the weight they are theoretically capable of sustaining. By taking into account the strength of iron, and imposing the conditions enumerated, we find from the data that the tension at the points of suspension is 1644.5 feet.

Then since the semi-span is 400 feet, and x in Table I. is taken at 100 units, each unit must be 4 feet, and the tension at the points of suspension, estimated in terms of the same unit, is $\frac{1644.5}{4}$, or 411.125. Now it appears from Table I. where x is uniformly 100, that when $t = 412$, $a = 400$ units, or 1600 feet, $y = 12.565$ units, or 50.260 feet, $s = 101.045$ measures, or 404.180 feet, and θ the angle of suspension equals $75° 49'$. The value of a being now determined, the values of all the other quantities may be determined for different values of x, that is, for different points along the curve. But as a in this table denotes 100 units, each unit here must be 16 feet, consequently each gradation of x must be 16 feet, and the whole semi-span $\frac{400}{16}$, or 25 units. Since s is given in the table for each gradation of x, the additional weights may be so adapted as to preserve the true form of the catenary. Thus, at

21 units of x, $s = 21.1547$
20 " " $s = 20.1335$

$$1.0212 \times 16 = 16.3392.$$

Consequently, whilst the ordinate extends one unit of 16 feet from the 20th to the 21st unit, the length of the curve will increase $16\frac{1}{3}$ feet very nearly, and the adjunct weight should be proportionally increased.

It appears from Table I. that for a given span, t the tension at the points of suspension, is least when y equals $\frac{1}{3}$ of the span nearly.

CAUSE. [L. *causa*, a cause]. Anything which operates to produce a result. The result of a cause is called its *effect*.

The terms *cause* and *effect* are used technically in mathematics almost synonymously with *antecedent* and *consequent*. It is a natural law deduced from universal experience, that the *effect* is proportional to the *cause* which produces it. This principle is called the *principle of cause and effect*, and is of extensive use in solving questions in the *rule of three*, and also in many other branches of mathematics. The principle of cause and effect forms the basis of the science of mechanics; in fact, upon it rests the entire subject of the application of mathematics to the physical sciences.

A cause or an effect may be either *simple* or *compound*: *simple*, when it involves but a single element: *compound*, when it involves two or more elements. A compound cause or effect is equal to the continued product of all its elements.

Numerical Value of a cause, effect, or any one of their elements, is the ratio obtained by dividing either by its unit of measure. We shall consider the numerical values only in the following discussion.

To illustrate the meaning of simple and compound causes and effects, let us consider a few examples.

1. If 5 yards of cloth cost 20 dollars, we may regard 5 as the cause, and 20 as the effect; in this case both are simple.

2. If 5 yards of cloth, 2 yards wide, cost 20 dollars, we may regard 5×2 or 10 as the cause, and 20 as the effect; in this case the cause is compound, and the effect simple.

3. If 10 men dig a trench 40 feet long, 4 feet wide, and 8 feet deep, we may regard 10 as the cause, and $40 \times 4 \times 8$ or 1280 as the effect; in this case, the cause is simple, and the effect compound.

4. If 10 men in 10 days dig a trench 40 feet long, 4 feet wide, and eight feet deep, we may regard 10×10 or 100, as the cause, and $40 \times 4 \times 8$ or 1280 as the effect; in this case, both cause and effect are compound.

These are the only possible combinations of simple causes and effects.

Causes are *similar* when they are of the same kind, or have the same unit; and effects are similar when they are of the same kind, or have the same units. The principle of cause and effect as applicable to the solution of questions in the rule of three, may be enunciated thus:

Any cause is to any similar cause as the effect of the first cause is to the effect of the second cause.

In all cases of the simple rule of three, we have given two causes and the effect of the first, to find the effect of the second, or we have given two effects and the cause of the first, to find the cause of the second. The question may then be solved by the following rule:

When two causes are given, write the first cause in the first term, the second cause in the second term, and the effect of the first cause in the third term.

When two effects are given, write the first effect in the first term, the second effect in the second term, and the cause of the first effect in the third term. Then multiply the second and third terms together, and divide the product by the first term; the quotient will be the required effect or cause.

1. If 8 hats cost 24 dollars, what will 110 hats cost?

Cause.		*Cause.*		*Effect.*
8	·	110	: :	24,

whence $\dfrac{24 \times 110}{8} = \$330$, effect or cost.

2. If 120 sheep yield 330 pounds of wool, how many sheep will yield 2640 pounds?

Effect.		*Effect.*		*Cause.*
330	:	2640	.	120,

whence $\dfrac{2640 \times 120}{330} = 960$, cause or No. sheep.

In double or compound rule of three, that is, when either of the causes, or the effects, or both, are compound, we may have given everything except one element of either a cause or an effect to find that element. The solution of all such cases can at once be effected by a slight modification of the preceding rule.

When the compound causes produce the same effects.

1. If 12 men consume a certain amount of provision in 7 days, how long will the same provisions last 21 men?

The first cause is compounded of 12 men and 7 days, and the second cause of 21 men, and an unknown number of days, say x days.

But since the effects are equal, the causes are equal. Hence, to find the unknown element: Divide the product of the elements of the cause, where all the elements are known, by the product of the known elements of the other cause, and the quotient will be the unknown element. In the example the unknown element is 4 days.

When the compound causes produce different effects.

1. If 10 men in 5 days of 7 hours each day, dig a trench 25 feet long, 8 feet wide, and 7 feet deep, in how many days of 12 hours each will 4 men dig a trench 12 feet long, 10 feet deep, and 8 feet wide.

Cause.	*Cause.*	*Effect.*	*Effect.*
$10\times5\times7$:	$12\times4\times x$: :	$25\times8\times7$:	$12\times10\times8$

in which we wish to find x, an element of the second cause; hence,

$$x = \frac{12 \times 10 \times 8 \times 10 \times 5 \times 7}{12 \times 4 \times 25 \times 8 \times 7} = 5 \text{ days.}$$

Hence, arrange the terms in the statement so that the causes shall compose the first couplet, and the effects the second, putting x in the place of the required element. If x falls in one of the extremes, make the product of the means the dividend, and the product of the extremes the divisor; but if x falls in one of the means, make the product of the extremes the dividend, and the product of the means the divisor.

By a similar application of the principle of cause and effect, a great variety of problems in *barter*, *fellowship*, *percentage*, &c., may be solved. The practical application of the principle has not only the advantage of leading to the simplest solutions of the questions proposed, but it also serves more clearly than any other method, to show the analytical relation of the elements of the problem solved.

CAUS'TIC CURVE. [Gr. καυστικος, from καιω, καυσο, to burn]. A curve of the higher geometry, which is always tangent to rays of light proceeding from a point, and deviated at a given surface. When deviated by reflection, the curve is called catacaustic, (which see). When deviated by refraction, the curve is called diacaustic. See *Diacaustic Curve.*

CEN'TI-GRAMME. L. *centum*, a hundred: Fr. *gramme*]. The hundredth part of a French gramme. See *Weights* and *Measures.*

CEN'TI-LĬ-TRE. [L. *centum*, a hundred;

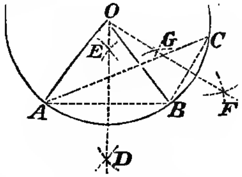
Fr. *litre*]. The hundredth part of a French litre. See *Weights* and *Measures.*

CEN-TIM'E-TRE. [L. *centum*, a hundred; Gr. *μετρον*, measure]. The hundredth part of a French metre. See *Weights* and *Measures.*

CEN'TRAL. [L. *centralis*, placed in the centre]. Appertaining to the centre. A diameter is a central line of a circle. A line drawn through the centres of two circles, in the same plane, is a central line of both circles, and is called the line of centres.

CEN'TRE. [Gr. *κεντρον*, centre; L. *centrum*, centre]. The centre of a plane curve is a point in the plane of the curve, which bisects every straight line drawn through it and terminated by the curve. If any curve has a centre, and the origin of a system of rectangular co-ordinates be taken at the centre, then for every point on one side of this centre whose co-ordinates are x and y, there will be another point diametrically opposite, whose co-ordinates are $-x$ and $-y$. And since the co-ordinates of both these points must satisfy the equation of the curve at the same time, it follows that the form of the equation must be such that it will not be changed by changing $+x$ into $-x$ and $+y$ into $-y$; that is, every term must be of an even degree with respect to x and y. Hence, in order to ascertain whether any curve has a centre, we transform it by changing the origin of co-ordinates, the new axes being parallel to the primitive ones, and see whether such values can be assigned to the arbitrary constants which enter the equation, as will reduce all of its terms to an even degree with respect to the variables. If such values can be assigned, the curve has a centre, and the new origin is at the centre. If such values cannot be assigned, the curve has no centre.

In curves of the second order, the ellipse and hyperbola have each one centre, whilst the parabola has none at a finite distance. It is shown that every diameter passes through the centre; hence, in order to construct the centre of an ellipse or hyperbola, *draw any two parallel chords in the curve and bisect them by a straight line; this will be a diameter. Construct, in like manner, a second diameter, and this will intersect the one already constructed at the centre.* If we attempt to apply this construction in the parabola, we shall find the diameters parallel, and the construction must fail.



In the circle, the diameters which bisect chords, are also perpendicular to them. Hence, to find the centre of any circle or arc of a circle, as ABC, draw any two chords, AB and BC, and bisect them by the perpendiculars FO and DO; the point in which the perpendiculars DO and EO intersect, is the *centre* required.

The centre of a surface is a point which bisects all straight lines drawn through it and terminated by the surface. When a surface is given by its equation, we can, by a course of proceeding entirely analogous to that used in discussing the subject of centres of curves, ascertain whether the surface has a centre. It is found that amongst surfaces of the second order, the ellipsoid and hyperboloid have centres, whilst the paraboloids have no centres at a finite distance. To find the centre of a surface, if it has one, draw three parallel chords and bisect them by a plane; this is a diametrical plane, and passes through the centre. In like manner, construct two other diametrical planes, and the point common to the three planes is the centre.

To find the centre of the sphere, we may find one diametrical plane and then find the centre of the circle, which it cuts from the surface, and this point will be the centre of the sphere.

The centre of a regular polygon is the centre of the inscribed or circumscribed circle. The centre of a solid is the centre of an inscribed or circumscribed sphere.

Centre of Symmetry, is that point of a figure about which the different parts are symmetrically arranged.

Centre of Curvature. The centre of curvature of any curve at any point, is the centre of the osculatory circle at that point. The locus of all the centres of curvature of a curve, is the evolute of the curve.

CEN′TRI-PLE. [L. from *centum*, a hundred, *plico*, to fold]. A hundred fold.

CEN′TU-RY. [L. *centuria*, from *centum*, a hundred]. A period of time equal to 100 years.

CHĀIN. [Fr, *chaîne*, a chain. L. *catena*]. An instrument used in surveying, for measuring horizontal distances. The chain most used in land surveying is that called Gunter's. It is 66 feet in length, and contains one hundred links, which are connected with each other by small rings. The length of each link, including a connecting ring, is 7.92 inches. Every tenth link is marked by inserting a piece of brass between it and the next, to aid in counting the links in any distance. The length of Gunter's chain is so chosen, that an area which is one chain in breadth, and ten chains in length, shall be equivalent to one acre; so that if we measure all the courses in chains and links, the resulting area will be expressed in square links, and may at once be converted into acres and decimals of an acre, by pointing off five places of decimals from the right. In using the chain, care should be taken to compare its length from time to time with a standard. In order to make this comparison the more readily, a distance of 66 feet should be accurately marked on some smooth surface, as the coping of a wall, and its extremities permanently marked. The comparison can then be easily made.

If it is found that a survey has been made with a chain, either too short or too long, the area, as found, may be reduced to the true area by multiplying it by the square of the ratio obtained by dividing the length of the chain employed, by 66 feet.

To find any linear dimension, when it has been measured with a chain either too long or too short, we multiply the measure found by the first power of the above ratio.

In making surveys for topographical purposes, it is often found convenient to employ a chain 50 feet in length, which is divided into 100 links, each of which is 6 inches in length.

CHĂNCE. [Fr. *chance*, chance]. In the heory of probabilities, the word chance is used to signify the occurrence of an event in a particular way, when there are two or more ways in which it may take place, and when no reason can be assigned why it should happen in one way rather than in another. For example, if a die be thrown up into the air, it will necessarily fall upon one of its six faces; but we can assign no reason why it should fall upon the face marked *one*, rather than on the face marked *two*, *three*, &c. We say, therefore, that the chance of its falling on any one face is the same as that of its falling on any other. Now, since there are six different faces, upon any one of which it *may* fall, we say that there are six chances in all, and as it can only fall on one, we say that the chance of its falling upon any designated face, is one out of six, or $\frac{1}{6}$.

The word chance is applied to events, to denote that they happen without any foreknown cause; or it is used to denote the possibility of an event, when nothing is known to hinder it. We say, a thing happens by chance, when we would simply indicate that we know nothing of its cause. We do not intend the term to imply that chance can be the cause of anything.

CHAR′AC-TER. [L. *caracter*; Gr. *χαρακτηρ*, from *χαρασσω*, to cut, to engrave]. A symbol employed to represent some quantity or some operation to be performed upon an expression. Thus, $\sqrt{}$ is a *character* to express that the square root of the quantity before which it is placed, is to be extracted. See *Notation.*

CHAR-AC-TER-IS′TIC OF A LOGARITHM. [Gr. *χαρακτηριστικος*, from *χαρακτηρ*, a mark or token impressed on a thing]. The logarithm of a number is composed of two parts, a whole number and a decimal fraction. The whole number is called the *characteristic*, and the decimal part is sometimes called the *mantissa.*

In the common system, the characteristic of the logarithm of a whole number is always 1 less than the number of places of figures in the integral number. When the number is a decimal fraction, the characteristic of its logarithm is always negative, and numerically 1 greater than the number of 0's which immediately follow the decimal point. When the number is a mixed decimal, the characteristic of its logarithm is the same as that of the entire part, without reference to

the decimal part. The characteristic of the logarithm of a vulgar fraction is equal to the number of places of figures in the numerator minus the number of places of figures in the denominator.

On account of the simplicity of these rules for finding the characteristic of the logarithm of a number taken in the common system, it is not customary to write the characteristics in the tables; this is one of the advantages of this system.

Let a denote the base of any system of logarithms, and let us write the series

$$a^n, \cdots a^2, \quad a^1, \quad a^0, \quad a^{-1}, \quad a^{-2}, \quad \cdots a^{-n}.$$
$$n, \cdots 2, \quad 1, \quad 0, \quad -1, \quad -2, \cdots -n.$$

It is plain from an inspection of this series that the logarithm of 1 is 0, and the logarithm of a is 1; and we see that the logarithm of any number between 1 and a will be found between 0 and 1; that is, will be a fraction less than 1: its characteristic is therefore 0. In like manner, the characteristic of the logarithm of any number between a and a^2 is 1; of any number between a^2 and a^3 is 2; and in general, the characteristic of the logarithm of any number between a^n and a^{n+1} is n. If the number falls between two negative powers of a, as for instance, between a^{-2} and a^{-3}, the characteristic is then -3, and the decimal part or mantissa of the logarithm is positive. The series above written indicates the characteristic, in all cases; hence, the characteristics of logarithms need not be written in the tables.

Characteristic Property of a magnitude is such a property as can only exist in that magnitude.

Thus, "the portion of a tangent line to a curve, which is intercepted between the asymptotes, is bisected at the point of contact," is a characteristic property of the hyperbola; "the squares of the ordinates, to any diameter, are always equal to the rectangles of the segments into which they divide the diameter," is a characteristic property of the circle. If we can show that any given magnitude possesses a characteristic property of any magnitude, it is equivalent to showing that these magnitudes are the same in kind.

CHÄRT. [L *charta*, a map]. A hydrographic map for the use of navigators, being a projection of some part of the earth's surface on a plane. Charts, as well as ordinary maps, may be constructed according to any of the methods of spherical projection; but the method first employed by Mercator, and called Mercator's projection, is generally preferred. In this projection, the meridians and parallels of latitude are represented by straight lines. The degrees of longitude on all the parallels of latitude are represented of equal length, and each equal to the length of a degree of longitude at the equator; the degrees of latitude increase from the equator towards the pole, so that in any latitude they shall bear the same ratio to the degrees of longitude that they do on the surface of the earth, at the corresponding latitude.

The great advantage of this chart, and the one which has caused its almost universal adoption, is that the rhumb or sailing course between two points is represented on the chart by a straight line. This enables the navigator to plot the path of his ship without difficulty.

The mathematical relation between the length of a minute of latitude and a minute of longitude, at any point of the earth's surface, may be enunciated as follows: "The length of a minute of latitude at any point of the earth's surface, is to the length of a minute of longitude at that point, as the radius of the equator is to the radius of the parallel of latitude through the point; that is, as 1 is to the cosine of the latitude, or as the secant of the latitude is to 1." This principle enables us to construct a blank chart representing the parallels of latitude and longitude corresponding to any given portion of the earth's surface. For this purpose, a table of meridional parts will be required, which may be found in any treatise on Navigation.

The blank chart may be thus constructed: Draw on the lower part of the paper a horizontal line, to represent the southernmost parallel of latitude which is to be represented in the chart. From a suitable scale of equal parts lay off upon this line a number of equal distances, each of which we will suppose to be equal to 60 equal parts of the scale, and through these points of division draw lines perpendicular to the first line; these will represent meridians which are one degree apart.

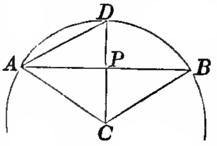
Find, from the table of meridional parts,

the meridional parts corresponding to the latitude of the parallel already drawn, and also the meridional parts corresponding to a parallel one degree farther north, and subtract the former from the latter; lay off this distance from the scale of equal parts on one of the meridians from the southernmost parallel, and mark the point thus found. Find, in like manner, the meridional difference of latitude corresponding to two degrees, and lay it off on the same meridian and from the same point. Find, also, the meridional differences of latitude corresponding to three, four, five, &c., degrees to the northern limit of the chart, and lay them off as before. Then, through the points of division found, draw lines parallel to the first lines drawn, and they will represent parallels of latitude one degree apart. We might, in a similar manner, construct a blank chart, in which the parallels represented should be nearer than one degree or farther apart. Having constructed the blank chart, it may be filled in by plotting down the principal points by means of their latitudes and longitudes, and then sketching in the coast lines, and marking such features as the nature of the chart may require.

Plane Charts. These have the parallels of latitude and longitude parallel to each other respectively, and everywhere as far apart as at the equator. They can only be used with any tolerable degree of accuracy in the immediate neighborhood of the equator, consequently they are not much used.

For other methods of making charts, see *Spherical Projection* and *Projection of Maps*.

CHORD. [L. *chorda*; Gr. χορδή, string or gut]. Of an arc of a curve, is a straight line joining its two extremities. In the circle, the chord of an arc possesses the following properties:



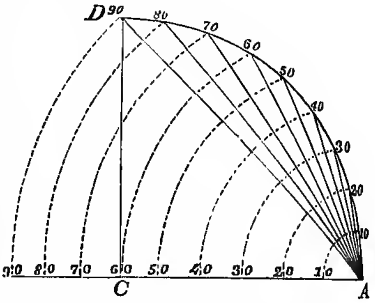
1st. A straight line drawn from its middle point to the centre, is perpendicular to it, and also bisects the subtended arc.

2d. Chords which are equally distant from the centre are equal to each other, and of two unequal chords the longer is nearer the centre.

3d. The chord of an arc is a mean proportional between the diameter and versed sine of the arc.

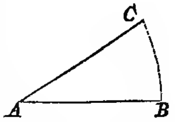
4th. The chord of an arc is equal to twice the sine of half the arc, or it is equal to the sine of half the arc described with double the radius.

Scale of Chords. This is a scale usually laid down upon the rules accompanying boxes of mathematical instruments. It may be constructed as follows:



With a radius AC describe a quadrant AD, and divide it into 90 equal parts; then through A, and each of the points of division let chords be drawn, and let these chords be laid off on a scale from A; the resulting scale is a scale of chords. It is used for laying off angles.

1. To lay off any angle, as 30°, from the line AB.

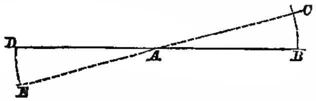


With A as a centre, and with a radius equal to the chord of 60° taken from the scale, describe the arc BC, then with B as a centre, and a radius equal to the chord of 30° taken from the scale, describe an arc cutting the first one in C. Draw AC, and the angle CAB will be equal to 30°. When greater accuracy is required, the chord of the arc may be computed and the distance taken from a scale of equal parts.

2. Suppose it were required to construct an angle of 31° 24′ 20″ by means of a table of natural sines.

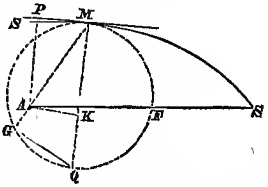
With A as a centre, and with a radius of ten parts taken from a scale of equal parts,

describe the arcs BC and DE. Look in a table of natural sines for the sine of 15° 42′ 10″ which is .2706134, remove the decimal



point one place to the right, since the radius used is 10, and multiply the result by 2; then with B and D as centres, and with a radius equal to $2 \times 2.706134 = 5.412268$ taken from the same scale of equal parts, describe arcs cutting BC and DE in the points C and E. Join C and E by a straight line, it will pass through A and make the angle CAB equal to 31° 24′ 20″. By determining the two points E and C, we are enabled to verify the accuracy of the construction.

Chord of Curvature. If SMS is any curve, which we will suppose referred to a



system of polar co-ordinates, A being the pole, AS the initial line of the system, and MFG the osculatory circle at the point M, then is the chord MG of this circle which passes through the pole A and the point of osculation M, called the chord of curvature at the point M. If we denote the radius vector of the point M by r, the perpendicular AP from the pole to the tangent at M by p, and the radius of curvature at M by R, we have

$$R = \frac{rdr}{dp}.$$

To deduce an expression for the chord of curvature, let us draw the chord MG and its supplement GQ; draw also through the pole the line AK perpendicular to the diameter of the osculatory circle through the point of contact. Then, from the similar right-angled triangles MKA and MGQ, we have

$$\text{MG} : \text{MQ} :: \text{MK} : \text{MA}, \text{ or}$$
$$\text{MG} : 2\text{R} :: p : r,$$ since

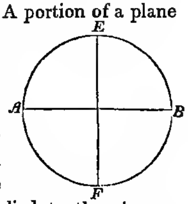
$\text{MK} = \text{AP} = p$; hence, $\text{MG} = \dfrac{2p\,dr}{dp}$.

This is an important element in astronomical investigations.

CI′PHER. [Fr. *chiffre*, cipher, figure]. In Arithmetic, the character 0; when it stands by itself, it signifies *no number;* in combination, it occupies a place in the arithmetical scale, and indicates that there are no units of that order in the number. If ciphers be annexed to an integral number, the effect is the same as if the number were multiplied by 10, for each cipher annexed. If ciphers be prefixed to an integral number, they produce no effect upon the number. In decimals, these effects are reversed.

To Cipher, is a common term, which is applied to the performance of any arithmetical operation, by pupils.

CIR′CLE. [L. *circulus*, from *circus*, anything of a round form]. A portion of a plane AEBF, bounded by a curved line, every point of which is equally distant from a point within called the *centre*. The bounding line is called the *circumference*. The



term circle, is often applied to the circumference or bounding line, but this is not strictly correct, for the circle is, properly speaking, the space included.

Any straight line AB drawn through the centre and terminated by the curve, is a *diameter.*

The circle is one of the elements of plane geometry, the right line being the other, and those constructions only are regarded as geometrical which can be made by the aid of these two elements. The circle, however, derives its chief importance from its application in trigonometry, to the measurement of angles. Its application in trigonometry depends upon the fact, that if circles of the same radii be described from the vertices of angles as centres, the arcs of the circles intercepted between the sides are always proportional to the angles. It is for this reason that the circle is almost always employed to compare angles with each other. For this purpose, the circumference of the circle is divided into four equal parts, each of which is called a *quadrant;* each quadrant is divided into 90 equal parts, called *degrees;* each

degree is divided into 60 equal parts, called *minutes;* each minute into 60 equal parts, called *seconds*, and so on according to the sexagesimal scale. See *Trigonometry*.

The following are some of the most important properties of the circle:

1. Every diameter divides the circle and the circumference into two equal parts, and generally, equal arcs are subtended by equal chords.

2. The circumference of a circle is equal to the length of a diameter multiplied by π, or 3.14159265 . . . . or 3.1416, which is generally used. Hence, the circumferences of any two circles are to each other as their diameters or as their radii.

3. The area of a circle is π multiplied by the square of the radius. Hence, any two circles are to each other as the squares of their radii, or as the squares of their diameters, or generally as the squares of any two homologous lines.

4. The area of a circle is less than that of any regular circumscribing polygon and greater than any regular inscribed polygon. It is equal to that of the limit both of circumscribed and inscribed polygons; that is, it is equal to either when the number of sides becomes infinite. An analogous relation exists between the circumference of the circle and the perimeters of the circumscribed and inscribed polygons.

5. The circle has the greatest area for a bounding line of the same length of any plane figure.

6. Various expressions have been deduced for the length of the circumference, when the diameter is 1, some of the most useful of which are subjoined.

If π denotes the length of the circumference when the diameter is 1, we have

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c. \cdots\right)$$

$$\pi = \sqrt{8}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \&c.\right).$$

$$\pi = 8\left(\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{3.5.7} + \frac{1}{5.7.9} - \frac{1}{7.9.11} + \frac{1}{9.11.13} - \&c.\right)$$

$$\pi = \sqrt{12}\left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c.\right).$$

$$\pi = 8\left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \frac{1.3.5}{4.6.8.11} - \&c.\right).$$

$$\pi = 4\sqrt{2}\left(\frac{2}{3} - \frac{1}{5.2} - \frac{1}{4.7.2^2} - \frac{1.3}{4.6.9.2^3} - \frac{1.3.5}{4.6.8.11.2^4} - \&c.\right).$$

$$\pi = 4\left(1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7} - \frac{1.3.5}{2.4.6.8.9} - \&c.\right).$$

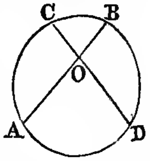
Many other formulas might be added. It is to be observed that π is equal to the numerical expression for the area when the radius is equal to 1.

The following curious expression for π is given by Wallis:

$$\pi = \frac{9}{8} \times \frac{25}{24} \times \frac{49}{48} \times \frac{81}{80} \times \frac{121}{120} \ \&c.$$

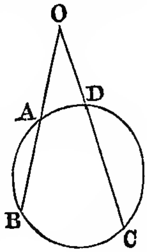
In which the numerators are the squares of the consecutive odd numbers, and the denominators less than the numerators by 1, the product being continued to infinity.

7. If two straight lines, AB and CD, cut the circumference and intersect each other within the circle, the angle DOB is measured by half the sum of the intercepted arcs

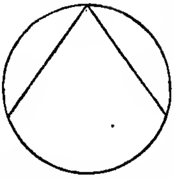


If the lines intersect each other without the circle, the angle is measured by half the difference of the intercepted arcs.

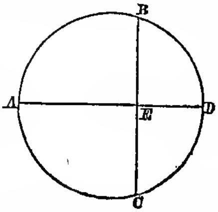
If they intersect at the centre, then are the intercepted arcs equal, and the angle between them is measured by either one of them.



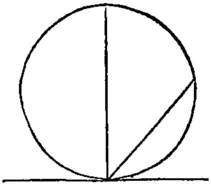
If they intersect on the circumference, the angle between them is measured by half the intercepted arc.



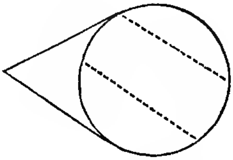
It follows from these principles, that if the intersecting lines are perpendicular to each other, that the sums of the opposite arcs are equal; that is, $BD + AC = AB + DC$.



If one of the lines is tangent to the circle, and the other passes through the point of contact, the angle between them is measured by half the intercepted arc. If both lines are tangent to the circle, the angle between them is measured by half the difference of the two arcs between the points of contact. If the tangents intercept equal arcs, they are parallel. A tangent at the middle point of an arc is parallel to the chord of the arc.



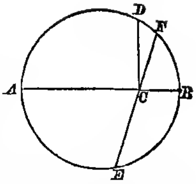
Two parallel straight lines intercept equal arcs; and conversely, straight lines drawn through the corresponding extremities of equal arcs, are parallel.



If from the same point without a circle, two tangents be drawn to the circle, they will be equal.

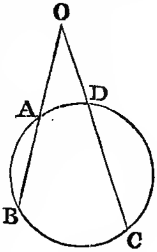
8. Any ordinate, CD, perpendicular to a diameter of a circle, is a mean proportional between the two segments, into which it divides the diameter; that is

$$DC = \sqrt{AC \times BC}.$$

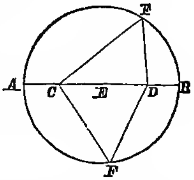


If a chord EF be drawn, cutting a diameter in a point, so that one segment EC is equal to the radius, then is EC an arithmetical mean between the segments into which it divides the diameter, and the remaining part of the chord CF is an harmonical mean between the same segments.

9. If two chords, OB and OC, be drawn through O, a point without the circle, then $OC \times OD \asymp OB \times OA$.

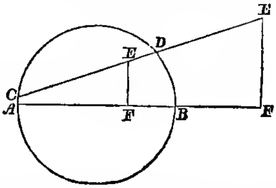


If two points C and D, be taken on the same diameter AB at equal distances from the centre, and from these points lines be drawn to any point F on the curve; then



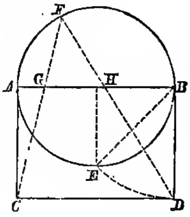
$$CF^2 + DF^2 \asymp AC^2 + BC^2 \asymp AD^2 + DB^2.$$

If a line EF, perpendicular to a diameter AB intersect a secant CE in the point E, then



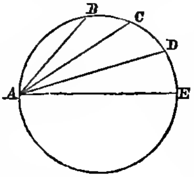
$$AF \times FB \asymp CE \times ED + EF^2.$$

If upon a diameter AB, of a circle, a rectangle AD be constructed, whose side BD is equal to BE, the chord of a quadrant, or the side of an inscribed square, and if lines be drawn



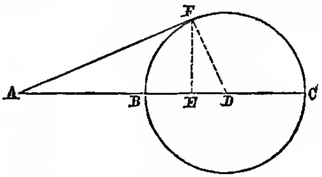
from any point F on the circumference to the points C and D, they will cut the diameter in the points G and H, so that $AH^2 + BG^2 = AB^2$.

10. If through a point A, on the circumference, chords AB, AC, AD, AE, &c., be drawn so as to intercept equal arcs, then



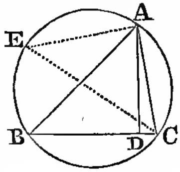
AB . AC : : AC : AB + AD : : AD : AC + AE : : &c.

If in any circle, BFC, an ordinate FE be drawn perpendicular to a diameter BC, and a tangent be also drawn at F, meeting



the diameter produced in A, then will DE, DB, and DA be in geometrical proportion.

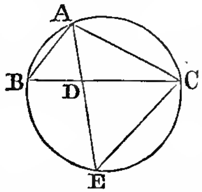
11. If a triangle ABC be inscribed in a circle, and a perpendicular AD be let fall from the vertex A upon the opposite side BC, and a diameter CE be drawn, then AB CE : : AD : AC; whence,



$$AB \times AC = CE \times AD.$$

If a triangle BAC be inscribed in a circle, and one of the angles A be bisected by the line AE, cutting the side BC in D; then

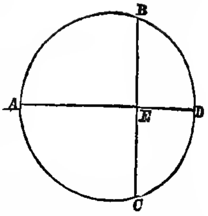
$$BA \times AC = AD^2 + BD \times DC.$$



If a quadrilateral be inscribed in a circle, the rectangle of its two diagonals is equal to the sum of the rectangles of the opposite sides.

If an equilateral triangle be inscribed in a circle, the square of either side is equal to three times the square of the radius.

If a square is inscribed in a circle, it is equal to twice the square of the radius.

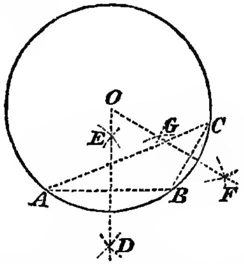


If two chords, AD and CB, of a circle, ACDB, are at right angles, and intersect at the point E, then is the sum of the squares of the four segments equal to the square of the diameter.

12. If the distance between the centres of two circles lying in the same plane is greater than the sum of their radii, they lie entirely external to each other; if it is equal to the sum of the radii, they are tangent externally, if it is less than the sum, and greater than the difference of the radii, they intersect each other in two points; if it is equal to the difference of the radii, they are tangent internally; if it is less than the difference of the radii, the one lies entirely within the other.

13. A circle can always be circumscribed about, or inscribed within, a regular polygon.

1. To pass a circle through three points A, B, and C:

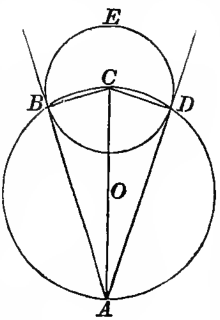


Draw the straight lines AB and BC and bisect them by the perpendiculars EO and FO; the point O, in which these intersect will be the centre, and the distance OC from O to either point will be the radius of the circle. If the circle is given, and it be required to find its centre, take any three points A, B, and C, on its circumference, and proceed as above; O will be the required point.

2. Through a point A, to draw a tangent to any circle.

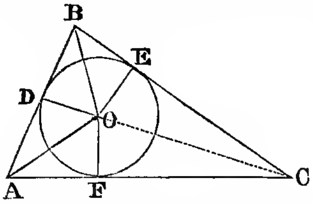
Draw a line from A to the centre C, and on this line as a diameter construct a circle ABD, cutting the given circle in B and D; join the points D and B with A, and the

lines AB and AD are the tangents required.



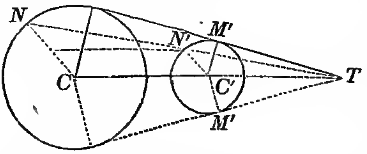
If A lies upon the circumference, but one tangent can be drawn to the circle through it, and that will be perpendicular to the radius through the point of contact. If A fall within the circle, no tangent can be drawn.

3. To inscribe a circle in a given triangle ABC.



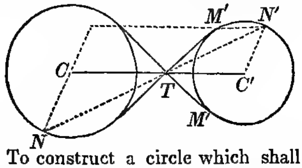
Draw the lines AO and BO, bisecting the angles A and B; the point O of intersection of these lines is the centre, and the perpendicular distance OF to either side AC, is the radius of the required circle. This problem is always possible.

4. To draw a straight line tangent to two circles in the same plane.

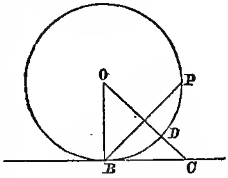


Draw CC′ through their centres, and prolong it; draw two parallel radii CN and C′N′ in the two circles; through N and N′ draw a straight line, and prolong it till it meets CC′, in T; through T draw TM′ tangent to one circle, and it will be tangent to the other. If the circles are equal, the point T will be at an infinite distance, and the tangent will be parallel to CC′.

If the parallel radii are drawn on opposite sides of CC′, (see next figure) the point T will fall between the centres as in the last figure. In either case, there will be two tangents. Hence, four tangents in all can be drawn to two circles in the same plane.

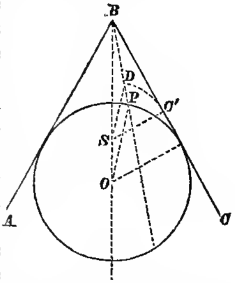


5. To construct a circle which shall pass through a given point P, and be tangent to a given straight line BC at a point B.



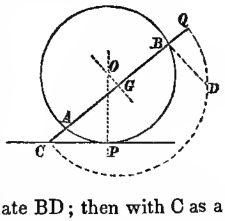
Draw PB, and bisect it by the perpendicular D O; erect at B a perpendicular to BC. The point O in which these lines intersect is the centre, and OB the radius of the circle.

6. To construct a circle which shall pass through a given point P, and be tangent to two given straight lines AB and BC.



Bisect the angle B by the straight line BO. Take any point S of BO, and through it draw a perpendicular SC′ to the line BC; with S as a centre, and SC′ as a radius, describe an arc C′D, cutting the straight line drawn through B and P in D, draw SD, and through P draw PO parallel to DS, O will be the centre, and the perpendicular distance from O to BC the radius of the required circle.

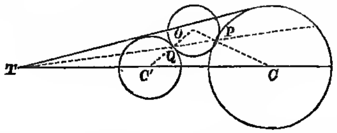
7. To construct a circle which shall pass through two points A and B, and be tangent to a given straight line CP.



Draw the straight line BA, and prolong it in both directions; make BQ equal to AC, and upon CQ as a diameter construct a semi-circle QDC; at B erect the ordinate BD; then with C as a centre, and BD as a radius, describe an arc

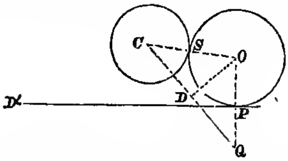
cutting CP in P. P is the point of contact. Draw PO perpendicular to CP, at P, and bisect AB by a perpendicular GO; the point O in which these lines intersect is the centre, and OP the radius of the required circle.

8. To construct a circle which shall be tangent to two given circles C and C′.



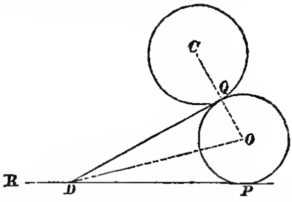
Draw a straight line BT tangent to the given circles, and produce it till it intersects the line of centres CT in T. Through T draw a secant TQP, and through the points of intersection Q and P, draw CP and C′Q; the point of intersection, O, of these lines is the centre, and OP or OQ the radius of the circle required. The problem admits of an infinite number of solutions.

9. To construct a circle which shall be tangent to a given circle C, and a given straight line D′P at a point P.



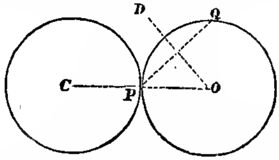
At P, erect a perpendicular OPQ, and on PO lay off a distance PQ equal to CS, the radius of the given circle. Draw QC and bisect it by a perpendicular DO; the point of intersection, O, of this line with the perpendicular PO, is the centre, and OP the radius of the required circle. There are two solutions.

10. To construct a circle which shall be tangent to a straight line DP, and to a given circle C at a point Q.



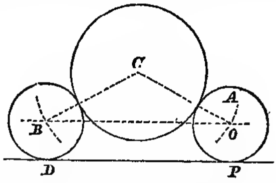
Draw DQ tangent to the given circle at the point Q, and produce it till it meets PD in D; draw DO bisecting the angle QDP, draw CQ, and prolong it till it intersects DO in O; O is the centre, and OQ the radius of the required circle. There are two solutions of this problem. The second solution is made by bisecting the angle QDR, and the given circle is tangent to the required circle internally.

11. To construct a circle which shall pass through a given point Q, and be tangent to a given circle C at a given point P.



Draw the radius CP, and produce it indefinitely; draw the line PQ, and bisect it by the perpendicular DO; the point of intersection, O, is the centre, and OP the radius of the required circle. The point Q might be within the required circle, in which case the required circle would be entirely within the given circle.

12. To construct a circle with a given radius, which shall be tangent to a given straight line DP, and to a given circle C.



Draw BO parallel to DP, and at a distance from it equal to tne given radius: with C as a centre, and a radius equal to the sum of the given radius, and that of the circle C, describe an arc cutting BO in the points B and O: then either of these points will be the centre, and the given line the radius of the required circle.

Many other problems might be added relating to circles, but a sufficient number have been given to indicate the general method of solving all problems of that nature.

Circles of the Sphere. The curve of intersection of any plane with a sphere is a circle. Different names are given to these circles according to the circumstances under which they are considered.

Every circle cut out by a plane passing through the centre, is a *great circle.* If the cutting plane does not pass through the centre, the curve of intersection is a *small circle.*

In spherical projections, the principal circles considered are as follows :

1. The *primitive circle* is the great circle, which is cut from the sphere by the primitive plane, or the plane on which the projection is made.

2. The *Equator*, a great circle, whose plane is perpendicular to the axis of the sphere.

3. The *Ecliptic*, a great circle, whose plane makes with the equator an angle of about $23\frac{1}{2}°$.

4. The *Meridians* are great circles, whose planes pass through the axis of the sphere. The two principal ones are, 1st. The *Equinoctial Colure*, which passes through the equinoctial points, or the points in which circumferences of the equator and ecliptic intersect; and 2d. The *Solstitial Colure*, whose plane is perpendicular to that of the equinoctial colure.

5. *Circles of Latitude*, are small circles whose planes are perpendicular to the axis: those particularly considered, are, 1st. The tropics which pass through the solstitial points, and are consequently about $23\frac{1}{2}$ degrees from the equator; the northern one is the tropic of cancer, and the southern one the tropic of capricorn. 2d. The polar circles which pass through the poles of the ecliptic, and are consequently as far from the poles of the equator as the tropics are from the equator. The northern one is called the *arctic circle*, and the southern one the *antarctic circle.*

6. The *horizon* of any point on the surface of the sphere, is that great circle whose plane is perpendicular to the radius through the point; all circles of the celestial sphere, whose planes are parallel to the horizon, are called circles of equal altitude or *almucantars.*

7. *Vertical Circles* are those great circles, whose planes are perpendicular to that of the horizon. The *prime vertical* is that whose plane is perpendicular to that of the meridian of the place.

Circle of Analysis. In analysis, the circle is given by an equation of the second degree between two variables; hence, it belongs to curves of the second order.

Its most general equation is

$$(y-b)^2 + (x-a)^2 = R^2.$$

In which x and y denote the general co-ordinates of all of the points of the circumference, a and b the co-ordinates of the centre, and R the radius.

If the origin of co-ordinates is taken as the vertex of a diameter, which diameter coincides with the axis of abscissas, $a = 0$, and $b = R$; and the equation of the circle becomes $y^2 = 2Rx - x^2$.

If the centre coincides with the origin of co-ordinates, $a = 0$, $b = 0$, and the equation becomes $x^2 + y^2 = R^2$.

This is the most ordinary form of the equation of the circle. From this equation, it may readily be shown that any ordinate is a mean proportional between the segments into which it divides the diameter. Since the equation of the ellipse reduces to the form of that of the circle, when we suppose the axes equal, we conclude that the circle is a particular species of the ellipse. We may readily show, from the equation above given, and from that of the right line, that a right line cannot cut a circle in more than two points.

In the Integral Calculus, the formula for the rectification of the circle is

$$Z = R\int\frac{dx}{\sqrt{R^2 - x^2}};$$

in which Z denotes the length of an indefinite portion of the arc. The integral can only be expressed by series.

The formula for the area of any portion of a circle included between any two ordinates, the axis of X and the curve, is

$$s = \int dx\sqrt{2rx - x^2},$$

in which s denotes the area, the origin of co-ordinates being at the vertex of a diameter. The integral can only be expressed by a series.

Circle of Curvature. See *Osculatory Circle.*

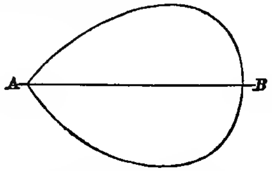
Circles of the Higher Orders. All curves, whose equations are of the form

$$y^{m+n} = x^m(a-x)^n,$$

have been called *circles*. If $m = 1$, and $n = 1$, the equation becomes that of the common circle.

If m is an odd number, the curves represented will be ovals. If $m = 3$, and $n = 1$,

the equation becomes $y^4 = x^3(a - x)$, and the curve is of the form of AB in the annexed figure.



When m is an even number, the curve has two infinite branches, and is, strictly speaking, an hyperbola.

CIR'CU-LAR. [L. *circularis*, circular]. Appertaining to a circle; thus, we speak of circular parts, circular segments, &c.

Circular Arc. Any part of the circumference of a circle. If r denote the radius, d the diameter, and c the circumference of the entire circle to which the arc belongs; and if z denote the length of any circular arc, s the sine of the arc, v the versed sine of half the arc, and m the number of degrees in the arc, we have the following formulas:

1. $z = rm \times 0.0174533.$

2. $$z = 2\sqrt{vd}\left\{1 + \frac{v}{2\cdot 3\cdot d} + \frac{3v^2}{2\cdot 4\cdot 5d^2} + \frac{3\cdot 5\cdot v^3}{2\cdot 4\cdot 6\cdot 7d^3} + \&c.\right\}$$

3. $$z = 2s\left\{1 + \frac{s^2}{3\cdot 3r^2} + \frac{3s^4}{5\cdot 2\cdot 4r^4} + \frac{3\cdot 5s^6}{7\cdot 2\cdot 4\cdot 6r^6} + \&c.\right\}$$

If we now denote the chord of the arc by c', and the chord of half the arc by c'', we have the following formulas which give good approximate results:

4. $z = 2d\sqrt{\dfrac{35}{3d - v}}$ nearly.

5. $z = \dfrac{2}{9}\cdot\left\{5d\sqrt{\dfrac{5v}{5d-3v}} + 4vd\right\}$ nearly.

6. $z = \dfrac{8c'' - c'}{3}$ nearly.

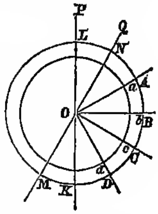
Circular Instruments. The name of any instrument used in surveying, or in navigation, for measuring angles in which the graduation extends around the entire circumference, or from 0° to 360°.

Sextants, octants, &c., are frequently employed for measuring angles, but in these the graduation is only carried around through a portion of the circumference. Experience has shown that when the instruments are of considerable size, entire circles afford much the most accurate results. The reflecting circle differs but little in principle from the sextant, but is considered a more reliable instrument for measuring angles.

Repeating Circle is a circular instrument so arranged, that by moving the axis of the telescope over successive portions of the graduated limb, corresponding to the angle to be measured, and reading only the multiple arc, all errors of graduation may be eliminated. The principle of repetition is independent of the instrument used, and may be advantageously applied to all circular instruments.

When applied to the reflecting circle, it becomes a *repeating, reflecting circle*; when applied to a theodolite, it becomes a *repeating theodolite*.

The following account of the application of the principle of repetition to circular instruments is taken from Sir J. Herschell:



"Let P, Q, be two objects, which we may suppose fixed for the purpose of explanation; and let KL be a telescope movable on O, the common axis of two circles, AML and abd, of which the former, AML, is absolutely fixed in the plane of the objects, and carries the graduations freely movable on the axis. The telescope is attached permanently to the latter circle, and moves with it. An arm, OaA, carries the index, or vernier, which reads off the graduated limb of the fixed circle. This arm is provided with two clamps, by which it can be temporarily connected with either circle, and detached at pleasure. Suppose now the telescope directed to P; clamp the index, OA, to the inner circle, and unclamp it from the outer and read off; then carry the telescope around to the other object Q. In doing so, the inner circle, and the index-arm which is clamped to

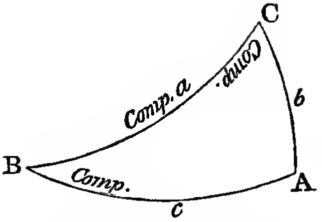
it, will also be carried around over an arc, AB, on the graduated limb of the outer, equal to the angle, POQ. Now clamp the index to the outer circle, and unclamp the inner, and read off. The difference of the readings will measure the angle, POQ. The reading will be liable to two sources of error: that of graduation, and that of observation, both of which it is our object to get rid of. To this end, transfer the telescope back to P without unclamping the outer circle; then, having made the bisection of P, clamp the arm to b, unclamping it from B, and again transfer the telescope to Q, by which the arm will now be carried with it to C over a second arc, BC, equal to the angle POQ. Now again, read off; then will the difference between this reading and the original one measure twice the angle POQ, affected with both errors of observation, but only with the same error of graduation as before. Let this operation be repeated as often as we please (say ten times); then will the final arc, ABCD, read off on the circle, be ten times the required angle affected by the joint errors of all the ten observations, but only the same constant error of graduation, which depends on the initial and final readings alone. Now the errors of observation, when numerous, tend to balance and destroy each other, so that, if sufficiently multiplied, their influence will disappear from the result.

"There remains, then, only the constant error of graduation, which comes to be divided in the final result by the number of observations, and is therefore diminished in its influence to one-tenth of its possible amount, or to less, if need be."

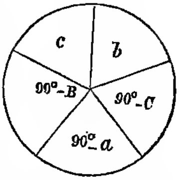
Circular Numbers. A name sometimes given to numbers whose powers terminate with the numbers themselves, as 5, 25, &c. The different powers of 5 always end in 5, and the different powers of 25 always end in 25.

Circular Parts of Napier. In a right-angled spherical triangle, the sides about the right angle, the complement of the hypothenuse, and the complements of the two oblique angles, are called Napier's circular parts.

If we designate the right angle by A, the oblique angles by B and C, and the sides opposite them by a, b and c, respectively, the



parts may be expressed circularly as in the annexed diagram.



The parts being arranged circularly, if any part be assumed as a middle part, it will have two adjacent and two opposite parts. Thus, if $90° - a$ be taken as the middle part, then $90° - B$ and $90° - C$ are adjacent parts, and b and c are opposite parts, and so on, when any part is taken as the middle part. By the aid of this convention, we are enabled to solve most of the cases of spherical trigonometry, by the aid of the following simple rules:

1. *The sine of the middle part is equal to the products of the tangents of the adjacent parts.*

2. *The sine of the middle part is equal to the product of the cosines of the opposite parts:* thus,

$$\sin(90° - a) = \tan(90° - B)\tan(90° - C),$$

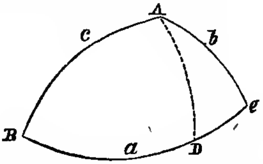
and

$$\sin(90° - a) = \cos c \cos b.$$

The only cases which cannot be brought under these rules are those in which the three angles or the three sides are given. They apply to all other cases of oblique-angled triangles, since each oblique-angled triangle may be divided into two right-angled triangles by an arc of a great circle drawn through one of its vertices, and perpendicular to the opposite side.

It has been observed that the two rules above given do not apply in the two cases when the three angles or the three sides are given. There is, however, an analagous rule which will enable us to solve these cases. Let us consider an oblique spherical triangle, and call the three sides and the supplements

of the three angles, circular parts; there will be six such parts. If any one of these six be assumed as a *middle* part, then are the other parts of the same denomination *opposite* parts: thus, if a side is taken as a middle part, the other sides are the opposite parts; if the supplement of one of the angles be taken as a middle part, the supplement of the other angles are opposite parts. The rule is as follows:



Select the angle A opposite the greater side, and let fall a perpendicular from its vertex upon the opposite side BC, dividing the angle A and the side BC into two segments. Take the supplement of the angle or the side opposite, as the middle part; *then will the rectangle of the tangents of the half sum and the half difference of the segments of the middle part be equal to the rectangle of the tangents of the half sum, and the half difference of the opposite parts.*

If, for example, the angle A is the middle part,

$$\tan \tfrac{1}{2} \text{A} \tan \tfrac{1}{2} (\text{DAC} - \text{DAB}) = \tan \tfrac{1}{2} (360° + \text{B} + \text{C}) \tan \tfrac{1}{2} (\text{B} - \text{C}).$$

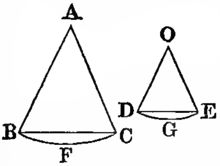
If the side BC $= a$ is the middle part,

$$\tan \tfrac{1}{2} a \tan \tfrac{1}{2} (\text{BD} - \text{DC}) = \tan \tfrac{1}{2} (b + c) \tan \tfrac{1}{2} (b - c).$$

Having determined the two segments of the middle part, the auxiliary triangles BAD and CAD can be solved by the rules first given.

Circular Sailing, is that performed on the arc of a great circle. In Mercator's sailing, the problems are solved by a solution of plane triangles: in circular sailing, they are solved by means of spherical triangles.

Circular Sector. A portion of a circle included between an arc of a circle and the radii drawn through its extremities thus, ABFC is a sector. Similar sectors are those which correspond to equal angles at the centre, as ABFC and ODGE. The angle at the centre is called the angle of the sector, or the sectoral angle.



If we designate by l the length of the arc of the sector, by n the number of degrees which it contains, and by S the area of the sector, r denoting the radius, we have the following formulas:

$$1.\quad S = \frac{1}{2} r \cdot l. \qquad 2.\quad S = \pi r^2 \times \frac{n}{360}.$$

Circular Segment. A portion of a circle included between an arc of a circle and its chord. To find the area of a circular segment, we have the following simple rule:

Multiply the square of the radius of the circle by half the difference of the arc of the segment and the sine of the angle at the centre, —the arc being less than a semicircle. If the arc is greater than a semicircle, subtract the product obtained above from the area of the entire circle. The length of the arc of the segment may be found by any of the methods already given under *Circular Arc.*

Circular Function. A function in which the relation between it and the independent variable is expressed by means of some of the trigonometrical lines, as the sine, tangent, &c. thus, in the expression $y = \sin x$, y is said to be a circular function of x; so, also, is y in the expression $x = \tan^{-1} y$.

CIR'CU-LĀ-TING DECIMAL. One in which one or more figures are continually repeated in the same order. Such are sometimes called repeating decimals. The figure, or set of figures which is continually repeated, is called the *repetend.* Circulating decimals are *pure* or *mixed; pure,* when the first figure after the decimal point is the first figure of the repetend; *mixed,* when one or more figures occur before the repetend commences.

A *single* repetend, is one in which only a single figure is repeated: thus, .333333 . . . Such repetends are expressed by putting a mark over the first figure: thus $.\dot{2}$ is the same as .22222 . . . and $.\dot{3}$ the same as .333333 . .

A compound repetend, is one in which the repetend consists of more than one figure, as .57235723 . . . These are distinguished by putting a mark over the first and last figures in the repetend, inclined in different directions; the above example may be expressed .\`5723′ . . .

The true value of a repeating decimal, supposed pure, is equal to a vulgar fraction whose numerator is the repetend and whose denominator is expressed by a number of nines equal to the number of figures in the repetend: thus, the value of .‵2 . . is $\frac{2}{9}$; and the value of .‵5723′ . . . is $\frac{5723}{9999}$. If the circulating decimal is *mixed*, it may be reduced to a vulgar fraction by taking that part which precedes the first figure of the repetend and reducing it to the form of a vulgar fraction, then add to this result a fraction whose numerator is the repetend, and whose denominator is expressed by as many nines as there are figures in the repetend, followed by as many 0's as there are places of figures between the decimal point and the first figure of the repetend: thus, 2.4‵18′ . . . is equal to $2\frac{4}{10} + \frac{18}{990} = 2\frac{23}{55}$.

Similar repetends are those which begin at the same number of places of figures from the decimal point. *Dissimilar* repetends are those which do not begin at the same number of places of figures from the decimal point.

Conterminous repetends are those which terminate at the same number of places of figures from the decimal point.

Similar and *Conterminous* repetends are those which both begin and end at the same number of places of figures from the decimal points. Thus, .3‵54′ . . and 2.7‵534′ . . are similar, but .‵253′ and 47‵52′ . . . are dissimilar. .1‵25′ . . . . and .‵354′ . . . are conterminous; 53.2‵753′ . . . and .4‵632′ . . . are both similar and conterminous.

Properties of Circulating Decimals.

1. Any decimal, having a finite number of places of figures, may be regarded as a circulating decimal, provided we regard the repetend as made up of 0's; thus, .35 may be written .3‵50′ . . or .35‵00′ . . or .35‵000′ . . , and so on.

2. Any circulating decimal having any number of figures, may be written as one having twice, three times, or any multiple of that number of figures, by simply taking the repetend twice, three times, &c., as a single repetend; thus, .25‵37′ . . . may be written

.25‵3737′ . . . or .25‵373737′ . . . &c.

Hence, two circulating decimals whose repetends have not the same number of places of figures, can be so written that they shall have the same number of places by the following rule: Find the least common multiple of the numbers of figures in the two repetends, and then reduce each decimal to equivalent decimals having this number of places of figures in the repetend. Thus, .13‵8 . . . and 7.5‵43′ . . . and .04‵354′ . . . , may be written respectively .13‵888888′ . . 7.5‵434343′ . . and .04‵354354′ . .

3. Any circulating decimal may be written under the form of a mixed circulating decimal having any number of places of figures between the decimal point and the first figure of the repetend. Thus, the circulating decimal ‵57′ may be written .5‵75′ . . or .57‵57′ . . or .575‵75′ . . . , &c. Hence, any two circulating decimals may be so written that their repetends shall be similar and conterminous.

4. If two or more circulating decimals, whose repetends are similar and conterminous, be added, their sum will be a circulating decimal, whose repetend is similar to and conterminous with each of the repetends of the given decimals.

5. If any circulating decimal be multiplied by any number whatever, the product will be a circulating decimal, whose repetend is similar to, and conterminous with, that of the given decimal.

These principles enable us to deduce simple rules for operating upon circulating decimals.

I. To add circulating decimals:

Make their repetends similar and conterminous as above explained, then write them down so that units of the same order shall fall in the same column: write as many figures of the next consecutive repetend as shall indicate with certainty how many are to be carried from one repetend to the other, and then add as in ordinary decimals, and point off the repetend so that it shall be similar to, and conterminous with, those of the given decimals.

II. To subtract one circulating decimal from another.

The rule is entirely analogous to that for addition, and may easily be supplied.

III. To multiply one circulating decimal by another.

Transform each into an equivalent vulgar fraction, and perform the multiplication by the rules for multiplying vulgar fractions to-

gether, then convert the resulting vulgar fraction into an equivalent circulating decimal.

IV. To divide one circulating decimal by another.

Transform each into an equivalent vulgar fraction and divide; after which, transform the result into a circulating decimal.

To find the number of places of figures in the repetend of a circulating decimal corresponding to any given vulgar fraction. First, reduce the vulgar fraction to its lowest term, then resolve the denominator into its prime factors. Now, since every fraction whose denominator is a multiple of either 2 or 5 can be expressed by a decimal fraction having a finite number of places of figures, and since all other vulgar fractions can be expressed by equivalent circulating decimals, we have the following rule:

Resolve the fraction into two factors: one of which is the given numerator divided by the product of all the prime factors which are equal to either 2 *or* 5; *the other being equal to* 1 *divided by the product of all the other factors of the given denominator. Then will the number of places of decimals which precede the first figure of the repetend be equal to the number of prime factors which are equal either to* 2 *or* 5, *that occur in the denominator of the first fraction.*

Again, divide a number expressed by a succession of 9's *by the denominator of the second fractional factor, until a remainder is found which is equal to* 0, *then will the number of* 9's *employed be equal to the number of places of figures of the repetend. Denote the first number found by* n, *and the second by* m.

Annex 0's *to the numerator of the given fraction, and divide by the denominator of the given fraction till a number of places of decimals is equal to* n + m; *point off* m *places from the right for the repetend, and* n *preceding decimal places, and the quotient will be the equivalent circulating decimal.*

1. To find a circulating decimal equivalent to

$$\frac{249}{29304}\text{; we have } \frac{249}{29304} = \frac{83}{9768}$$
$$= \frac{83}{2 \times 2 \times 2} \times \frac{1}{1221},$$

here $n = 3$. If we divide 9999 ... by 1221, we shall have to use six 9's before we get 0 for a remainder; hence, $m = 6$. If we divide 83000 ... by 9768, and continue the operation to 9 places of decimals, and point off, according to the rule, we shall have

$$\tfrac{83}{9768} = \cdot 008'497133' \ldots$$

We see, therefore, that any vulgar fraction may be transformed into an equivalent circulating decimal.

CIR-CUM'FER-ENCE. [L. *circumferentia*, from *circum*, around, and *fero*, to carry]. The curved line which bounds a plane curvilinear area. In ordinary language, the use of the term is restricted to the line which bounds or limits the area of a circle. The characteristic property of the circumference of a circle is, that every point of it is equally distant from a point within called the centre.

The length of the circumference of any circle is equal to π multiplied by the diameter, or by twice the radius. If the diameter is taken equal to 1, the circumference is equal in length to 3.14159265358979323846264338 32795028841971 . . . . $= \pi$, which in practical operations, where great accuracy is not required, we take equal to 3.1416.

The circumferences of different circles are to each other as their radii, or as their diameters, or as any two homologous lines.

CIR-CUM-FE-REN'TOR. An instrument employed in surveying for the purpose of measuring horizontal angles. It is nearly the same as the surveyor's compass, (see *Compass*), except that the graduation is continued from 0 round to 360°. The method of using it will be apparent from the description of the compass.

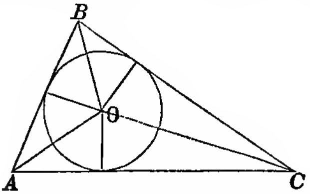
CIR-CUM-SCRĪBE'. [L. *circumscribo*, from *circum*, about, and *scribo*, to draw]. To limit, to bound, to confine, to inclose within limits.

Circumscribed Figure. A figure drawn around another, so that all its sides or faces shall be tangent to the second figure, which is then called an *inscribed figure.*

Circumscribed Polygon. A polygon is said to be circumscribed about a curved figure, when all its sides are tangent to the curved line which bounds the curvilineal figure. The term is usually applied to figures circumscribed about a circle. A triangle can always be circumscribed about a circle, which shall be similar to any plane triangle, as follows:

Let DEF be any circle, and O its centre.

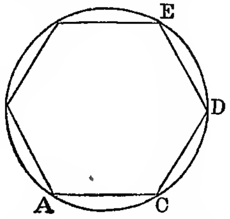
Draw through O the radii OE, OD, and OF, respectively, perpendicular to the sides of the



given triangle, and at the points E, D, and F, draw tangents to the circumference ; these will, by their intersection, form a triangle, ABC, which will be similar to the given triangle.

When a quadrilateral can be circumscribed about a given circle, which shall be similar to a given quadrilateral, the construction is entirely analogous to that for the triangle. In general, a regular polygon of any number of sides can be circumscribed about a circle, as follows : divide the circumference into as many equal parts as there are sides in the required polygon, and at the points of division let tangents be drawn to the circle. These will, by their intersection, form a regular polygon, which will be circumscribed about the circle. The area of every circumscribed polygon is greater than that of the circle, and the areas of polygons circumscribed about the same or equal circles, are to each other as their perimeters. The limit of the area of a regular circumscribed polygon, that is, the area of a polygon having an infinite number of sides, is equal to the area of the circle.

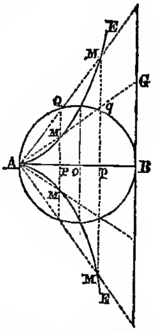
If all of the sides of a polygon are chords of a circle, the circle is said to be circumscribed about the polygon.



The circle ACDE is circumscribed about the polygon ACDE.

Circumscribed Polyhedron. A polyhedron is said to be circumscribed about a sphere when all of its faces are tangent to the surface of the sphere. If all of the vertices of the polyhedral angles of a polyhedron are in the surface of a sphere, the sphere is said to be circumscribed about the polyhedron. The solidities of polyhedrons, circumscribed about the same or equal spheres, are to each other as their surfaces.

CIS'SOID OF DIOCLES. [Gr. κισσος, ivy, and ειδος, form]. A curve first employed by Diocles, whose name it bears, for the purpose of solving two celebrated problems of the higher geometry, viz : to *trisect a plane angle*, and to *construct two geometrical means between two given straight lines.*



Let AB be a diameter of any given circle, and let PQ and pq be any two ordinates, taken at equal distances from its extremities, A and B. If we draw a straight line through A, and either of the points, q or Q, and produce it till it cuts the other, produced if necessary, the point of intersection M, will, in its different position, trace out a curve called a *cissoid*. The circle AB is called the generating circle, and the diameter AB is the axis of the curve. The cissoid consists of two infinite and symmetrical branches, AE and AE, having a cusp point at A, and having the straight line drawn tangent to the generating circle at B, for a common asymptote.

If A be taken as the origin of a system of rectangular co-ordinates, the axis of X coinciding with the axis of the curve, the equation of the curve may readily be deduced. Denote the co-ordinates of any point M, by x and y respectively, and we shall have, if we denote the length of the diameter AB by a.

$$AP : Ap :: PM : pq, \qquad \text{or}$$

$$x : a - x :: y : \sqrt{x(a-x)}; \qquad \text{whence,}$$

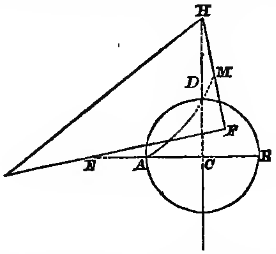
$$y = \frac{x\sqrt{x(a-x)}}{a-x}; \quad \text{or,} \quad y^2 = \frac{x^3}{(a-x)},$$

which is the equation of the curve.

From the method of constructing points,

it is plain that the curve bisects each semicircle of the generating circle, and that the parts AM and qG of the straight line AG are equal.

The curve may be constructed mechanically. Produce CA to E, making AE equal to CA, or to the radius of the generating

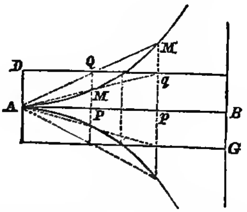


circle. Produce the ordinate CD, through the centre, indefinitely. Take a right angled ruler, HFE, right angled at F, whose side HF is equal to AB, the diameter of the generating circle, and move it so that the side FE shall constantly pass through the point E, whilst the angular point H continues upon CDH, then will M, the middle point of HF, describe the cissoid.

The area of the entire space included between both branches of the cissoid, and their common asymptote, is equal to three times the area of the generating circle. The volume of the solid generated by revolving this area about its axis, is infinite.

If instead of using a circle as the generating curve, we had employed any other curve, of which AB is a portion of the axis, the curve generated is *cissoidal;* therefore, there is an infinite variety of cissoidal curves.

If the generating line is the perimeter of a rectangle, the cissoidal curve generated will have for its equation $y = \frac{bx}{a - x}$, in which



b is equal to AD, the semi-side of the rectangle DG. This is the equation of an hyperbola; hence, each branch of the curve is a portion of an hyperbola. If, instead of taking the ordinates pq and PQ at right angles to the axis, they be taken oblique to it, we shall have still another system of *cissoidal curves.*

If we take a cissoid as a generating curve, the cissoid generated will be the generating circle of the cissoid. If we reverse the axis of the cissoid, taking the cusp at B, and generate a cissoidal curve, its equation will be $y^2 = \frac{x^5}{(a - x)^3}$. If we again reverse the secondary cissoid, and generate a tertiary cissoid, its equation will be $y^2 = \frac{x^7}{(a - x)^5}$; and so on.

CLAMP. A contrivance for fastening two parts of an instrument together, as the vernier plate and limb of a theodolite, or the vane, and rod of a leveling staff. Clamps have a great variety of forms, but the general principle involved is nearly the same in all. A piece of metal of suitable form is firmly attached to one of the parts to be fastened together, and this bears a screw which, on being tightened, increases the friction and prevents the parts from sliding upon each other. When an instrument is clamped, there is generally an arrangement by means of which slight motion can be imparted, consisting of a screw working tangentially and called from this circumstance a tangent screw.

CLAMP SCREW. The screw by means of which the parts of an instrument are firmly connected together, at pleasure.

CLĂSS. [L. *clasis*, a collection]. A scientific division or arrangement. A group of things possessing some common attribute or attributes.

CLAS-SI-FI-CĀ'TION. The operation of grouping objects together according to some law of arrangement. As an example of the mode of classification sometimes employed in mathematics, let us examine the analytical classification of surfaces.

All surfaces are divided into two classes, *algebraic* and *transcendental. Algebraic* surfaces are those whose equations can be expressed by means of the ordinary operations of algebra. *Transcendental* surfaces are those whose equations cannot thus be expressed.

These are again subdivided into other classes: thus, algebraic surfaces are divided into orders according to the degree of their equations.

The different orders are each again subdivided in species: thus, surfaces of the second order are divided into three species; viz., *ellipsoids*, *hyperboloids* and *paraboloids*, depending upon the nature of the sections made by secant planes. Surfaces of the second order may also be divided into species, dependent upon the number and position of their centres: thus, the *first* species has *one* centre at a finite distance; this embraces the ellipsoids and the hyperboloids; the *second* species embraces those which have but one centre, and that at an infinite distance; this species includes the paraboloids; the *third* species embraces those which have an infinite number of centres at a finite or an infinite distance; this species embraces all of the cylinders having any of the conic sections for bases.

The species are again subdivided into varieties, dependent upon the relations existing between the constants which enter their equations, or upon the nature of their plane sections: thus, the ellipsoids have for varieties, the ellipsoids of revolution; that is, the oblate and prolate spheroid, the sphere, the point, and the imaginary surface.

This classification of magnitudes greatly facilitates the discussion of their properties. In the above classification, if it were desired to describe any surface, as the sphere, for example, we should say that it was the second variety of the first species of the second order of algebraic surfaces. The above example of the method of classification serves to show the general principles of classification, though the same surfaces might be differently classed, the classification being based on other principles. For different purposes, different classifications may be adopted; hence, we often find the same magnitudes ranged in different classes.

CLŌSED CURVE. A curve which, counting from any point, returns upon itself, as the circle, the ellipse, or an oval.

CO-EF-FĬ″CIENT. [L. *con*, with, together, and *efficio*, to effect, to bring about]. A number written before a quantity, to show how many times it is to be taken additively: thus, in the expression $3ax$, 3 is the co-efficient of the quantity ax, and indicates that ax is to be taken three times additively. In the expression just given, if we regard x as the quantity taken, $3a$ is the co-efficient, and it shows that x is to be taken as many times as there are units in $3a$. Hence, we see that a co-efficient may be *numerical*, or it may be *literal*, or it may be *mixed*; that is, it may be expressed by means of both figures and letters.

Custom has, however, given a more enlarged signification to the word co-efficient than that which has just been explained. In its most general sense, it is nearly synonymous with *factor*, and may be either positive or negative, entire or fractional, real or imaginary. If an expression contains any unknown or variable quantity, the factor which is to be multiplied into it may be considered a co-efficient of that quantity. In strictness, the co-efficient is the co-efficient of the quantity which is multiplied by it; but it is usual to speak of it as *the co-efficient of the term or of the quantity*, as may be most convenient. When a term contains no variable or unknown quantity, if it can be resolved into two factors, it is allowable to speak of either factor as a co-efficient of the other.

When no co-efficient is written, the co-efficient 1 is always understood: thus, a is the same as $1a$.

Every equation which contains but one unknown quantity, and which is *entire* with respect to that unknown quantity, may be reduced to the form

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} + \cdots$$
$$+ Tx + U = 0;$$

and when so reduced, the co-efficients P, Q, &c., T, U, have some remarkable properties. It may be observed that U is regarded as a co-efficient, being the multiplier of x^0. The co-efficient of the first term being 1.

The co-efficient P, *of the second term is equal to the algebraic* sum of the roots of the equation, taken with their signs changed.

The co-efficient Q, of the third term, is equal to the algebraic sum of the different products of the roots of the equation taken in sets of two.

The co-efficient R, of the fourth term, is

equal to the algebraic sum of the different products of the roots taken three in a set, with their signs changed; and so on for successive co-efficients.

The absolute term U, or the co-efficient of x^0, is equal to the continued product of all the roots taken with their signs changed. See *Equation*. For *Differential Co-efficients* and *Indeterminate Co-efficients*, see the respective articles under these headings.

COIN. [Fr. *coin*]. Money stamped; a piece of metal, as gold, silver, copper, or some alloy converted into money by stamping upon it certain characters. The coinage of money can only be legal when executed under the authority of government. In the United States the Constitution provides that gold and silver only shall be a legal tender for debts due. The different coins established by law, which are of gold, silver and copper, are as follows:

Gold Coins. Double eagle, $20; eagle, $10; half eagle, $5; $3 piece; quarter eagle, $2,50; and gold dollar.

Fineness or Standard. Of 1000 equal parts, by weight, 900 must be pure gold, and the remaining 100 of silver and copper, the silver in no case to exceed one-half of the alloy.

Weight. The weight of the eagle is 258 grains Troy, and that of the other coins in the same proportion, according to their values.

Silver Coins. Dollar; half dollar; quarter dollar; dime; half dime; three cent piece.

Fineness or Standard. Of 1000 equal parts, 900 must be pure silver, and 100 pure copper.

Weight. The weight of the dollar is $412\frac{1}{2}$ grains Troy, and that of the other coins in the same proportion, according to their values.

The gold and silver coins are legal tenders for the amounts which they severally represent.

Copper Coins. The copper coins are the cent and the half cent, and are of pure copper. The weight of the cent is 168 grains Troy, and of the half cent 84 grains.

The value of foreign coins, which circulate as money, is fixed by law.

Including the legal value, a foreign coin may be said to have four different values:

1. The *intrinsic value*, which is determined by the amount of pure metal which it contains.

2. The custom house, or legal value, fixed by law.

3. The mercantile value, which is the amount it will sell for in the market.

4. The exchange value, which is the value assigned to it, in buying or selling bills of exchange between one country and another.

For example, the gold sovereign of England is worth at the mint $4.861, in terms of our gold eagle. Its legal value is $4.84. Its commercial value, in Wall-street, varies from $4.83 to $4.86, seldom reaching either limit.

The exchange value of the sovereign or English pound, is $\$4.44\frac{4}{9}$; this was the legal value before the change in our standard.

CO-IN-CĪDE′. [L. *con*, with, together, and *incido*, to fall upon]. To fall upon. Two magnitudes are said to be coincident, when, being placed one upon the other, they are made to agree with each other throughout, in which case they are said to be equal.

COL-LEC′TION. [L. from *con*, with, and *lego*, to gather]. A group of particulars gathered together. A number is a collection of equal things, each of which is a unit of the collection. See *Number*.

COL-LI-MĀ′TION. [L. *con*, with, and *limes*, a limit]. The line of collimation of a telescope is a line drawn through the optical centre of the eye and object lenses. This line is indicated by cross hairs fixed at the common focus of these lenses.

COL-ŪRE′. [Gr. κολουρος, from κολος, mutilated, and ουρα, a tail. So named, because a part is always below the horizon]. Two meridians of the celestial sphere, one of which passes through the axis of the ecliptic, and is called the *solstitial colure*; the other, called the *equinoctial colure*, is perpendicular to the first, and passes through the equinoctial points. The points in which the solstitial colure intersects the ecliptic are called solstitial points, and they mark the extreme northern and southern limits of the sun's annual path.

COM-BI-NĀ′TIONS. [Fr. *combinaison*, union. L. *con*, with, and *bini*, two and two] In Algebra, the combinations of m factors are the different results obtained by multiplying

them together in every possible manner, when taken in sets of n, n being less than m. If m designate the whole number of factors, and n the number which is taken in a set, and if z denote the whole number of combinations, we have the formula

$$z = \frac{m(m-1)(m-2)\cdots(m-n+1)}{1 \cdot 2 \cdot 3 \cdots\cdot n} \cdots (1).$$

That is, *the whole number of combinations of* m *factors, taken* n *in a set, is equal to the continued product of the natural numbers, from* m *down to* m − n + 1 *inclusively*, divided by *the continued product of the natural numbers, from* 1 *up to* n *inclusively*.

If z' denote the number of combinations of m factors, taken in sets of $m-n$, we shall have, by simply changing n into $m-n$, in the preceding formula,

$$z' = \frac{m(m-1)(m-2)\cdots\cdot(n+1)}{1 \cdot 2 \cdot 3 \cdots\cdot (m-n)} \cdots\cdot (2).$$

If now we divide equation (1) by (2), member by member, and arrange the factors in both terms of the quotient, we shall have

$$\frac{z}{z'} = \frac{1\cdot 2\cdot 3\cdots(m-n)\times(m-n+1)\cdots\cdot}{1\cdot 2\cdot 3\cdots \qquad n\cdot(n+1) \qquad \cdots\cdot}$$

$$\frac{(m-1)m}{(m-1)m} = 1.$$

Hence, since $\frac{z}{z'} = 1, \quad z = z'$.

That is, *the number of combinations of* m *factors, taken in sets of* n, *is equal to the number of combinations of* m *factors, taken in sets of* m − n. This principle is of much importance in deducing the binomial formula.

The applications of the principle of combinations are various. One of the most important is its application to the doctrine of chances or probabilities. It also forms the basis of the demonstration of the binomial formula for entire exponents.

COM-MEN′SU-RA-BLE QUANTITIES, [L. *con*, with, and *mensura*, a measure]. Are those which may be exactly expressed by means of a common unit. Thus, a foot and a yard are commensurable, since both can be expressed in terms of an inch; the first being equal to 12 inches, and the second being equal to 36 inches. The numbers $2\frac{5}{8}$ and $3\frac{7}{9}$ are commensurable, because the first is equal to $\frac{189}{72}$, and the second is equal to $\frac{272}{72}$; hence may both be referred to the common fractional unit $\frac{1}{72}$. In general, any numbers which are entire, or which can be expressed by a vulgar fraction, are *commensurable*; all other numbers are incommensurable. All denominate quantities which can be expressed by any number of entire parts of the same unit, or by means of a vulgar fraction, are commensurable. All radicals are incommensurable by any unit, either 1, or a part of 1.

COM′MON DIVISOR. [L. *communis*, common, and *divisor*, a divider]. Any quantity is a common divisor of two other quantities, which will divide both of them without a remainder. Thus, 4 is a common divisor of 12 and 16; also $3a^2 - b$ is a common divisor of $9a^4 - 6a^2b + b^2$ and $12a^4 - 4a^2b$.

GREATEST COMMON DIVISOR, of two numbers, is the greatest number which will exactly divide both.

The *greatest common divisor* of two polynomials, is the greatest algebraic expression, with respect to co-efficients and exponents, that will exactly divide them both. The greatest common divisor of any number of polynomials, is the greatest algebraic expression that will exactly divide them all. The rule for finding the greatest common divisor of two polynomials depends upon the following principles:

1st. The greatest common divisor of two polynomials contains all the prime factors which are common to both, and does not contain any other factors.

2d. The greatest common divisor of two polynomials is the same as that which exists between the least polynomial and their remainder, after division.

From these two principles the following rule is deduced:

I. Suppress the monomial factors common to all the terms of the first polynomial; do the same with the second polynomial; and if the factors so suppressed have a common factor, set it aside as forming a factor of the common divisor sought.

II. Prepare the first polynomial in such a manner that its first term shall be divisible by the first term of the second polynomial, both being arranged with reference to the same letter: apply the rule for division, and continue the process till the greatest exponent of the leading letter in the remainder is at

least one less than it is in the second polynomial.

Suppress, in this remainder, all the factors that are common to the co-efficients of the different powers of the leading letter; then take this result as a divisor, and the second polynomial as a dividend, and proceed as before.

III. Continue the operation till a remainder is obtained, which will exactly divide the preceding divisor; this last remainder, multiplied by the factor set aside, will be the greatest common divisor sought; if no remainder is found, which will exactly divide the preceding divisor, then the factor set aside is the greatest common divisor sought.

The rule for finding the greatest common divisor of two numbers, may at once be deduced from this.

In applying the rule, if it is found that the co-efficients of the different powers of the leading letter have the polynomial factor, or factors, they may be suppressed and treated in the same manner as the monomial factors. If one polynomial contains a letter which does not enter the other, the greatest common divisor will be the same as that which exists between the co-efficients of the different powers of this letter and the second polynomial. If each polynomial contains a letter which is not in the other, arrange them both with respect to this letter, and the greatest common divisor of all the co-efficients of the letters in the two polynomials will be the common divisor sought.

To find the greatest common divisor of several polynomials, find that of the first and second polynomial, and then that of this result, and the third polynomial, and so on, till the last: the final result will be the greatest common divisor of all the polynomials.

COMMON MEASURE. The common measure of two quantities is the same as the common divisor.

COMMON MULTIPLE OR COMMON DIVIDEND. A common multiple of two quantities, or their common dividend, is a quantity which they will both divide without a remainder. Thus, 24 is a common multiple of 6 and 4; so is 12, so is 48, &c.

The least common dividend of two or more quantities is the least quantity which they will all divide without a remainder. Thus 12 is the least common dividend of 4 and 6.

The rule for finding the least common dividend of any number of quantities, depends upon this principle, viz.:

The least common dividend of any two quantities contains as factors all the prime factors which enter both quantities, each raised to a power equal to the highest power of that factor in either of the given quantities, and does not contain any other factor.

Hence, to find the least common dividend of any number of quantities: *Resolve them into their prime factors; some of these may be equal; then take all the prime factors which enter all of the given quantities, and raise each to a power equal to the highest power which that factor enters any of the given quantities; multiply these resulting expressions together, and the product will be the least common dividend required.*

1. Find the least common dividend of the numbers 45, 50, and 60. Resolving these into their prime factors, we find

$$45 = 3^2.5,\ \ 50 = 2.5^2,\ \text{and}\ 60 = 2^2.3.5.$$

Hence, the least common dividend is

$$2^2.3^2.5^2 = 900.$$

2. Find the least common dividend of

$$a^2 + 2ab + b^2,\ a^2 - b^2,\ a^2 - 2ab + b^2,$$

and $\qquad a^3 + 3a^2b + 3ab^2 + b^3$.

These become severally, when resolved into their prime factors,

$$(a + b)(a + b),\ (a+b)(a - b),\ (a - b)(a-b),$$

and $\qquad (a + b)(a + b)(a + b)$;

hence, their least common dividend is

$$(a + b)^3 (a - b)^2 = (a + b)(a^2 - b^2)^2 \ldots$$

If it is not convenient to resolve the quantities into their prime factors, the least common dividend of two quantities may be found by multiplying them together and dividing the product by their greatest common divisor. This amounts to the same thing as pursuing the course indicated above. Or we may, if we please, divide either of the quantities by their greatest common divisor, and then multiply the quotient by the other quantity, which is but another method of arriving at the same result. To find the least common dividend of several quantities, we may find the least common dividend of the first and second; then find the least common dividend of this result and the third; then the least common dividend of this result and the fourth, and so

on to the last: the final result is the least common dividend of all the given quantities.

COM'PASS. [Fr. *compas*, a compass]. A name given to instruments contrived to indicate the direction of the magnetic meridian, and also to determine the angle contained between that meridian and any horizontal line. It is named according to the different purposes for which it is used: *surveyor's compass*, *mariner's compass*, *azimuth compass*, *variation compass*, &c. Each particular application requires some change in the detail of arrangement, but the general principle is the same in all.

Surveyor's Compass. The surveyor's compass, or circumferentor, is an instrument used in surveying, for the purpose of measuring horizontal angles, where great accuracy is not required. It consists essentially of a compass-box of brass, usually about six inches in diameter, which rests upon an arm of brass about fourteen inches in length; at the two extremities of this arm are two vertical sights or openings, so arranged that their plane shall pass through the centre of the compass-box. At the centre of the compass-box, is a small pin terminating in a sharp and hardened point, upon which the magnetic needle is carefully poised, so as to admit of free horizontal motion. To diminish friction as much as possible, a cap of agate or other hard mineral is let into the lower surface of the needle, and a conical hole is drilled into it to receive the hardened point of the pin. Just within the rim of the compass-box is a graduated circle, whose centre is at the central pin, and the length of the needle is so regulated that it may turn freely within the circle, and yet have its pointed ends as near as possible to it. The needle and graduated circle are covered by a glass plate, and the needle is so adjusted that when not needed for use, it can be thrown off the pin, to prevent wearing off the point. The sights are arranged with two openings, the one a narrow, and the other a broad one, the broad opening being above the narrow one in one sight, and below it in the other. Stretched across the broad opening is a fine hair lying in the direction of the narrow slit above or below it. The plane of these sights cuts the graduated circle in two points, and from these points the graduation is begun and carried in each direction around to 90°.

There is also a vernier plate attached to the box, so that smaller portions of an arc may be measured than one of the smallest divisions of the circle.

The whole instrument thus described, when needed for use, is mounted upon a tripod, to which it may be firmly fastened; when not in use, the needle is raised from its pivot, and the compass packed in a box.

The compass is used either in field surveying or in filling in the details of a trigonometrical survey. In either case, its use is essentially the same, viz., to measure the angle between two *courses* directly, or to determine the magnetic bearing of each, thus giving data for computing the angle; the latter is most frequent.

To determine the bearing of any course, plant the instrument at one extremity of the course, and a flag at the other extremity; then, having made the compass-box truly horizontal, direct the line of sights in the direction of the flag: the reading of the arc at either extremity of the needle will be the bearing. In recording, the bearing is preceded by the letter N or S, according as the course makes northing or southing, and followed by the letter E or W, according as the course makes easting or westing. Having the bearings of two courses, the angle between them may be determined by the following rules: If the courses are both run from the same point, the letters of the bearings are allowed to remain; but if they are run in the usual way, as in running around a field, the letters of one of the courses must be changed, viz., N to S, S to N, E to W, and W to E.

1st. If the meridional letters are alike, and the letters of departure are also alike, *the difference of the bearings is the angle between the courses.*

2d. If the meridional letters are alike, and those of departure are unlike, *the sum of the bearings is the angle between the courses.*

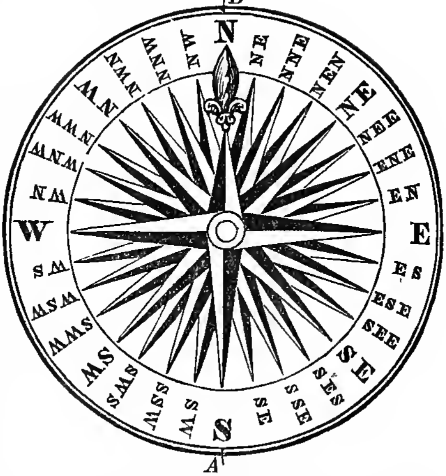
3d. When the meridional letters are unlike, and those of departure alike, the *angle between the courses is equal to* 180° *minus the sum of the bearings.*

4th. When the meridional letters are unlike, and the letters of departure also unlike,

the angle between the courses is equal to 180° *minus the difference of the bearings.*

Mariner's Compass. In the mariner's compass there is a circular card attached to the upper surface of the needle, having its centre exactly over the central pin. The circumference of this card is divided into 32 equal parts, called *points ;* each point is subdivided into four equal parts, called quarter points. The direction of the needle coincides with the line marked NS. The diameter marked EW, at right angles to NS, indicates the East and West points. The points of the Compass are read thus : beginning at the north point, and going around to the east, we say, *north and by east, north north-east, north-east and by north, north-east ;* and so on around the card as indicated by the letters.

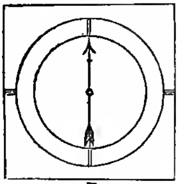
B



A

The compass box is suspended within a larger box by means of two concentric brass circles, called gimballs, the outer one being fixed both to the outer box and to the inner circle, which carries the compass box by two sets of points, the axes of motion being at right angles to each other. By means of this arrangement the inner circle of the compass box and card always retain a horizontal position, notwithstanding the rolling of the ship.

A



B

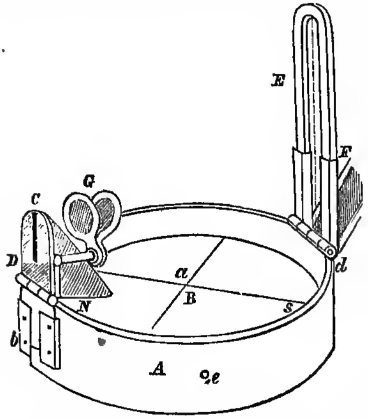
On the interior of the compass box in which the card swings are two marks A and B, which lie in a line passing through the centre of the card, and the compass is so mounted on shipboard that this line is parallel to the keel. Consequently, if B be placed towards the bow of the vessel, the point which it marks on the card will show the compass course of the ship. For NS is always on the magnetic meridian. The course is generally read to quarter points, each quarter point being equal to 2° 48′ 45″.

Azimuth Compass, is so named from its being used for measuring the magnetic azimuth of one of the heavenly bodies, for the purpose of determining the variation of the needle. Its construction is like that of the mariner's compass, in that it carries a circular card, but is unlike it in many other particulars. The card, instead of being graduated to points and quarter points, is divided into degrees and quarter degrees. It has also sights, and is often constructed so as to be attached to a tripod.

This little instrument, besides being used for the purpose already indicated, is very valuable for making reconnoissances, or for making a sketch map of a country. The following is a description of a very convenient form of this instrument, as manufactured by Troughton and Sims, and by them called the prismatic compass :

A represents the compass box ; B the card which moves with the needle about the agate centre, on which it is suspended ; C is a prism through which the observer looks in making an observation ; E is the sight vane, bearing a vertical hair ; on looking through the prism, when properly adjusted, this hair and the figures on the graduated card are seen together, the former by direct view and the latter by reflection ; F is a mirror, turning about a hinge *d*, for the purpose of reflecting an image of the heavenly body, when used as an azimuth compass ; G is a collection of colored glasses, for modifying the light when the sun is observed ; they turn about a hinge, and one or more may be interposed according to the brightness of the sun's

rays; e is a small spring for throwing the agate centre off the pin on which it rests.



The prism C moves up and down in a slide, to suit the eye of the observer in reading the figures on the card, and turns about a hinge D. The sight vane E also has a motion about a hinge, so that the instrument may be rendered more portable. There is also a cover, which can be placed over the glass when not in use. The arc is graduated around from 0 to 360°, the 0 point being so arranged that it shall be under the prism when the line of sight is directed along the magnetic meridian towards the north. The instrument having been arranged for use by fixing the prism at the proper distance from the card and raising the sight-vane, its use is very simple.

If the magnetic azimuth (from the north point) of any object is required, hold the instrument horizontally and turn it around till the vertical hair bisects the object; the reading of the card after it has come to rest, is the required azimuth. If it is required to measure the horizontal angle subtended by two objects, direct the line of sight to the left hand object, and take the reading; then turn it around till the line of sight passes through the second object, and again take the reading; if the 0 point has not passed under the prism, the difference of these readings is the angle sought: if it has, then take the greater reading from 360°, and add the difference to the lesser reading, and the sum will express the value of the required angle.

Variation Compass. A compass of very delicate construction, made use of to show the small daily variation of the magnetic needle. Its needle is made very light and much longer than the ordinary compass, and is very carefully balanced upon a fine point, the bearing surface of the needle being formed of agate. As it is not required that the needle shall turn around the whole circumference, the box, instead of being circular, is oblong, so as to admit of a deviation of only 20 or 25 degrees from the middle line. A vernier with a magnifier is usually applied in order to estimate the changes of position of the needle with the greatest possible precision.

Compasses, or pair of Compasses. A mathematical instrument used for describing circles, measuring and dividing lines, &c. They are more commonly called dividers, from this latter use which is made of them.

The common compasses consist of two legs, sharp at one end and connected at the other by a hinge, by means of which the legs may be opened or closed, so as to place their sharp points at any distance from each other less than the sum of the lengths of the legs. In some compasses the points are both fixed; in others one of them is arranged so that it may be taken out and replaced by a pen or pencil. There are in use various kinds of compasses, adapted to the various uses for which they are intended.

Triangular Compasses. These have three legs, the third leg being attached to the other two by a double joint, so that it can be moved in any direction, with respect to the other two. It is used to transfer three points, as the three vertices of a triangle, from one drawing to another. They are particularly useful in transferring maps.

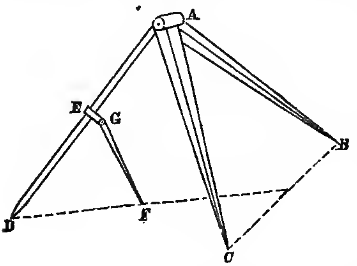
Beam Compass, consists of a straight beam or bar, bearing two arms at right angles to it, one of which is fixed, and the other movable. These arms are so arranged that sharp points, pens or pencils, may be screwed into them. The movable arm is provided with a clamp screw, by means of which it may be fastened to the beam, and sometimes is provided with a vernier and tangent screw so that the point which it bears may be set at any distance from the fixed point. The beam or limb is

accurately graduated into equal parts, the 0 of the scale coinciding with the fixed point. The beam compass is used for describing circles with a greater radius than would be convenient to describe with the ordinary compasses.

Bow Compasses, are very small compasses, having their legs terminating in a bow of steel which serves as a handle. The distance between the points is regulated by a screw which connects the legs. Their use is to describe very small circles.

Elliptical Compasses, are compasses contrived to describe ellipses. They are of various constructions, one of the simplest as well as most elegant of which, we shall describe.

The instrument consists of three legs AB, AC and AD, the last of which turns about a hinge at A, so that its plane of motion is at right angles to the plane of the other two.



The leg AD is cylindrical, and a fourth leg EF, carrying a pencil or pen, and so fitted as to slide freely along it. This leg EF has a hinge joint at G, so that it may be set to make any inclination with AD. The manner of using the instrument is this: having the axes of the ellipse given, the leg AD is placed at the centre, and the points C and B are set in a line parallel to the conjugate axis; then the leg EF is turned about the hinge at G till the distance from the point F to the axis is equal to the semi-conjugate axis. It is then secured at this angle by a clamp screw at G. The leg EF is then turned around AD till the point comes into the transverse axis, and by moving the legs AB and AC, the point F is brought to one extremity of the transverse axis; then the point F is carried around AD, sliding along it so that the point F shall always remain in the plane of the paper. It will trace out the ellipse. For, the point F being always at the same distance from AD, will be upon the surface of an oblique cylinder, of which AD is the axis; and being always in the plane of the paper, it must describe an ellipse.

Proportional Compasses. Those in which the joint lies not at one end of the legs, but at some intermediate point. The joint is made movable, so that when the compasses are opened, the distance between the points of the legs at one end may be any fractional part of the distance between the points of the legs at the other end. The joint is moved by means of a groove in each leg, and is secured by a clamp screw.

Proportional compasses are used in reducing drawings from one scale to another, but on account of their liability to get out of order, they are not of much use in practice.

Besides the compasses already described, there are a great variety of others; indeed a complete catalogue of them would exceed our limits.

COM-PAT'I-BLE VALUES. [Fr. *compatible*, from L. *con*, with, and *peto*, to seek]. If a group of n equations containing n unknown quantities, be so combined as to eliminate all the unknown quantities but one, the resulting equation containing that one is called the *final equation*. If the final equation be solved, its roots are called *compatible* values of that unknown quantity, because these are the only values of that quantity which are compatible with the simultaneous existence of the group of given equations.

COM'PLE-MENT. [L. *complementum*, from *con*, with, and *pleo*, to fill]. In general, is what is wanted to complete some quantity or thing.

Complement of an Arc, in Trigonometry, is what remains after subtracting the arc from 90°: thus, 30° is the complement of 60°, and the reverse. If the given arc exceeds 90°, its complement is negative: thus, the complement of 105° is $-15°$, and the reverse.

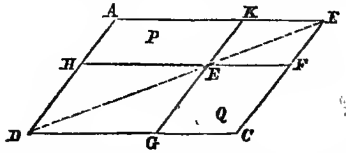
The complement of an angle, is what remains after subtracting it from a right angle.

Arithmetical Complement of a number. What remains after subtracting it from a unit of the next higher order: thus, 4 is the arith-

metical complement of 6, and 24 is the arithmetical complement of 76.

The arithmetical complement of a logarithm, is what that logarithm wants of 10.

COMPLEMENTS OF A PARALLELOGRAM. The lesser parallelograms formed by drawing lines



parallel to the sides of the given parallelogram through the same point of the diagonal. In the parallelogram AC, P and Q are complements. In every case these complements are equivalent.

COMPLEMENT OF LIFE. A term used in discussing life annuities by DeMoivre and others. It is the difference between 86 years and any given age: thus, 30 is the complement of 56, and the reverse.

COM'PLEX FRACTION. [L. *complexus*, connection of things]. A fraction having a fraction or mixed number in either the numerator or denominator, or in both: thus,

$$\frac{(\frac{3}{4})}{14}, \quad \frac{2}{47\frac{1}{2}}, \quad \frac{(\frac{3}{5})}{(\frac{4}{7})}, \quad \frac{42\frac{5}{6}}{87\frac{4}{5}},$$

are complex fractions.

COM-POS'ĬTE NUMBER. One which can be divided by some other number greater than 1, in opposition to a *prime* number, which cannot be thus divided: thus, 12, 15 and 27, are composite numbers, whilst 11, 47, 89, are not composite.

The following table shows all of the composite, as well as all of the prime numbers to 100, and is often useful in numerical computations. The composite numbers are resolved into their prime factors.

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	2^2	5	$2 \cdot 3$	7	2^3	3^2
1	$2 \cdot 5$	11	$2^2 \cdot 3$	13	$2 \cdot 7$	$3 \cdot 5$	2^4	17	$2 \cdot 3^2$	19
2	$2^2 5$	$3 \cdot 7$	$2 \cdot 11$	23	$2^3 \cdot 3$	5^2	$2 \cdot 13$	3^3	$2^2 \cdot 7$	29
3	$2 \cdot 3 \cdot 5$	31	2^5	$3 \cdot 11$	$2 \cdot 17$	$5 \cdot 7$	$2^2 \cdot 3^2$	37	$2 \cdot 19$	$3 \cdot 13$
4	$2^3 \cdot 5$	41	$2 \cdot 3 \cdot 7$	43	$2^2 \cdot 11$	$3^2 \cdot 5$	$2 \cdot 23$	47	$2^4 \cdot 3$	7^2
5	$2 \cdot 5^2$	$3 \cdot 17$	$2^2 \cdot 13$	53	$2 \cdot 3^3$	$5 \cdot 11$	$2^3 \cdot 7$	$3 \cdot 19$	$2 \cdot 29$	59
6	$2^2 3 \cdot 5$	61	$2 \cdot 31$	$3^2 \cdot 7$	2^6	$5 \cdot 13$	$2 \cdot 3 \cdot 11$	67	$2^2 \cdot 17$	$3 \cdot 23$
7	$2 \cdot 5 \cdot 7$	71	$2^3 \cdot 3^2$	73	$2 \cdot 37$	$3 \cdot 5^2$	$2^2 \cdot 19$	$7 \cdot 11$	$2 \cdot 3 \cdot 13$	79
8	$2^4 \cdot 5$	3^4	$2 \cdot 41$	83	$2^2 \cdot 3 \cdot 7$	$5 \cdot 17$	$2 \cdot 43$	$3 \cdot 29$	$2^3 \cdot 11$	89
9	$2 \cdot 3^2 \cdot 5$	$7 \cdot 13$	$2^2 23$	$3 \cdot 31$	$2 \cdot 47$	$5 \cdot 19$	$2^5 \cdot 3$	97	$2 \cdot 7^2$	$3^2 \cdot 11$

To use this table, find the tens of the given number in the left-hand column, then look along the corresponding horizontal line till you come under the number of units at the top of the table; the number thus found is the given number resolved into its prime factors.

A careful study of this table will show some remarkable properties of composite numbers.

First. Every multiple of 11 is found in the diagonal of the table, running from the upper left angle.

Second. 3^2 enters into any number of the second diagonal, drawn from the other angle of the table; and parallel to this, in each alternate line, the other 3's of the table are all found.

Third. The 5's and 2's are found in every vertical column in which they occur at the top; and under 0, they both occur in every line.

COM-PO-SĬ''TION. [L. *compono*, to lay together]. In a general sense, means, *uniting* or *joining* several different things, so as to form one whole, called a compound.

PROPORTION BY COMPOSITION. Quantities are compared by *composition*, when the sum or difference of the antecedent and consequent is compared with either antecedent or consequent. If four quantities are in proportion, they will be in proportion by composition; that is, if

$$a : b :: c : d, \text{ then}$$
$$a \pm b : a :: c \pm d : c.$$

COMPOSITION OF EQUATIONS. The operation of finding an equation when its roots are given. Suppose the required equation to contain but one unknown quantity, which denote

by x. Then, if there are m roots given of the form a, b, c, &c., the equation from which they may be derived is of the form

$$x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0,$$

in which the highest exponent of x is equal to the number of roots, and goes on decreasing by 1 in each term to the right. The values of the co-efficients of the different terms observe the following law: The co-efficient of the first term is 1. That of the second term is equal to the algebraic sum of the roots taken with a contrary sign. That of the third term is equal to the algebraic sum of the different products of the roots taken in sets of two. That of the fourth term is equal to the algebraic sum of the different products of the roots taken three in a set, with its sign changed, and so on, for the successive co-efficients. The co-efficient of x^0, or the absolute term, is equal to the continued product of all the roots taken with its proper sign, if the number of roots is even, and with its sign changed if the number is uneven.

It follows from this rule for the composition of the equation, that if one of the roots is 0, the absolute term will be 0, and the converse is equally true. If the co-efficient of the second term is 0, the sum of the positive roots is numerically equal to the sum of the negative ones: the converse of this is also true.

The rule given above is very useful in theory, but in practice, when there are more than three roots, its application is tedious. The following method presents little difficulty: Subtract each root in succession from x, forming factors of the form $x - a$, $x - b$, $x - c$, and so on. Multiply these results together, and place their continued product equal to 0, and that will be the required equation. If the method of multiplication by detached co-efficients is employed, this operation becomes a very simple one.

COM-POUND′. [L. *compono*, *con*, with, and *pono*, to set or put]. The result of a composition of different things. The term stands opposed to simple.

COMPOUND ADDITION, *Compound Subtraction*. *Compound Multiplication*, and *Compound Division*, are names given to these several operations when the numbers to be added are expressed in a varying scale. See *Arithmetical Scale*, and the several articles, *Addition*, *Subtraction*, *Multiplication*, and *Division*.

COMPOUND FRACTIONS. A fraction is compound when it is a fraction of a fraction, or several fractions connected by the word *of*; thus, $\frac{1}{2}$ of $\frac{1}{4}$, $\frac{1}{7}$ of $\frac{1}{8}$ of 4.

COMPOUND INTEREST is, when the interest on a sum of money, becoming due and not paid, is added to the principal and then interest computed on this amount, as on a new principal, and so on.

The amount originally placed at interest is called the *principal*. Let us denote the principal by p, the rate per cent per annum by r, the time which the money is at interest in years by t, also let S denote the amount which will be due at the end of t years. Then will the following formulas express the relations existing between these several quantities:

$$S = p(1 + r)^t \text{ or } \log S = \log p + t \log (1 + r).$$

If any three of the four quantities, S, p, t, and r, be given, the other one can be found. It may happen that interest is added to principal oftener than once a year, as at intervals of six or of three months. In such case, t denotes the number of these intervals, and r the rate per cent for one of the periods of time.

By the aid of these formulas, a table may be computed for finding by inspection the amount of one dollar at compound interest for any number of years at any rate per cent.

COMPOUND NUMBER. A number constructed according to a varying scale, as 3 *cwt.*, 1 *qr.*, 5 *lb.*; more properly, a denominate number See *Denominate Numbers*.

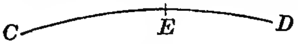
COMPOUND RATIO. The product of two or more ratios; thus, $\frac{cd}{ab}$, is a ratio compounded of the simple ratios $\frac{c}{a}$ and $\frac{d}{b}$.

COM-PŬTE′. [L. *computo*, *con*, with, and *puto*, to lop or prune]. To reckon by the aid of characters.

COM-PU-TĀ′TION. [L. *computatio*, computing, calculating]. The operation of computing or reckoning; the practical application of the rules of a science to individual examples.

CON′CĀVE. [L. *concavus*, *con*, with, and

cavus, hollow]. A term employed in speaking of the inner surface of a hollow body, and by analogy extended to lines. A line has its concavity at any point on the side which is opposite to the tangent drawn at the point; that is, it is on that side towards which the line bends or curves. Thus, the line CD is

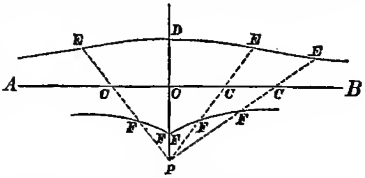


concave on the side E, and convex on the opposite side. In like manner, a curved surface is concave on the side towards which it bends, and is convex on the opposite side. Thus the surface of a sphere is concave towards its centre, and convex on the side opposite to it.

CON-CEN'TRIC. [L. *concentricus*, *con*, with, and *centrum*, centre]. Having the same centre: it is opposed to eccentric, having different centres. The term is usually applied to spheres or circles, but sometimes, by analogy, to other surfaces and lines. Two spheres are concentric which have the same centre and unequal radii. Two circles are concentric which have the same centre and unequal radii. Two cylinders having circular bases, are concentric when their bases are parallel, and when they have a common axis. In this case, every section made by a plane parallel to the bases, cuts out two concentric circles. Every oblique plane cuts out similar ellipses, which may be called concentric, since they have a common centre. Hence, we may call similar ellipses concentric when they have a common centre, and when their axes lie in the same direction.

CONCH'OID. [L. *concha*, a shell, and Gr. εἶδος, form]. A curve of the fourth order, first made use of by Nicomedes, who invented it, and whose name it bears, for the purpose of trisecting an angle, and for constructing two geometrical means between two given straight lines, and ultimately to construct a cube double a given cube. It is well adapted to this end, since it admits of an easy mechanical construction. It is also used in architecture as a bounding line of the meridian section of columns.

It may be constructed by points as follows: Let AB be any straight line, and P a point not upon the line: then if straight lines PE be drawn, cutting AB in the points C, and

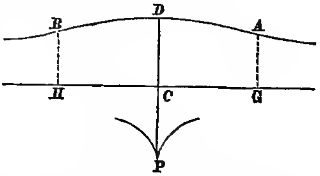


distances CE and CF be laid off from the points of intersection equal to a given line CD: the curve traced through the points thus determined, is the conchoid. That branch which is most remote from P, is called the first or superior conchoid, and the other branch is the second or inferior conchoid. Both branches are infinite in extent, and AB is their common asymptote. The line AB is called the directrix, and P is the pole of the curve. The line $CD = EC = EF$ is sometimes called the modulus of the curve.

If we denote the length of the modulus by a, the distance PC by b, and take AB as the axis of abscissas, and the line PD at right angles to it through the pole, as the axis of ordinates, we find for the equation of the curve,

$$x^2 = \frac{(b+y)^2(a^2-y^2)}{y^2},$$

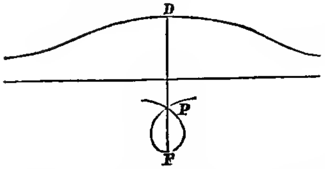
which is of the fourth degree.



If $a = b$, the inferior branch will pass through the pole P, which will then be a cusp point of the first species. The superior branch will have two points of inflexion, A and B, whose ordinates AG and HB are equal to $a(\sqrt{3}-1)$. If $a < b$, there will be two points of inflexion in each branch, as in the last figure.

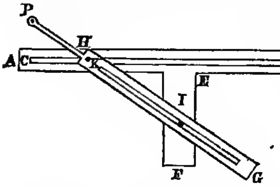
If $a > b$, the point P is a multiple point, and there will be an oval part of the inferior branch lying between P and F, as in the annexed figure. If P be taken as a pole, and PD as the initial line, the polar equation of the conchoid is

$$r = \frac{b}{\cos v} \pm a.$$



The entire area included between the curve DE and its asymptote, is infinite, but the solid generated by revolving it about this line as an axis, is finite and equal to a hemisphere whose radius is a, together with a cylinder whose base is measured by πa^2, and whose altitude is $\frac{\pi b}{2}$.

The mechanical construction of this curve may be made as follows:

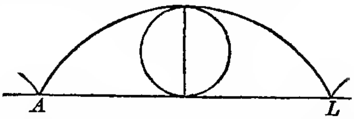


In the ruler AB a groove is cut, so that a smooth pin firmly fixed in a movable ruler PG, may slide freely along it: a pin is fixed firmly at I, and a pencil at P. If the ruler be moved so that the pin K may slide along the groove CD, and is the same time firmly pressed against the pin I, the pencil at P will mark out the superior branch of the conchoid.

A second pencil might be arranged to trace the inferior branch of the conchoid. For the practical application of this curve, and also the conchoid, in trisecting an angle, or in doubling a cube, see the articles,—*Trisection of an Angle*, and *Duplication of the Cube.*

CON′CRĒTE. [L. *concretus*, concrete; *con*, with, and *cresco*, to grow]. A concrete quantity is one that carries with it the idea of matter. Concrete stands opposed to the term abstract. An abstract quantity is a mere mental conception, and may have for its representative a number, a letter, or a geometrical figure. A concrete quantity is a physical object, or a collection of such objects, and may likewise be represented by numbers or letters, in which case the numbers or letters express simply the number of physical units that compose the quantity, the unit being a physical substance. Thus, 3 is an abstract number, and may be the number of any things; but 3 pounds immediately suggests the idea of some ponderable substance. A portion of space, bounded by a surface, all of whose points are equally distant from a point within, is an abstract magnitude; but if we conceive this space to be filled with matter, the idea becomes concrete, and we have the idea of a physical sphere or globe.

CON-CUR′RENCE. [L. *con*, with, and *curro*, to run]. When two lines have a common point, which is neither a point of tangency, nor a point of intersection,—that point is called a *point of concurrence.*

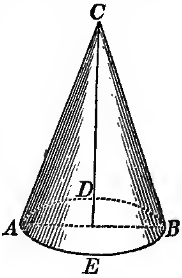


For example, the points A and L, in which the cycloid meets the axis, are points of concurrence. The vanishing point of a system of straight lines, which are parallel, is a point of concurrence of their perspectives.

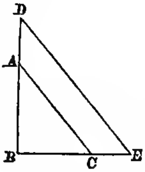
CON-CUR′RENT, meeting, but not intersecting, and not tangent.

CON-DĬ″TION. [L. *conditio*, condition; from *condo*, to build, or make]. See *Equation of Condition.*

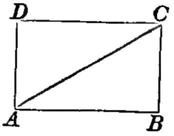
CŌNE. [L. *conus*, a cone; Gr. κωνος, a cone]. In Elementary Geometry, a *cone* is a solid which may be generated by a right-angled triangle CAD, revolving about one of the sides, CD, adjacent to the right angle. The side CD, which remains fixed, is called the *axis*, and its length measures the *altitude* of the cone. The side AD, perpendicular to the axis, generates a circle called the *base*, and the hypothenuse, CA, generates a curved surface, which is called the *lateral* or *convex surface of the cone.* The length of the hypothenuse is the measure of the *slant height* of the cone.



Any plane, ACB, passed through the axis of the cone, is called a *meridian* plane, and cuts out a meridian isosceles triangle, ACB, whose vertical angle is equal to twice the vertical angle of the generating triangle, and whose altitude is equal to that of the cone. When the vertical angle, ACD, of the generating triangle is less than 45°, the vertical angle, ACB, of the meridian triangle, is less than 90°, and the cone is said to be *acute;* when the vertical angle of the generating triangle is greater than 45°, that of the meridian triangle is greater than 90°, and the cone is *obtuse;* when the vertical angle of the generating triangle is equal to 45°, that of the meridian triangle is 90°, and then we have a *rectangular* or *right-angled* cone.



If two similar triangles, ABC and BDE, be revolved about their homologous sides, AB and DB, the cones generated are similar.



If, in a rectangle BD, we draw a diagonal AC, dividing it into two right-angled triangles; then, if the first be revolved about the side DC, as an axis, and the second about the perpendicular side CB, as an axis, the two cones, so generated, are called *conjugate cones.* If the one is acute, the other must necessarily be obtuse. If one is rectangular, the other will be so also. This relation is of importance in the discussion of conjugate hyperbolas.

An expression for the lateral or convex surface of a cone may be deduced from that of the right pyramid. In the right pyramid, the area of the convex surface is equal to the perimeter of the base multiplied by half the slant height, and this measure is true, whatever may be the number of sides of the base; but when the number of sides becomes infinite, the base becomes a circle, the pyramid becomes a cone, and the slant height of the left pyramid becomes that of the cone; hence, the *convex surface of a cone is equal to the circumference of the base multiplied by one-half of the slant height;* or, denoting the surface by S, the radius of the base by r, and the slant height by h, we have the formula

$$S = \pi r h.$$

In similar cones, $\pi r h$ is proportional to the area of the generating triangle: hence, their convex surfaces are to each other as the areas of these triangles. If the cones are similar, the areas of the generating triangles are to each other as the squares of their homologous sides; hence, the convex surfaces of similar cones are to each other as the squares of their altitudes, or as the squares of the radii of their bases.

If the cones are conjugate, the areas of their generating triangles are equal; and the convex surfaces of conjugate cones are to each other as their altitudes.

To find the entire area of the surface of a cone, it is necessary to add to the area of the convex surface, that of the base; the expression for this area is

$$S = \pi r (h + r).$$

An expression for the solidity of a cone may be deduced as follows:

The solidity of a right pyramid is equal to the area of its base multiplied by one-third of its altitude, and this measure is true, whatever may be the number of sides of the base; but if the number is infinite, the pyramid becomes a cone, and the base of the pyramid becomes the base of the cone,—their altitudes being the same; hence, *the solidity of a cone is equal to the area of its base multiplied by one-third of its altitude.* Denoting the solidity or volume by V, the radius of the base by r, and the altitude by h, we have the formula

$$V = \pi r^2 \times \tfrac{1}{3} h, \quad \text{or,}$$

$$V = \tfrac{1}{3} \pi h r^2.$$

We see that the solidities of two cones are to each other as the products of their bases and altitudes.

If the cones are similar, their bases will be to each other as the squares of their altitudes; or, their altitudes will be to each other as the radii of their bases; hence, the solidities of similar cones are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.

If the cones are conjugate, we have the following expressions for their solidities:

$$1.\ S = \tfrac{1}{3}\pi h r^2.$$

$$2.\ S' = \tfrac{1}{3}\pi r h^2.$$

whence $$3.\ \frac{S}{S'} = \frac{\frac{1}{3}\pi h r^2}{\frac{1}{3}\pi h^2 r} = \frac{r}{h}.$$

That is, they are to each other as the radii of their bases; or, as their altitudes. If the conjugate cones are rectangular, their solidities are equal.

CON''GRU-OUS. [L. *congruus*, concordant]. Two numbers, p and q, are congruous with respect to a third number n, when their difference is exactly divisible by that number; that is, when the expression $\frac{p-q}{n}$ is a whole number. Thus 12 and 7 are congruous with respect to 5, because $\frac{12-7}{5} = 1$, 27 and 12 are also congruous with respect to 5, because $\frac{27-12}{5} = 3$, and so on. The numbers considered may be either positive or negative, but they must be entire.

Two numbers, p and q, are incongruous with respect to a third number n, when their difference is not exactly divisible by that number; thus, 7 and 2 are incongruous with 4, because $\frac{7-2}{4} = 1\frac{1}{4}$.

When two numbers are congruous with respect to a third, either one is called a *residual* of the other with respect to the third; thus, 13 and 7 are residuals of each other with respect to 5.

All the residuals of any given number p, with respect to any number n, are of the general form $p + nx$, x being a whole number.

Of m successive numbers, differing from each other by 1, and also from another number n, by 1, one of the m is necessarily congruous with n, with respect to m, and only one.

Thus, of the numbers 11, 12, 13, 14, 15, 16, 17, and the number 8, only 15 and 8 are congruous with respect to $m = 7$.

If two numbers are congruous with respect to a composite number, they are also congruous with respect to its prime factors, and also with respect to the products of any number of these factors; thus, the numbers 57 and 9, which are congruous with respect to 24, are also congruous with respect to 2, 3, 6, and also with respect to 12, 8 and 4.

If, in an expression of the form

$$ax^n + bx^m + cx^h + dx^q + \&c.,$$

in which a, b, c, &c., are whole numbers, if we substitute in succession for x, two numbers congruous with respect to a third number, the results will also be congruous with respect to the same number.

Congruous numbers possess many other curious properties, which, with those already enumerated, have been successfully applied by Gauss and other writers in the investigation of the properties of numbers.

CON'IC SECTION. The curve of intersection of a plane and conic surface. Although the term, from its signification, is applicable to the intersection of a plane with any conic surface, custom has restricted its use to the designation of those curves which can be cut from a right cone with a circular base. Since the equation of such a cone is of the second degree, whilst that of a plane is of the first degree, it follows that the equation of the curve of intersection, when referred to axes in its own plane, is of the second degree between two variables.

Analytical investigation has also shown that every equation of the second degree, between two variables, represents one of the conic sections. Every equation of this kind may be reduced to the form

$$ay^2 + bxy + cy^2 + dy + ex + f = 0,$$

in which x and y are rectangular co-ordinates of every point of the curve represented.

The curves which may be represented by the different cases of this general equation, are divided into three classes:

1st. When $b^2 - 4ac < 0$, the curve is called an *ellipse*.

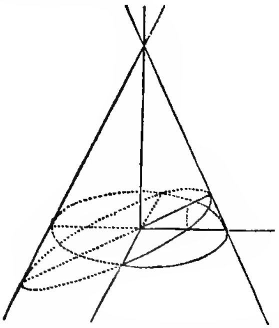
2d. When $b^2 - 4ac = 0$, the curve is called a *parabola*.

3d. When $b^2 - 4ac > 0$, the curve is called an *hyperbola*.

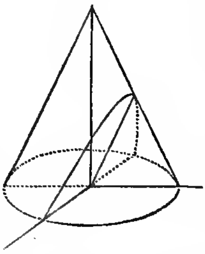
These classes have each several species, which we shall discuss in their order. But first we shall explain the manner of passing the cutting plane so as to cut out each class of curves.

First. To cut out the *ellipse*, the plane

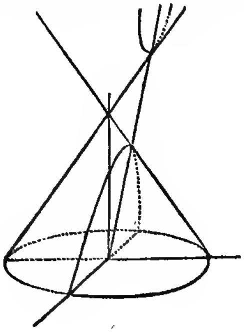
must be passed so as to make a less angle with the plane of the base, than one of the elements makes. In this case all the elements of the conic surface are cut in one nappe, so that the curve is a closed one, returning into itself.



Second. To cut out the parabola, the plane is passed so as to make with the plane of the base an angle equal to that made by one of the elements. In this case, all the elements in one nappe, except one, are cut. The curve has but one branch, which is infinite in extent, and does not return upon itself. The element which is not cut is the one which is parallel to the plane of the section.



Third. To cut out the hyperbola, the cutting plane is passed so as to make with the plane of the base a greater angle than one of the elements of the conic surface makes. In this case, all the elements except two are cut; half of them in one nappe and half in the other nappe. The curve has two branches, infinite in extent, lying in opposite directions and on different nappes. Neither branch returns into itself. The two elements which are not cut are those which lie in a plane through the vertex of the cone and parallel to the plane of the section.



The same sections may be cut from an oblique cone with either a circular or elliptical base, as follows:

First. Pass a plane in any manner so as to cut all the elements; the curve of intersection will be an *ellipse*.

Second. Pass a plane so as to be parallel to any one element, and parallel to but one; the curve of intersection will be a *parabola*.

Third. Pass a plane so as to be parallel to any two elements, and the curve of intersection will be an *hyperbola*. If the two elements which are not cut be projected upon the cutting plane by lines parallel to the line drawn through the centre of the hyperbola and the vertex of the cone, these projections will be the asymptotes of the curve.

In cutting any of those sections from the right cone with a circular base, it may be observed that all sections whose planes are parallel, are *similar* curves.

It has been observed that the *analytical characteristic* of the ellipse is

$$b^2 - 4ac < 0;$$

if in addition, $b = 0$, and $a = c$, the ellipse will become the circle; if

$$(bd - 2ae)^2 - (b^2 - 4ac)(d^2 - 4af) = 0,$$

the ellipse becomes a point; if

$$(bd - 2ae)^2 - (b^2 - 4ac)(d^2 - 4af) < 0,$$

the curve is imaginary. Hence, the *circle*, the *point*, and the *imaginary curve*, are particular cases of the ellipse.

To cut the *circle* from the right cone with a circular base, the cutting plane may have any position parallel to the plane of the base. To cut the *point*, the plane must pass through the vertex, and may make any angle with the plane of the base which is less than the angle made by one of the elements. As to the imaginary curve, it is to be remarked, that the conditions which make the ellipse imagi-

nary make the conic surface also imaginary; and as the cutting plane is always real, there can, in this case, be no curve, or it will be *imaginary*, which agrees with the analysis.

The analytical characteristic of the parabola is

$$b^2 - 4ac = 0.$$

If, in addition to this,

$$bd - 2ae = 0,$$

the parabola reduces to two straight lines, which are parallel; they will be real and distinct when

$$d^2 - 4af > 0;$$

they will coincide when

$$d^2 - 4af = 0;$$

and they will be imaginary when

$$d^2 - 4af < 0;$$

Hence, *two parallel straight lines* form a particular case of the parabola, and they may be either *separate*, *coincident*, or *imaginary*. To cut these from the cone, we first conceive the vertex to be removed to an infinite distance, the base remaining fixed. In this case the cone becomes a right cylinder, as the elements become parallel. If now a plane be passed parallel to one of the elements, cutting the base in two points, it will cut out the two separate parallel straight lines; if the cutting plane be moved parallel to its first position, and from the axis, the elements cut out will approach, and when the plane becomes tangent to the cylinder, the lines will be *coincident*; if the plane be still moved parallel to its first position, and from the axis, it will fulfill the condition for cutting out two parallel straight lines, but the section will be imaginary. The two coincident straight lines may also be obtained from the ordinary curve, by passing the plane so that it will pass through the vertex and be tangent to the cone.

The analytical characteristic of the hyperbola is

$$b^2 - 4ac > 0.$$

If in addition, $b = 0$ and $a = -c$, the hyperbola is *equilateral*; if

$$(bd - 2ae)^2 - (b^2 - 4ac)(d^2 - 4af) = 0,$$

the hyperbola becomes *two straight lines, which intersect each other*. Hence, the *equilateral hyperbola* and *two straight lines which intersect*, are particular cases of the hyperbola.

To cut the two straight lines which intersect from the cone, the plane must be passed through the vertex, and may make any angle with the plane of the base that is greater than that made by one of the elements. The equilateral hyperbola can only be cut from a rectangular or from an obtuse cone. To cut it out, we first pass a plane through the vertex, which shall cut out two elements that are at right angles to each other, then any parallel plane will cut out an equilateral hyperbola of which these two lines are a particular case.

To cut out a pair of conjugate hyperbolas, we take two conjugate cones and pass planes parallel to the axes, at distances proportional to the sines of the vertical angles.

Every pair of *sub-contrary* sections are *similar curves*; and in a right cone with a circular base all planes that make the same angle with the axis, however they may be situated, cut out *similar sections*.

Every plane section of any surface of the second order, is one of the conic sections. The surfaces of the second order are, *the ellipsoid*, *the hyperboloid*, *the paraboloid*, and their different cases.

The equation of the ellipsoid referred to its centre and axis is

$$a^2b^2z^2 + a^2c^2y^2 + b^2c^2x^2 = a^2b^2c^2,$$

in which a, b, and c, denote the semi-axes We shall suppose, in the following discussion, that $a > b > c$, and shall speak of the several axes as the *axis a*, the *axis b*, or the *axis c*, meaning thereby the axes whose length is designated by $2a$, $2b$, and $2c$, respectively.

The sections of the ellipsoid by a plane are all ellipses, or their varieties. If the cutting plane, in any case, be moved from the centre, continuing parallel to its first position, the sections will continually diminish, till finally the plane becomes tangent to the surface, when the section becomes a point; if the plane be moved still further from the centre, the curve of intersection becomes imaginary, or the plane does not cut the surface. If the cutting plane is parallel to the *mean axis b*, and makes with the plane of the axes ab an angle whose tangent is

$$\pm \frac{c}{a}\sqrt{\frac{a^2 - b^2}{b^2 - c}},$$

the section will be a circle. Hence, from the ellipsoid we may cut every species of ellipse.

The equation of the hyperboloid of two nappes is

$$a^2b^2z^2 + a^2c^2y^2 - b^2c^2x^2 = -a^2b^2c^2.$$

If a plane be passed in any manner, so as to cut both nappes, the curve of intersection is an hyperbola. If it be passed so as to cut but one nappe, the curve cut out is an ellipse. If the cutting plane be moved towards the centre, continuing parallel to its first position, the curve will grow smaller and smaller, pass through the point, and finally become imaginary. The section will be a circle under the same conditions as have been explained in speaking of the ellipsoid.

The equation of the hyperboloid of one nappe, referred to its centre and axes, is

$$a^2b^2z^2 + a^2c^2y^2 - b^2c^2x^2 = a^2b^2c^2.$$

Any plane passed tangent to the surface, will intersect it in two straight lines, which cut each other; this is a particular case of the hyperbola. All planes parallel to a tangent plane, cut out hyperbolas. All other planes cut out ellipses, which become circles under the conditions already mentioned in the preceding cases.

Since the radical expression

$$\pm \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2-c^2}}$$

admits of two values, it follows that there are two systems of parallel planes which cut circles from the three surfaces already mentioned, except in some cases, which we now propose to consider.

First. In the ellipsoid. If $a = b$, the expression becomes 0, or the cutting plane has but one position, and that is parallel to the plane of the axes ab. This is as it should be, for the supposition makes the ellipsoid one of revolution about the axis c. If $b = c$, the expression becomes *infinite*, or the cutting plane is perpendicular to the axis a. This should be so, for the supposition makes the ellipsoid one of revolution about the axis a.

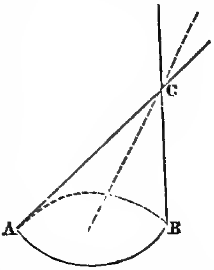
If $a = b = c$, the expression becomes $\frac{0}{0}$, or indeterminate, and the ellipsoid becomes a sphere. This shows that every section of the sphere is a circle. A similar discussion may be had with respect to the hyperboloids.

In the case of the elliptical paraboloid, every plane parallel to the axis cuts out a parabola; all other planes cut out ellipses which may become *circles*, *points*, or *imaginary curves*, under the various suppositions that may be made with respect to the direction of the cutting planes.

In the hyperbolic paraboloid, every plane tangent to the surface cuts out two straight lines, which intersect. This is a particular species of hyperbola, and any plane parallel to a tangent plane cuts out an hyperbola. All planes which are not parallel to a tangent plane cut out parabolas. The discussion of the particular cases of these surfaces will show that the sections in all cases are conic sections, but the limits of the present article forbid any attempt at developing all of the curious results which might be discovered by a complete discussion of the problem. Enough has been given to show the manner of proceeding in the cases which may arise.

For a particular description of the peculiar properties of the several conic sections, the reader is referred to the several articles, *ellipse, hyperbala*, and *parabala*.

CONIC SURFACE. In higher Geometry, a surface which may be generated by a straight line, moving in such a manner that it shall always touch a given curve, and pass through a given point. The given curve, as AB, is called the *directrix;* the given point, as C, is called the *vertex;* the straight line moved, is called the *generatrix*, and any one of its several positions, an *element* of the surface.



The generatrix is supposed to extend indefinitely on both sides of the vertex; whence, it appears that the surface is composed of two parts meeting at the vextex; these parts are called *nappes;* the one nearest the directrix being the *lower nappe*, and the other one the *upper nappe*.

Every section of a conic surface, by a plane not passing through the vertex, is called a *base;* consequently, any conic surface may have an infinite number of bases. It is to be observed, however, that if with either of these bases as a directrix and the given vertex, a conic surface be generated, it will be identical with the given surface, and it is for this reason that any conic surface may be re-

garded as having a plane curve for its directrix. It is customary throughout the mathematical course to use the terms *base* and *directrix* of a conic surface as synonymous; also, for simplicity of expression, the term *cone* is often used when the conic surface only is meant.

If the base of a cone has a centre, the straight line passing through this point and the vertex, is called the *axis* of the cone or surface. If the base is a circle, and the axis perpendicular to its plane, the cone is *right*, otherwise it is *oblique*. Since any cone has an infinite number of bases variously inclined to each other, each of which may have a centre, it follows that the same cone may have an infinite number of different axes; it follows therefore, that the term, *axis of a cone*, is indefinite, unless the particular base to which the cone is referred be given. In ordinary mathematical language, the term cone is applied to the *right cone with a circular base*, and when the term *axis of a cone* is employed without explanation, it is intended to express the axis of a right cone with a circular base.

The nature and properties of a conic surface depend, 1st. Upon the nature and extent of the base assumed as a directrix; and 2d. Upon the position of the vertex with respect to the base: Hence, when these two elements are given or known the surface also is known, and by suitably varying them, every possible variety of conic surface may be obtained.

If, whilst the base remains fixed, the vertex be moved towards it until it finally coincides with it, all the elements will approach and finally coincide with the plane of the base, giving *a portion of a plane as one of the extreme cases of a conic surface:* If, on the other hand, the vertex be moved from the plane of the base until it becomes infinitely distant, each element will recede from the plane of the base, till finally they will all become parallel to each other, and we shall have, for *the other extreme case, a cylindrical surface*. This observation will be found of importance in the analytical discussions of the conic sections.

Conic surfaces are classed in orders, according to the degrees of their equations; or what is the same thing, according to the degrees of the equations of their bases.

If the base of a cone is of the 2d, 3d, 4th, &c., order, the cone itself is also of the 2d, 3d, 4th, &c., order.

Of all the different orders, the second is the most important, and amongst those of the second order, the right cone with a circular base holds by far the most conspicuous place, in a practical point of view.

We shall explain the method of finding the equation of any conic surface whatever, and then deduce the particular equation of the right cone with a circular base.

Since every cone may be regarded as having a plane base, we may take the co-ordinate plane XY to coincide with it, in which case the general equation of the base of any cone,. will be

$$f(x, y) = 0 \cdots (1).$$

If we denote the co-ordinates of the vertex by x', y' and z', the equation of any straight line passing through it, will be

$$x - x' = a(z - z') \cdots\cdot (2), \text{ and}$$
$$y - y' = b(z - z') \cdots\cdot (3).$$

If, now, equations (2) and (3) be solved with reference to x and y, respectively, we shall have

$$x = az + (x' - az') \cdots\cdot (4), \text{ and}$$
$$y = bz + (y' - bz') \cdots\cdot (5);$$

in which the absolute terms $x' - az'$ and $y' - bz'$, are the co-ordinates of the point in which the straight line pierces the plane XY. If these, therefore, be substituted for x and y, respectively, in equation (1), we shall have

$$f(x' - az', y' - bz') = 0 \cdots (6),$$

which is the equation of condition that the generatrix [equations (2), (3),], shall pierce the plane XY, in the directrix.

Now, for each couple of values of a and b, which will satisfy this equation of condition, the values of x, y and z, in equations (2) and (3), will denote the co-ordinates of every point of an element of the surface, and for all the couples of values of a and b, which will satisfy equation (6), the values of x, y and z, in equations (2) and (3), will represent the co-ordinates of every point of every element of the surface. Hence, if we find expressions for a and b, in terms of x, y and z, from equations (2) and (3), and substitute them for a and b in equation (6), the resulting equation will express a relation between the

co-ordinates of every point of the surface; that is, it will be the equation of the surface.

Finding the values of a and b,

$$a = \frac{x - x'}{z - z'}, \qquad b = \frac{y - y'}{z - z},$$

and substituting in (6), we get

$$f\left(x' - \frac{x - x'}{z - z'}z', \quad y' - \frac{y - y'}{z - z'}z'\right) = 0, \text{ or}$$

$$f\left(\frac{x'z - xz'}{z - z'}, \quad \frac{y'z - yz'}{z - z'}\right) = 0 \cdot \cdot \cdot \cdot (7),$$

which may be made the equation of any conic surface whatever, by attributing a suitable form to the function indicated by f, and by giving suitable values to x', y' and z'.

To find the equation of a right cone, with a circular base, assume the centre of the base at the origin of co-ordinates and denote the distance from it to the vertex, which will be on the axis of z by h, and the radius of the base by r: we shall have $x' = 0$ and $y' = 0$.

Making these substitutions in equation (7), and recollecting that the equation of a circle referred to its centre and rectangular axes, is $x^2 + y^2 = r^2$, we shall have

$$\frac{h^2 x^2}{(z - h)^2} + \frac{h^2 y^2}{(z - h)^2} = r^2, \quad \text{or}$$

$$\frac{h^2}{r^2}(x^2 + y^2) = (z - h)^2,$$

the equation of a right cone with a circular base.

If we now denote the angle which any element makes with the plane of the base by v, which gives $\frac{h}{r} = \tan v$, the equation becomes

$$(x^2 + y^2)\tan^2 v = (z - h)^2,$$

a more common form.

In this equation v and h are arbitrary constants, and by attributing suitable values to them, every possible right cone may be obtained. The value of v may be varied by varying either h or r If h, supposed positive, is increased, whilst r remains constant, the cone becomes more and more acute, until when h is infinite the cone becomes a cylinder. If h is diminished, the cone becomes more obtuse, till finally, when h becomes 0, the cone reduces the other extreme case of a plane coinciding with the plane XY. If h and r vary together so as to preserve the ratio $\frac{h}{r}$ constant, the only effect is to raise or depress the entire cone, without changing the inclination of its elements.

CON′IC-AL. [L. *conicus;* Gr. *κωνικος*. See *Cone*]. Having the shape of a cone—appertaining to a cone.

CON′ICS. A name often given to that branch of mathematics which treats of the nature and properties of the conic sections. See *Conic Sections*.

CON′JU-GĀTE. [L. *conjugatus*, *con*, with, and *jugo*, to yoke]. United according to a peculiar law, as conjugate diameter, conjugate cones, conjugate hyperbolas, &c.

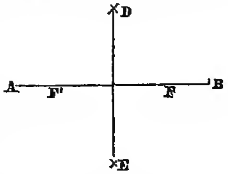
CONJUGATE AXIS, of a conic section, is that axis which is perpendicular to the transverse axis.

In the ellipse, it is limited by the curve, and its length can never exceed that of the transverse axis.

In the hyperbola, it does not cut the curve, and it may be less than, equal to, or greater than the transverse axis; these several cases arising when the curve is acute, equilateral or obtuse.

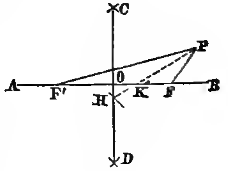
The parabola has no conjugate axis at a finite distance.

The conjugate axis of the ellipse may be readily constructed when the transverse axis and the foci are given. Let AB represent the transverse axis, and F and F′ the foci. Then, with the foci as centres, and with a radius equal to half of the transverse axis CB, describe arcs of circles cutting each other in D and E; the straight line DE joining these points, is the axis required.



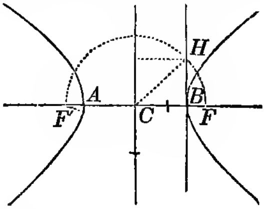
If we have the foci and one point of the curve given, the conjugate axis may be constructed as follows:

Let F and F′ be the foci, and P a point of the curve. Draw an indefinite straight line AB through F and F′, also a straight line CD perpendicular to it, through the middle point



of the line FF′. Draw also the two lines PF and PF′. With P as centre, and with a radius equal to half of PF and PF′ describe an arc cutting CD in H. Join HP, and the portion of this line PK between P and the line AB will be equal to the semi-conjugate axis. Lay this distance off from P, to C and D, and the line CD will be the axis required.

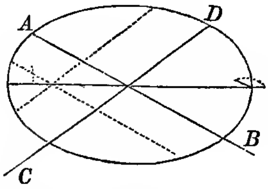
In the hyperbola, the conjugate axis may be constructed when the transverse axis and the foci are given. Let AB be the transverse axis, and F and F′ the foci; at B erect



BH perpendicular to AB, and from C, the middle point of AB, as a centre, and with CF as a radius equal to CF′, describe an arc cutting BH in H: the line BH is equal to the semi-conjugate axis.

Conjugate Cones. Two cones are conjugate, when their axes are at right angles to each other, and when they are tangent to each other along two elements which lie in the plane of their axes.

Conjugate Diameters. Two diameters of a conic section are said to be conjugate, when each is parallel to the chords of the curve which the other bisects, as AB and DD′. In the ellipse and hyperbola, there are



an infinite number of pairs of conjugate diameters, every diameter having one conjugate. The axes are the only pair of conjugate diameters, in either curve, which are at right angles to each other. In the ellipse, the angle between any two conjugate diameters can never be less than 90°: in the hyperbola it never can be greater than 90°.

The parabola has no conjugate diameters; all the diameters in that curve being parallel to each other.

If we designate by a' and b' the lengths of a pair of semi-conjugate diameters, we have for the equation of the ellipse referred to its centre and conjugate diameters,

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2;$$

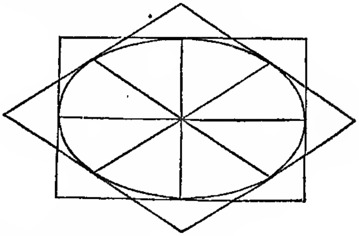
and for the hyperbola,

$$a'^2 y^2 - b'^2 x^2 = - a'^2 b'^2.$$

It will be seen from these equations and from the definition given for conjugate diameters, that the curve is so divided as to have a sort of symmetry with respect to both. This species of symmetry has been called oblique symmetry, and consists in the diameter bisecting a system of chords parallel to a given straight line. The parabola has this sort of symmetry with respect to any diameter and the tangent and its vertex.

The analytical properties of these curves, when referred to conjugate diameters, are entirely analogous to those obtained when they are referred to the axes. Any analytical expression in the former case may be derived from the corresponding one in the latter case, by simply changing a into a' and b into b', and recollecting that the new axes are oblique.

If we designate the angle which the diameter a' makes with the transverse axis by α, and the angle which the diameter b' makes with the same axis by α', we shall have the following analytical relation between a, b and a', b', in terms of α and α':

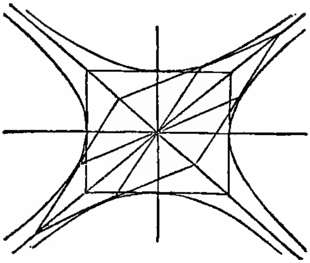


For the ellipse,

$$a^2 \tan \alpha \tan \alpha' + b^2 = 0$$
$$a' b' \sin (\alpha' - \alpha) = ab$$
$$a'^2 + b'^2 = a^2 + b^2$$

For the hyperbola,

$$a^2 \tan \alpha \tan \alpha' - b^2 = 0$$
$$a' b' \sin (\alpha' - \alpha) = ab$$
$$a'^2 - b'^2 = a^2 - b^2$$



From the first equation of each group, we are able to find the angle between any two conjugate diameters, when either α or α' is given; or when the angle between the diameters is known, we can find the angle which each makes with the transverse axis. The second equation of each group shows that the parallelogram constructed by drawing lines through the extremities of each diameter parallel to its conjugate, is always constant for the same curve.

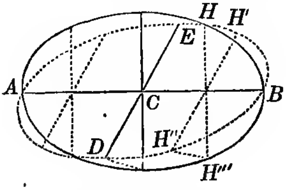
The third equation of the first group shows that the sum of the squares of any pair of conjugate diameters of the ellipse is constant; and the third equation of the second group shows that the difference of the squares of any pair of conjugate diameters of the hyperbola is constant.

In the ellipse, the only equal conjugate diameters are those which coincide in direction with the diagonals of the rectangle constructed upon the axes, except in the circle, where every diameter is equal to its conjugate.

In the hyperbola, there are no equal conjugate diameters except in the equilateral hyperbola, in which every diameter is equal to its conjugate. It appears from the foregoing discussion. that an ellipse or hyperbola may be constructed when we have given the lengths of a pair of conjugate diameters, and the angle included between them.

The ellipse may be constructed by points when a pair of conjugate diameters is given, as follows:

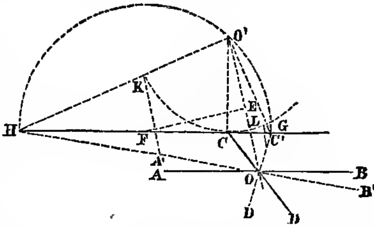
Let AB and ED be two conjugate diameters; revolve ED around the centre C, until it becomes perpendicular to AB, and then construct an ellipse on AB and ED as axes, by known methods. Take any double ordi-



nate of this curve, as HH''', and revolve it about the point in which it intersects AB, till it becomes parallel to ED; the extremities H' and H'' are points of the required ellipse. In a similar manner any number of points may be found. Having a sufficient number of points, trace a curve through them, and it will be the ellipse required.

The hyperbola may be constructed, by points, in a manner entirely similar.

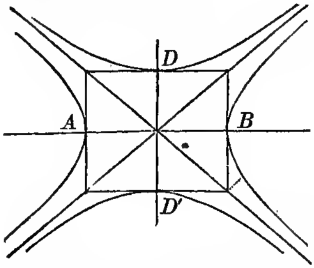
If we wish to construct the axes of an ellipse, having a pair of conjugate diameters given, we can do so as follows:



Let AB and CD be any pair of conjugate diameters. Through C draw HG parallel to AB, and at C erect a perpendicular to GH and make it equal to OB. With the extremity of this perpendicular as a centre, and with a radius equal to OB, describe an arc of a circle KCL. Draw the straight line OO', and bisect it by the perpendicular EF, cutting GH in F. With F as a centre and FO' as a radius, describe a circle, which will pass through O. Through the points G and H, in which it cuts the line GO. draw the lines GO, GO' and HO, HO'; draw LC' and KA' parallel to OO'; OC' and OA' will be the semi-axes.

Conjugate Hyperbolas. Two hyperbolas are conjugate when the conjugate axis of

the one is the transverse axis of the other, and the reverse, as AB, DD′.



If the equation of an hyperbola is

$$a^2 y^2 - b^2 x^2 = -a^2 b^2,$$

that of its conjugate is

$$a^2 x^2 - b^2 y^2 = -a^2 b^2.$$

Conjugate hyperbolas have common asymptotes, and those diameters which terminate in a given hyperbola have conjugates terminating in the conjugate hyperbola. If an hyperbola is acute its conjugate is obtuse, and the reverse; if an hyperbola is equilateral, its conjugate is also equilateral.

Conjugate Hyperboloids. Two hyperboloids are conjugate when they have the same set of axes, but do not coincide.

In this case one of the hyperboloids must have one nappe, and its conjugate two nappes. The axis which pierces the hyperboloid of two nappes will not pierce that of one nappe, whilst the two axes which do not pierce the hyperboloid of two nappes, both pierce that of one nappe. The surfaces approach each other in every direction as they recede from the centre, and become tangent to each other at an infinite distance. If one becomes a surface of revolution, the other also becomes a surface of revolution, having the same axis. This particular species of conjugate hyperboloids may be generated by revolving a pair of conjugate hyperboloids about either axis.

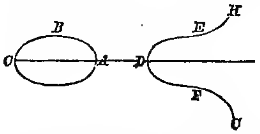
Conjugate Planes. In a surface of the second order, three planes are said to be conjugate when each bisects a system of chords of the surface parallel to the other two. In the ellipsoid and hyperboloid, there are an infinite number of systems of conjugate planes. Every central place has two conjugate planes. The properties of conjugate planes are analogous to those of conjugate diameters in the ellipse and hyperbola.

Conjugate Points of a curve are those which are expressed by the same equation, but have no consecutive points. They are sometimes called isolated points, and are considered as belonging to the curve, because their co-ordinates satisfy its equation when substituted for the variables. Conjugate points may be regarded as particular species of oval branches, which have become points, in consequence of a particular supposition made upon the arbitrary constants. We have considered the case in which the ellipse becomes a point; this is the simplest case of a conjugate point. In this case there is no other branch of the curve.

If we consider the curve whose equation is

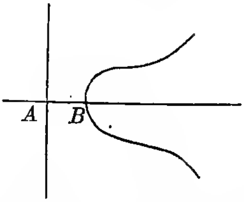
$$y = \pm\sqrt{x(x-a)(x-b)},$$

in which a and b are both positive, we shall have, when $a < b$, a curve with two branches; an oval branch and a parabolic branch, separated by an interval AD. If now we sup-



pose a to diminish, the oval branch will grow smaller, till finally, when $a = 0$, it will become a point coinciding with A, the origin of co-ordinates; this is a conjugate point. Under the supposition made, the equation of the curve becomes

$$y = \pm x\sqrt{x-b}.$$



We see that $x = 0$ and $y = 0$, satisfy the equation of the curve, but that all other values of x less than b give imaginary values for y, which shows that there is no point of the curve between A and B, AB being equal to b.

To determine whether a given curve has any isolated points, we have to ascertain whether there are any real values of x and y, which will satisfy the equation of the curve, and at the same time render the first differential co-efficient of the ordinate imaginary; if there are any such values, each pair corresponds to an isolated point.

In the case last considered, we have

$$\frac{dy}{dx} = \frac{3x^2 - 2b}{2\sqrt{x-b}};$$

$x = 0$ is the only value which will render $\frac{dy}{dx}$ imaginary, and at the same time give a real value for y; hence the origin of co-ordinates is a conjugate point, and is the only one.

CŌ′NOID. [Gr. κωνοειδης, from κωνος, a cone, and ειδος, form]. A warped surface which may be generated by a straight line moving in such a manner as to touch a straight line and curve, and continue parallel to a given plane. The straight line and curve are called directrices, the plane is called a plane directer, and the moving line is the generatrix. Any position of the generatrix is an element of the surface. If the rectilinear directrix is perpendicular to the plane director, the surface is a right conoid, and the directrix is called the line of *striction*, because the elements are compressed along this line. The line of striction being perpendicular to the plane director, is perpendicular to every element; and since it intersects them all, the shortest distance between any two will be measured on it.

If the line of striction is vertical, and if the curvilinear directrix is a helix, lying upon the surface of a cylinder whose axis coincides with the line of striction, the conoid becomes a particular species of helicoid often used in architecture, for finishing the lower surface of spiral stairways.

The term conoid has been used to designate the solid generated by revolving any one of the conic sections about its axis. If the parabola be revolved about its axis, the solid generated is called a parabolic conoid, or more properly, a paraboloid. If an ellipse or hyperbola be revolved about its axis, the solid would be called an elliptic or hyperbolic conoid.

CON-SEC′U-TĬVE. [L. *con*, with, and *sequor*, to follow]. Following another thing immediately. Thus, two points of a line are consecutive when they lie together; that is, if we suppose the line to be generated by a point moving according to a fixed law, the position which the generatrix first assumes after leaving any position, is a consecutive point.

Consecutive Elements. In a surface generated by a right line, any two consecutive positions of the generatrix are said to be *consecutive elements*.

To illustrate: let us consider the case of a right cylinder with a circular base. If a plane be passed through any element, it will, in general, cut from the surface a second element. If the plane be revolved about the first element as an axis, the second element will finally approach the first, and eventually pass into it. At this instant, the plane becomes tangent to the surface, and the two elements are said to be consecutive. In this case, they are really coincident, but for the purposes of demonstration they are regarded as separate elements, the distance between them being infinitely small. Or, if we regard the circular base as a regular polygon, having an infinite number of sides, the two elements, passing through two adjacent vertices, are consecutive. This amounts to the same thing as considering the lines coincident.

Consecutive Points. If we regard a curve as being generated by a moving point, the first position which it assumes, after leaving any given position, is said to be *consecutive* with it. If we draw a straight line intersecting a curve in two points, and then revolve it about the first point, the second will finally approach, and eventually coincide with, the first; just at the instant of coincidence, the two points are said to be *consecutive*, and the straight line is tangent to the curve.

Again, if we consider a curve as a polygon of an infinite number of sides, each being infinitely small, the vertices of two adjacent angles are called consecutive points. For all practical purposes, consecutive points are coincident points; but for purposes of demonstration, it is convenient to regard them as separate and distinct.

CON′SE-QUENCE. [L. *con*, with, and *sequor*, to follow] A conclusion deduced from an argument, or train of reasoning.

CON'SE-QUENT. [L. *consequens*, following]. The second term of a ratio, so called because its value is consequent upon a knowledge of the first term which is then called an antecedent. If we have the ratio $a : b$, which may be written $\frac{b}{a}$, the term b is the consequent, a being the antecedent.

If the value of a ratio is given, and the antecedent is known, the consequent may be found by multiplying the ratio by the antecedent; thus, if $\frac{b}{a} = r$, we have $b = ar$.

A proportion, being an expression of equality between two equal ratios, must have two consequents, viz.: the second and fourth terms. A geometrical progression being a continued proportion, each term must be a consequent of the preceding, and also an antecedent of the following term.

CON'STANT QUANTITY. [L. *constans*, fixed, determined]. A quantity whose value always remains the same in the same expression. Thus, in the equation of the circle,

$$x^2 + y^2 = R^2,$$

the quantity, R, remains the same for the same circle, and is therefore constant. It differs, however, for different circles; hence, constants may be either *absolute*, or *arbitrary*. *Absolute* constants are those whose values are absolutely the same under all circumstances; thus, the number 7 is an absolute constant; the length of the equatorial diameter of the earth is also absolutely constant. An *arbitrary* constant is one to which any reasonable value may be assigned at pleasure; thus, in the equation $x^2 + y^2 = R^2$, we may give to R any value from 0 to ∞, and thus cause the circle to have any area from 0 to ∞: here R is an arbitrary constant.

In analysis, an arbitrary constant is often introduced, and afterwards such a value is assigned to it as will cause the expression to satisfy some reasonable condition. To illustrate the use of the arbitrary constant, let us consider the case of the elimination of an unknown quantity from two given equations. Take the equations

$$ax + by + c = 0 \ldots (1),$$

and $$dx + ey + f = 0 \ldots (2).$$

If we multiply both members of (1) by k, k being entirely arbitrary, and then add the resulting equation to equation (2), member to member, there will result

$$(ka + d)x + (kb + e)y + (kc + f) = 0 \ldots (3).$$

If we make $k = -\frac{d}{a}$, (3) will become

$$\left(e - \frac{bd}{a}\right)y = \frac{dc}{a} - f,$$

in which x has been eliminated.

The judicious use of arbitrary constants is one of the most powerful instruments of analytical research.

The employment of arbitrary constants, in integral expressions, affords a beautiful illustration of their power in mathematical investigations.

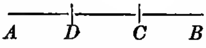
CON-STRUCT'. [L. *construo; con*, with, *struo*, to dispose or set in order]. To put the parts of a thing together in their proper order.

CON-STRUC'TION. [L. *constructio*, making, building]. The operation of constructing or of putting together according to known principles.

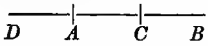
The *construction of an expression*, or *of an equation*, is the operation of finding a geometrical figure whose parts shall be respectively represented by the quantities in the equation, and in which the relation between them shall be the same as that expressed by the equation.

1. To construct the value of x in the equation $x = a + b$.

Draw an indefinite straight line AB. From A, as an origin of distances, lay off to the right a distance AC equal to a; from C lay off still to the right a distance equal to b; then will AD be equal to x, and is the distance required.



If b is negative, the last distance, equal to b, must be laid off from C to the left; AD will, as before, represent the value of x.

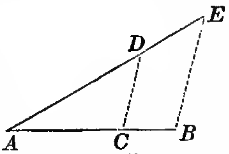


If b is numerically greater than a, and negative, the point D will fall to the left of A, and in accordance with the rule for inter-

preting negative results, x will be negative, and still equal to AD, numerically.

If there are more than two linear terms, we continue to lay off distances from the last point determined, these being measured to the right when the term is positive, and to the left when it is negative; the distance from the origin to the last point determined is equal to the value of x, being positive when the last point falls to the right of the origin, and negative when to the left.

2. To construct the value of x in the equation $x = \frac{ab}{c}$.



Draw two straight lines AE and AB, intersecting each other at A. From A on AB lay off a distance AC $= c$, also AB $= a$. From A on AE lay off AD $= b$; draw DC, and through B, draw BE parallel to CD; then is AE $= x$.

3. To construct the value of x in the equation $x = \frac{abc}{df}$. Place the expression under the form $x = \frac{ab}{d} \times \frac{e}{f}$, and construct as before the value of $\frac{ab}{d}$; call this g: then will $x = \frac{ge}{f}$; construct $\frac{ge}{f}$ in the same manner, and the line obtained will be equal to x.

4. If we have an equation of the form

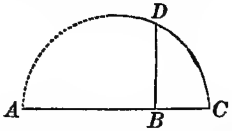
$$x = \frac{abc + dfg}{hm},$$

it can be placed under the form

$$x = \frac{abc}{hm} + \frac{dfg}{hm}.$$

Each term in the second member can be constructed as explained in (3), and the value of x finally determined as in (1).

5. To construct the value of x in the equation $x = \sqrt{ab}$, draw an indefinite straight line AC and lay off from A, AB $= a$, and



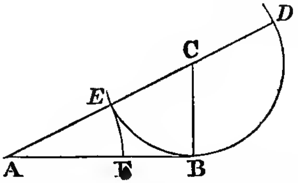
from B, BC $= b$; upon AC as a diameter, describe a semicircle, and at B erect an ordinate BD perpendicular to AC; BD will be equal to x. If we complete the circle and produce the ordinate below AC, till it intersects the circle, the prolongation will be equal to $-\sqrt{ab}$. In this case the construction gives both values of $x = \pm\sqrt{ab}$, as it should. In general, when there are two or more values of an expression, the geometrical construction ought to give them all.

6. To construct the roots of the four forms of equations of the second degree. The four forms are

1. $x^2 + 2ax = b^2$. 2 $x^2 - 2ax = b^2$.
3. $x^2 + 2ax = -b^2$. 4. $x^2 - 2ax = -b^2$.

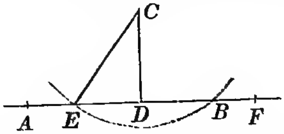
and their roots are respectively,

1. $x = -a \pm \sqrt{a^2+b^2}$. 2. $x = a \pm \sqrt{a^2+b^2}$.
3. $x = -a \pm \sqrt{a^2-b^2}$. 4. $x = a \pm \sqrt{a^2-b^2}$.



Draw AB $= b$, and at B erect BC perpendicular to AB. and equal to a; with C as a centre, and CB as a radius, describe the circle EBD; prolong AC to D. Then is $+$ AE the first, and $-$ DA the second root of the 1st form. Also $+$ AD and $-$ EA are the roots of the second form. The roots of these forms are respectively equal with contrary signs, as they should be.

Again, draw AF $= 2a$, and at its middle point D erect DC perpendicular to it, and equal to b; with C as a centre and a radius equal to a, describe a circle, cutting AF in E and B. Then are $-$ FB or $-$ EA and

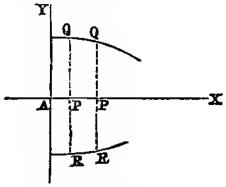


$+$ AB, the two roots of the third form. Also $+$ AE and $-$ BA are the two roots of the fourth form. The roots of the third and fourth forms are equal with contrary signs, as they should be. If $a = b$ the circle is tangent to AF in D, and the roots of each form become equal. If $a < b$ the circle does not

touch AF, and all of the roots of the third and fourth forms are imaginary.

These principles serve to show the method of proceeding in order to construct all expressions which are of the first and second degrees. There is an infinite variety of constructions which may arise, but the elementary principles here laid down are such as are most frequently applicable in the solution of determinable problems.

7. Having given the equation of any plane curve, $y = f(x)$. Draw two straight lines AX and AY at right angles. Assume any value for x and substitute it for x in the



equation of the curve, and deduce the corresponding value or values of y; lay off the distance AP equal to the assumed value of x, and at its extremity erect a perpendicular to AX, and make it equal to the deduced value of y; above AX if y is positive, below it if negative. The extremity Q or R is a point of the curve; in like manner any number of points may be constructed. Having determined a sufficient number of points, draw a curve through them, and it will be the curve required.

For more extended rules for constructions, see *Application of Geometry to Algebra*, *Construction of Curves from Equations*, *Construction of Roots of Cubic Equations*.

CONTACT. [L. *contactus*, from *contingo*, to touch]. Two curves are said to have a contact at a common point, when they have a common tangent at this point. The contact of a right line and curve is the same as simple tangency, but curved lines may have a more intimate contact. A complete discussion of the nature and order of contact can only be obtained by means of the Calculus.

The following are the analytical characteristics which distinguish the different orders of contact:

1. Two curves are said to have a contact of the first order, when they have a point in common, and the first differential co-efficients of the ordinates of the two curves, taken at this point, equal to each other. This is simple tangency.

2. They have a contact of the second order when they have a point in common, and the first and second differential co-efficients of the ordinates of the two curves, taken at the point, equal.

3. They have a contact of the third order, when in addition to the previous condition they have also the third differential co-efficients of the ordinates of the two curves, taken at the common point, equal.

4. Generally, two curves have a contact of the n^{th} order when they have a common point, and the first n successive differential co-efficients of the ordinates of the two curves, taken at the point, are respectively equal to each other.

Having given two curves, we may ascertain whether they have any contact, and if they have, we can determine the order of contact by the following method:

Combine the equations of the curves and find the values of x and y; for every pair of real values there will be a common point. Next, to ascertain whether this point is a point of contact, differentiate the equations of both curves, and find the differential co-efficients of the ordinates, and in these substitute for x and y their values corresponding to the common point; if the results are equal the curves have a contact of the *first* order at least: differentiate the equations again and find the second differential co-efficients of the ordinates and substitute in them the values of x and y, already found; if the results are again equal, the curves have a contact of the second order at least. Continue this operation of differentiation and substitution until two differential co-efficients of the ordinate, taken at the common point, are found, which are not equal; then the number of successive differential co-efficients taken at the common point, which are respectively equal, will denote the order of contact.

Having given a curve by its equation and a second curve in kind, that is, having given the form of its equation, it is possible to assign to this last curve an order of contact with the given curve at any assumed point of it, which will be denoted by the number

of arbitrary constants which enter its equation, less 1. No higher order of contact can be *assigned*, though it may happen that the conditions which make the two curves have the assigned order of contact, may cause them to have a higher order of contact. The method is as follows :

Assume the abscissa of any point of the given curve ; substitute it for x in the equation of the curve, and deduce the corresponding value of y. The assumed and deduced values will be the co-ordinates of the assumed point. Substitute these values for x and y in the equation of the curve given in kind, and the resulting equation will be the equation of condition that the assumed point shall be common to the two curves. Differentiate the equations of both curves, and deduce the expressions for the first differential co-efficients of the ordinates ; substitute in these for x and y, the co-ordinates of the common point, and place the results equal to each other ; the resulting equation, with the preceding, will be the equations of condition that the curves shall have a contact of the first order.

Differentiate the equations again, and find the second differential co-efficients of the ordinates of the curves ; substitute as before, and place the results equal ; the equation which results will express the additional condition, that the two curves shall have a contact of the second order. Continue this operation of differentiating, substituting and equating, till as many equations of condition are found as there are constants in the equation of the second curve ; then combine these equations, find the values of these constants, and substitute them for the constants in the equation of the second curve ; the resulting equation will be that of the curve which has the required order of contact with the given curve at the assumed point. Such a curve is said to be osculatory to the given curve at the given point.

Since the most general equation of the circle contains but three arbitrary constants, it follows that the circle cannot be made to have a higher order of contact than the second, with any given curve at a given point. It may be observed, however, that if the given point is one at which the normal divides the given curve symmetrically, the conditions which make the circle osculatory, will give it a contact of the third order. See *Osculatory* and *Osculatrix*.

CON′TENTS. [L. *contentus*, *con*, and *teneo*, to hold]. The contents of a plane figure is the same as its area. Numerically, it is the number of times which the figure contains some given area assumed as the unit of surface. For the contents of some of the principal plane figures, see *Mensuration*.

Contents of a Solid, is the same as it volume. Numerically, it is the number of times which the solid will contain some particular solid assumed, as the unit of volume. See *Mensuration*, and *Volume*.

CON-TIG′U-OUS. [L. *contiguus*, *con*, with, and *tango*, to touch]. Contiguous angles, are those which have a common vertex and one common side, but the other sides not in the same straight line. The latter condition distinguishes them from adjacent angles. See *Angle*.

CON-TIN′UED FRACTIONS. [*continuo*, *con*, with, and *teneo*, to hold]. A continued fraction, is a fraction whose numerator is 1, and whose denominator is a whole number plus a fraction whose numerator is 1 and whose denominator is a whole number plus a fraction, and so on :

Thus, the fraction written in the margin, is a continued fraction. The separate fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c., which make up a continued fraction, are called *integral fractions*. The number of integral fractions, in a continued fraction, may be finite or it may be infinite ; in the former case the true value of the fraction may be found, in the latter we can only approximate to the true value.

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d, \&c.}}}}$$

If we stop at any integral fraction and neglect all which follow, the resulting fraction is called an *approximating fraction*. If we stop at the first integral fraction and neglect all which follow, the result is an approximating fraction of the first order ; if we stop at the second, the result is one of the second order ; and generally, if we stop at the n^{th} integral fraction, and neglect all

that follow, the result obtained is an approximating fraction of the n^{th} order.

Every approximating fraction of an odd order is greater than the true value of the continued fraction, whilst every approximating fraction of an even order is less than the true value. Hence, the true value of a continued fraction always lies between any two consecutive approximating fractions. The difference between two consecutive approximating fractions, is equal to ± 1 divided by the product of their denominators, and since the denominator of each approximating fraction is greater than that of the preceding one, this difference is always less than ± 1, divided by the square of the denominator of the first one. Whence, we see, that if we take any approximating fraction as the true value of the continued fraction, the error committed will be less than 1, divided by the square of its denominator.

The value of any approximating fraction may be found from the two which immediately precede, by the following rule.

The numerator of the n^{th} approximating fraction is formed by multiplying the numerator of the $(n-1)^{\text{th}}$ approximating fraction by the denominator of the n^{th} integral fraction and adding to the product the numerator of the $(n-2)^{\text{th}}$ approximating fraction; and the denominator is formed by the same law, from the denominators of the two preceding approximating fractions.

Continued fractions arise in various ways, and are of use in solving certain kinds of problems, amongst which may be mentioned, the solution of problems in indeterminate analysis by means of whole numbers; they are also useful in getting approximate values for fractions whose terms are expressed in very large numbers; and in many other cases.

To convert an irreducible vulgar fraction into a continued fraction: divide the greater term of the fraction by the less, and the last divisor by the first remainder, and so on, till a remainder is found equal to 0; the several quotients will be the denominators of the successive integral fractions: thus, to reduce $\frac{65}{149}$ to a continued fraction, the operation is thus performed:

```
65)149(2
   130
    19)65(3
       57
        8)19(2
          16
           3)8(2
             6
             2)3(1
               2
               1)2(2
                 2
                 0
```

whence $\frac{65}{149} = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}}}$

And the several approximating fractions are

$\frac{1}{2}$, $\frac{3}{7}$, $\frac{7}{16}$, $\frac{17}{39}$, $\frac{24}{55}$ and $\frac{65}{149}$,

any one of which may be taken as an approximate value of the given fraction.

If we convert the ratio of the diameter to the circumference of a circle, $\frac{31415926535}{10000000000}$ into a continued fraction, and then find the successive approximating fractions, we shall get for them,

$\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{103993}{33102}$, &c.

These approximate values of π are of frequent application.

To apply the principles of continued fractions to the solution of indeterminate equations of the first degree, in whole numbers, let us take the equation $ax + by = c$, in which a, b and c are whole numbers, and a and b prime with respect to each other. Convert the fraction $\frac{a}{b}$ into a continued one, and find the successive approximating fractions, the last of which will be $\frac{a}{b}$. Designate the one preceding the last by $\frac{a'}{b'}$. If we subtract this from the preceding one, the numerator of the difference will be equal to ± 1, whence

$$ab' - ba' = \pm 1;$$

and multiplying both members by c, we obtain

$$a \times (\pm b'c) + b \times (\mp a'c) = c,$$

and comparing this with the given equation, we see that $x = \pm b'c$, and $y = \mp a'c$, will satisfy the given equation; and, furthermore, that every value given by the formulas

$$x = \pm b'c - bt, \quad \text{and} \quad y = \mp a'c + at,$$

will also satisfy it in whole numbers, t being any whole number whatever. The upper sign is to be used when $\frac{a}{b}$ is of an odd order,

and the lower one when it is of an even order.

Let us take, as an example, the equation

$$65x + 149y = 8.$$

Then will

$$\frac{a}{b} = \frac{65}{149}, \quad \text{and} \quad \frac{a'}{b'} = \frac{24}{55}.$$

The smallest values of x and y, in whole numbers which will satisfy the given equation, are

$$x = b'c = -440, \quad \text{and} \quad y = +192$$

The other values, in whole numbers, are

$$x = -589, \qquad y = +257.$$
$$x = -738, \qquad y = +322.$$
$$x = -887, \text{ \&c.}, \qquad y = +387, \text{ \&c.}$$

Continued fractions are also employed in solving exponential equations (see *Exponential Equations*); also in extracting roots by approximation, and in the solution of numerical equations of the higher degrees.

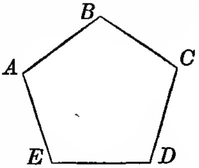
Continued Product, of any number of quantities, is the result obtained by multiplying the first by the second, that product by the third, that result by the fourth, and so on to the last.

Continued Proportion. Any number of quantities are said to be in continued proportion, when the ratio of any term to the succeeding one is constant. In this case, any term is a mean proportional between the preceding and succeeding ones. The terms of a geometrical progression are in continued proportion.

CON-TIN′U-OUS FUNCTION. [L. *continuus*, uninterrupted]. Is one in which the difference between any two consecutive states is less than any assignable quantity. In such a function, if we suppose the independent variable to pass through every possible state, from one given value to another, the function will pass, by insensible degrees, through every state from the first to the last. A function which follows this law, is said to be subject to the *law of continuity*.

Every function of a single variable which involves only positive and entire exponents, is subject to the law of continuity. Upon this principle is based a great portion of the general theory of equations.

CON-TöUR′, [Fr. *contour*, outline], of a plane figure, in Plane Geometry, is the same as its perimeter or bounding line. In the figure AD, the broken line ABCDE is the contour.



Apparent Contour of a body in Perspective, is the line of contact of the body and an enveloping visual cone. See *Perspective.* If a visual plane be passed tangent to the body, the point of contact is a point of the apparent contour. The perspective of the line of apparent contour is the bounding line or contour line of the perspective.

Contour of Ground, in surveying, is a term used in speaking of the surface of any part of the earth with respect to its undulations and accidents. See *Topography.*

Line of Contour, in topographical surveying, is a line in which a horizontal plane intersects the surface of a portion of ground to be surveyed.

CON-TRAC′TION. [L. *contractio*, *con*, with, and *traho*, to draw]. The process of shortening any operation. There are many cases in which operations may be greatly contracted without at all impairing the accuracy of the results. This is particularly true in the operation of multiplying and dividing decimals, when there are a great many decimal places in the numbers to be operated upon, and only a limited number is required in the result. Contractions are also used in the Square Root.

Contraction in Multiplication. Write down the multiplicand, and under it write the multiplier; but instead of writing the figures in their proper order, write the units' place under the last decimal place of the multiplicand, which is to be retained, and dispose of the remaining figures in an inverted or contrary order to that in which they are usually placed; then, in multiplying, reject all the figures at the right of that by which you are multiplying, and arrange the products so that the right-hand figure of each shall fall in the same vertical column; observing to add to the first figure on the right of every line the number that would have been carried, had you not neglected the places on the right, and also carrying 1 when this product exceeds 4,

2 when it exceeds 14, 3 when it exceeds 24, and so on. Then take the sum, and point off the required number of decimal places, and the result will be the product required. The reasons for carrying, as indicated, are obvious.

1. Multiply 34.17165 by 78.3333, retaining only five places of decimals in the product.

```
 34.171650   multiplicand.
    3333.87  multiplier inverted.
 239201550
  27337320
   1025150
    102515
     10251
      1025
2676.77811   product.
```

It should be observed that the last figure of the product may not be correct; it is therefore best to retain, through the operation, one more decimal place than is needed and then to reject it after the operation is completed.

As a second example, multiply .546768 by .671686, retaining only 7 places of decimals in the product.

```
 5467680   multiplicand.
 686176.0  multiplier inverted.
 3280608
  382738
    5468
    3281
     437
      33
.3672565   product.
```

which is certainly true to 6 places of decimals.

Contraction in Division. Take as many of the left hand figures of the divisor as shall be equal to one more than the number of integral and decimal places to be retained in the quotient; commence the division as usual; consider each remainder as a new dividend, and in dividing it, leave off one figure from the right of the divisor, observing to carry for the increase of the figure cut off, as directed in multiplication. When there are not so many places of figures in the divisor as are required in the quotient, begin the operation as usual, and continue it till the number of figures in the divisor exceeds by 1 the number remaining to be found in the quotient, then begin the contraction.

1. Divide 2508.928 by 92.41035, carrying the quotient to 4 places of decimals. This requires 6 places in all in the quotient.

```
9241035)2508.9280(27·1498
         6607210
          138485
           46075
            9111
             794
              55
```

The quotient is certainly correct to 3 decimal places.

3. Contraction in Square Root. Proceed as in the ordinary method until half or one more than half of the required number of places of figures in the root are found; then for the remaining places divide the last remainder by the corresponding divisor, by the preceding rule.

Example. Extract the square root of 14876.2357 to nine places of figures.

```
        14876.2357 | 121.96
         1
    22 | 48
         44
   241 | 476
         241
  2429 | 23523
         21861
 24386 | 166257
         146316
 24392 | 1994100 | 8175
           4274
           1835
            128
              6
```

Whence the required root is 121.968175.

CON'TRA HARMONICAL PROPORTION. Three terms or quantities are said to be in contra harmonical proportion, when the difference between the first and second is to the difference between the second and third, as the third is to the first.

CON-VERGE'. [L. *convergo*. *con*, with, and *vergo*, to incline]. To tend or incline towards the same point. Two straight lines converge when they will meet if sufficiently produced.

CON-VERG'ING SERIES. A series in which the greater the number of terms taken

the nearer will their sum approximate to a fixed value, which value is the true *sum of the series*.

The object of developing a function into a series is, generally, to obtain approximate numerical values for the function, corresponding to particular values of the variable which enters it. This can only be attained when the series is converging; it is, therefore, important to be able to ascertain, in any given case, whether a series is converging. No general rule can be laid down, but the following are some of the cases in which the series will be converging:

1. When the terms of a series are alternately plus and minus, each term being numerically less than the preceding one, the series is converging. The error committed in taking the sum of any finite number of terms for the true sum of the series, is numerically less than the succeeding term.

2. Every decreasing geometrical progression is a converging series. The error committed by taking the sum of n terms for the true sum of the series, is equal to the first term multiplied by the n^{th} power of the ratio, and divided by 1 minus the ratio. If the ratio is small and n is large, this error will be quite inappreciable.

3. A series, all of whose terms have the same sign, and in which the ratio of each term to the succeeding one goes on diminishing, finally becomes less than one, and then goes on diminishing continually, is converging.

It is plain that after the term whose ratio to the following one is less than 1, the sum of any number of terms will be less than the sum of the same number of terms of a geometrical progression having this term for the first term and this ratio for a ratio, and we have already shown that the sum of such a progression has a finite limit. The error committed in taking the sum of any number of terms for the true sum of the series (provided that number of terms includes the one at which the ratio becomes less than 1), is always less than the sum of a decreasing geometrical progression whose first term is the term next following the last one taken, and ratio the ratio of this term to the following one.

4. A series, all of whose terms have the same sign, each being smaller than the preceding, will be converging when r, the ratio of any term to the succeeding one, goes on increasing, provided this ratio is always less than 1. The error committed in taking the sum of any number of terms for the true sum of the series, will be less than the sum of a geometrical progression whose first term is the term next following that which immediately follows the last one taken, and whose ratio is the greatest value of the ratio of any term to the succeeding one.

5. A series will not be converging when the ratio of any term to the succeeding one is always greater than 1, either constant or variable; that is, no series whose terms go on continually increasing, can be a converging one. If the ratio is 1, the series cannot converge, and in the case mentioned in article 4, it is to be observed that the series will not be converging when the varying ratio has 1 for its limit. Thus, the series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \text{ \&c.,}$$

is not a converging one, because the successive ratios of the consecutive terms are

$$\frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{4}, \ \frac{4}{5}, \ . \ . \ . \ . \quad \text{\&c.,}$$

whose limit is 1. See *Series*.

CON'VERSE. [L. *con* and *versor*, to be turned]. One proposition is the converse of another when the conclusion in the first is employed as a supposition in the second, and the supposition in the first is the conclusion in the second. Thus, the proposition in geometry that "If two sides of a plane triangle are equal, the angles opposite to them are equal," is the converse of the proposition "If two angles of a plane triangle are equal, the sides opposite them are equal." Both propositions require separate proof; for it does not follow because a proposition is true, that its converse is also true. For example: it does not follow because the axes of an ellipse are conjugate diameters, that a pair of conjugate diameters will necessarily be axes of the ellipse.

CON'VEX. [L. *convexus*, arched]. The opposite of concave. Protuberant outwards, as the outer surface of a sphere.

If we regard a hollow sphere or globe, its

outer surface is convex, whilst its inner surface is concave.

By means of the differential calculus, we are able to determine whether a given line has its convexity or concavity, at a particular point, turned towards, or from the axis of X.

Differentiate the equation of the curve twice, and find an expression for the second differential co-efficient of the ordinate. Substitute in this for x and y the co-ordinates of the given point; if the sign of the result is the same as that of the ordinate of the given point, the curve is convex at that point towards the axis of X; if they have contrary signs, it is concave towards the axis of X.

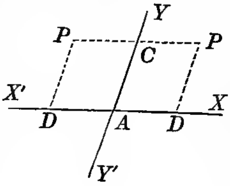
The convex surface of a cone or cylinder is that surface which is generated by the right lined generatrix. See *Cone*, *Cylinder*.

CO-OR'DI-NATES. [L. *con*, with, and *ordinatus*, from, *ordino*, to regulate]. Elements of reference, by means of which the relative positions of points may be determined, either with respect to each other, or to certain fixed objects of reference. These elements, the objects to which reference is made, and the method of making the reference, constitute what is called a *system of co-ordinates*. There may be any number of systems, but two only are of sufficient importance to require notice in this place, viz.: the *rectilineal system* and the *polar system*.

I. The rectilineal system may be employed for the purpose of showing the relative positions of points, all of which lie in the same plane, or of points which are situated in any manner in space.

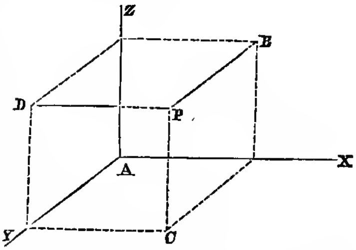
Rectilineal System in a Plane. In this system the relative positions of points are determined by referring them to two straight lines, intersecting each other, by means of their distances from these lines measured on lines parallel to them.

The lines to which points are referred, are called *co-ordinate axes*, their point of intersection is the *origin of co-ordinates*, and the lines drawn through any point parallel to them are the *rectilineal co-ordinates* of the point.



In the annexed figure, YY' and XX' are the *axes*, A the origin, and CP, DP the co-ordinates of the point P; YY' is the axis of ordinates, XX' that of abscissas, DP is the ordinate of P, and CP is its abscissa. The co-ordinates are always estimated from the axis towards the point; so that if we agree to consider distances, estimated upwards from XX' as positive, those estimated downwards must be regarded as negative. If we agree to consider distances estimated to the right of YY' as positive, those estimated to the left must be regarded as negative. If the axes of the rectilineal system are perpendicular to each other, the system is called *rectangular*; otherwise it is *oblique*.

Rectilineal System in Space. In this system, the relative positions of points are determined by referring them to three planes which intersect each other. These planes are called *co-ordinate planes*, their intersections, taken two and two, *co-ordinate axes*; and their common point of intersection the *origin of co-ordinates*.



In the annexed figure, the planes YAX, YAZ, and ZAX, are the co-ordinate planes, the lines AX, AY and AZ, are the co-ordinate axes, and the point A is the origin. The distances BP, CP and DP, of the point P from the co-ordinate planes, measured on lines parallel to the co-ordinate axes, are the co-ordinates of the point P. If the co-ordinate planes are perpendicular to each other, the system is said to be *rectangular*, if not it is *oblique*.

It has been agreed to consider all distances estimated upwards from the planes YAX *positive*; hence, all distances downward must be *negative*. All distances estimated to the right from the plane YAZ, are regarded as *positive*; hence, all distances to the left must be considered as *negative*. All distances to the point from the plane YAZ, are considered

as *positive;* hence, all distances backward from that plane, must be taken as *negative.*

This convention, in regard to signs of the co-ordinates of points in the two rectilinear systems, enables us to express the relation between the co-ordinates of all the points of a magnitude in the same general expression.

TRANSFORMATION OF CO-ORDINATES. It is often convenient to change the reference of points from one rectilinear system of co-ordinates to another; this is effected by means of certain formulas called *formulas for passing* from one system to another.

In a plane system, let x and y denote the co-ordinates of any point referred to the primitive system, and x' and y' the co-ordinates of the same point referred to the new system. Designate also the co-ordinates of the new origin referred to the primitive system by a and b, the angle included between the new axis of X, and the primitive axis of X by a, and that between the new axis of Y, and the primitive axis of X by a', we shall have the following formulas:

For passing from any system to a parallel system,

$$x = a + x' \ldots (1), \quad y = b + y' \ldots (2).$$

For passing from a rectangular to an oblique system,

$$y = b + x' \sin a + y' \sin a' \ldots (3),$$
$$x = a + x' \cos a + y' \cos a' \ldots (4),$$

For passing from an oblique to a rectangular system,

$$x = a + \frac{x' \sin a' - y' \cos a'}{\sin (a' - a)} \cdots (5),$$

$$y = b + \frac{y' \cos a - x' \sin a}{\sin (a' - a)} \cdots (6).$$

For passing from a rectangular system to another rectangular system,

$$x = a + x' \cos a - y' \sin a \ldots (7).$$
$$y = b + x' \sin a + y' \cos a \ldots (8).$$

To use these formulas: Having the equation of a magnitude referred to one system, and wishing its equation referred to another system, substitute for x and y their values taken from the formulas for passing from the given to the required system; the result wlll be the required equation.

To pass from a rectangular system in space to an oblique system, we have these formulas:

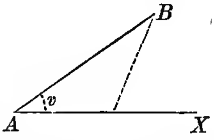
$$\left.\begin{array}{l} x=a+x'\cos X+y'\cos X'+z'\cos X'' \\ y=b+x'\cos Y+y'\cos Y'+z'\cos Y'' \\ z=c+x'\cos Z+y'\cos Z'+z'\cos Z'' \end{array}\right\} \ldots (9),$$

in which x, y, and z, are the co-ordinates of any point referred to the primitive system; x', y', z', the co-ordinates of the same point referred to the new system; X, X', X'', the angles which the new axes make respectively with the primitive axis of X; Y, Y', Y'', the angles which they make with the primitive axis of Y; Z, Z', Z'', the angles which they make with the primitive axis of Z; and a, b, c, the co-ordinates of the new origin referred to the primitive system. Their use is the same as that indicated in discussing the preceding formulas.

POLAR SYSTEM OF CO-ORDINATES. The polar system may be employed to determine the relative positions of points in a plane, or of points situated in any manner in space.

POLAR SYSTEM IN A PLANE. In this system points are referred to a fixed line of the plane, and a fixed point of the line, by means of an angle and a distance.

Let AX be the fixed line, and A the fixed point, and let B be any point in the plane. Draw BA. AX is called the *initial line*, AB, designated by r, the *radius vector*, the angle BAX, designated by v, the *variable angle*, and A the *pole:* r and v are polar co-ordinates. If, now, we suppose r to have every possible value from 0 to ∞, and v to have every possible value from 0 to 360°, the point B will, in succession, coincide with every point in the plane.



The formulas for passing from a rectangular system to a polar system, are

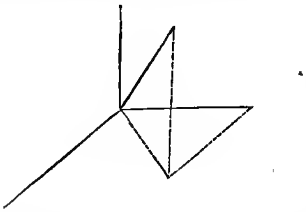
$$x = a + r \cos v, \text{ and } y = b + r \sin v;$$

their use is the same as already indicated.

The Polar System in space.

In this system, points are referred to a fixed plane, a fixed straight line of that plane, and a fixed point of that line, by means of the distance of the points from the fixed point or pole, the angle which this distance or *radias vector* makes with the fixed plane, and the angle which the projection of the ra-

dius vector on the plane makes with the fixed line. Designating the radius vector by r, the first angle by u, and the second by v, we



have, for passing from a system of rectangular co-ordinates in space to a polar system in space, the following formulas:

$$x = r\cos v \cos u,$$
$$y = r\sin v\cos u, \quad \text{and}$$
$$z = r\sin u.$$

Co-ordinates Trigonometrical. See *Trigonometrical Co-ordinates*.

COR'OL-LA-RY. L. *corollarium*, a coronet, from *corolla*, a crown]. An obvious consequence of one or more propositions. Thus, from the proposition, "If two sides of a plane triangle are equal, the angles opposite them are equal," the corollary may at once be deduced, that "If the three sides are equal, the three angles are also equal."

CO-SĒ'CANT. The secant of the complement of an angle. See *Trigonometry*.

CO'SĪNE. The sine of the complement of an angle. See *Trigonometry*.

The following formulas show the analytical equivalents of the cosine of an arc.

$$\operatorname{Cos} a = \frac{\sin a}{\tan a} = \sin a \cot a = \sqrt{1-\sin^2 a}$$
$$= \frac{1}{\sqrt{1+\tan^2 a}} \cdots\cdots (1).$$

$$\operatorname{Cos} a = \frac{\cot a}{\sqrt{1+\cot^2 a}} = \cos^2\tfrac{1}{2}a - \sin^2\tfrac{1}{2}a$$
$$= 1 - 2\sin^2\tfrac{1}{2}a \cdots\cdots (2).$$

$$\operatorname{Cos} a = 2\cos^2\tfrac{1}{2}a - 1 = \sqrt{\frac{1+\cos^2 a}{2}}$$
$$= \frac{1-\tan^2\frac{1}{2}a}{1+\tan^2\frac{1}{2}a} \cdots\cdots\cdots\cdots (3).$$

$$\operatorname{Cos} a = \frac{\cot\frac{1}{2}a - \tan\frac{1}{2}a}{\cot\frac{1}{2}a + \tan\frac{1}{2}a}$$
$$= \frac{1}{1+\tan a\tan\frac{1}{2}a} \quad (4).$$

$$\operatorname{Cos} a = \frac{2}{\tan(45^\circ + \frac{1}{2}a) + \cot(45^\circ + \frac{1}{2}a)} \quad (5)$$

$$\operatorname{Cos} a = 2\cos(45^\circ + \tfrac{1}{2}a)\cos(45^\circ - \tfrac{1}{2}a)$$
$$= \cos(60^\circ + a) + \cos(60^\circ - a) \quad (6).$$

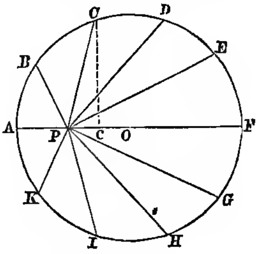
COS'MO-LABE. [Gr. κοσμος, world, and λαμβανω, to take]. An instrument resembling the astrolabe, formerly used for measuring the angles between heavenly bodies. It was also called a *pantacosm*.

COS-MOM'E-TRY. [Gr. κοσμος, world, and μετρον, a measure]. The art of measuring the world or sphere in terms of degrees.

CO-TAN'GENT. The tangent of the complement of an angle. See *Trigonometry*.

CO-TES'IAN THEOREM. A theorem first demonstrated by Cotes, and of great use in the integration of certain differentials. It is also sometimes employed in other branches of analysis. It may be enunciated as follows:

In order to find the factors of the binomials $a^n + x^n$ and $a^n \sim x^n$, when n is a whole number; with O as a centre, and with a radius equal to a, describe a circle, and suppose its circumference to be divided into as many equal parts as there are units in $2n$, at the points A, B, C, &c. Then, on the radius AD, produced if necessary, take OP equal to x, and from the point P draw straight lines to



each point of division. Then, if we take the factors alternately, we shall have

$$PB \times PD \times PF \times \cdots\cdots = a^n + x^n,$$

and also

$$PA \times PC \times PE \times \cdots\cdots = a^n \sim x^n;$$

that is, $a^n - x^n$ when P is within the circle, and $x^n - a^n$ when P is without the circle. For example, let $n = 5$; divide the circumference into ten equal parts, as in the figure, we shall then have the following relations:

$OA^5 + OP^5 = BP \times DP \times FP \times HP \times KP$ and $OA^5 - OP^5 = AP \times CP \times EP \times GP \times IP$.

The values of these several factors may be computed trigonometrically in terms of x, because we know the distance AO, and the values of the several arcs AB, BC, &c.

For instance, since tho arc AC is 72°, we shall have, by letting fall the perpendicular Cc upon OA,

$PC^2 = Cc^2 + x^2 - 2x\cos 72° + \cos^2 72°$; but

$Cc^2 = \sin^2 72°$; whence, by reduction,

$PC^2 = a^2 - 2x\cos 72° + x^2$; in like manner

$$PE^2 = a^2 + 2x\cos 144° + x^2,$$

and so on for the other factors.

Generally, if we make $a = 1$, the general factors of $1 - x^n$, or of $x^n - 1$, will be given by the formula

$$x^2 - 2\cos\frac{2k\pi}{n}x + 1 = 0,$$

in which k is a whole number, and prime with respect to n.

COUN'TER-REVOLUTION. A revolution opposed to a former one, and restoring things to their former state. In Descriptive Geometry, we often revolve a plane or line for the purpose of making a particular construction, after which we return to the normal state of affairs, by making a counter-revolution. See *Revolution*.

COUPLE. [L. *copula*, a tie, band]. Two things of a kind taken together.

CŌURSE. [L. *cursus*, from *curro*, to run]. A direction in which motion is performed. In Navigation, the course of a ship is the angle which the track of the vessel makes with the meridian; it is sometimes reckoned in degrees, sometimes in points or quarter points.

In Surveying, a *course* is any line measured upon the ground, usually from one compass station to the next. In speaking of a course, we usually understand the length of the course expressed in chains and links, or sometimes in feet. The angle which it makes with a magnetic meridian, is the *bearing* of the course

CO-VERS'ED SINE. The versed sine of the complement of an arc or angle.

CROSS. [L. *crux*, a cross]. An instrument used in surveying, and usually called the surveyor's cross. It is employed for the purpose of laying off offsets perpendicular to the main course. It consists essentially of two pairs of sights fixed at right angles to each other, so that when one pair is directed along a course, the other will point out a line at right angles to it. The method of using this instrument requires no explanation.

Two lines are said to *cross* each other in space when they are so situated, that if two straight lines be drawn parallel to them respectively, through any point, these two lines will not coincide. The lines in space are considered as making the same angle with each other as is made by their parallels.

CROSS MULTIPLICATION. See *Duodecimals*.

CU'BA-TURE. [From *cube*]. The operation of finding an expression for the volume of an indefinite portion of a solid. If the solid is one of revolution about the axis of X, the formula for the volume of any indefinite portion, that is, of a portion included between any two planes perpendicular to the axis is,

$$v = \int \pi y^2 dx,$$

in which v denotes the *volume*, y and x being the co-ordinates of every point of the meridian curve.

To find an expression for the volume of an indefinite portion of a given solid of revolution, solve the equation of the meridian curve; find the value of y in terms of x, and substitute it for y in the integral formula, and perform the integration indicated.

Then, to get an expression for a definite portion, take the integral between the limits corresponding to the limiting planes. In the paraboloid of revolution, we have, from the equation of the meridian curve,

$$y^2 = 2px, \text{ whence,}$$

$$v = \int \pi 2px dx = \pi p x^2 + C;$$

and, integrating between the limits $x = 0$ and $x = a$, we have

$$v'' = \pi p a^2.$$

If we denote the ordinate corresponding to $x = a$ by b, we have $b^2 = 2pa$; hence,

$$v'' = \tfrac{1}{2}a \times \pi b^2,$$

or, the volume of a paraboloid is one half that of the circumscribing cylinder.

CUBE or HEXAHEDRON. [Gr. κυβος; L. *cubus*, a cube]. In Geometry, a regular polyhedron bounded by six equal squares. The cube is selected as the unit of measure for all volumes, and for this purpose, that cube is employed whose edges are each equal to the linear unit. The volume of any cube is numerically equal to the product obtained by taking one of its edges three times as a factor.

Cube of a Number or Quantity, in Algebra, is the product obtained by taking the number or quantity three times as a factor: thus, the cube of 3, is $3 \times 3 \times 3$, or 27; the cube of a, is $a \times a \times a$, or it may be written a^3.

The cubes of numbers possess some remarkable properties, the principle ones being as follows:

1. All cubes of numbers are of the form $4n$, or $4n \pm 1$, in which n is a whole number.
2. All cubes of numbers are of one of the forms $9n$, or $9n \pm 1$, in which n is a whole number.
3. If any cube of a number be divided by 6, the remainder will be equal to the remainder obtained after dividing the number itself by 6; that is, the difference between the cube of any number and the number itself, is divisible by 6, or $a^3 - a = 6n$.
4. Neither the sum nor difference of two cubes can be the cube of a number.
5. The sum of any number of consecutive cubes is a square, whose square root is equal to the sum of the cube roots of all the cubes: thus,

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$
$$= (1 + 2 + 3 + 4 + 5)^2.$$

6. The terms of the third order of differences of a series of cubes are all equal to each other, each being 6: thus,

cubes	1,	8,	27,	64,	125,	216,	343,	512,	
1st or. dif.	7,	19,	37,	61,	91,	127,	169,		
2d or. diff's.	12,	18,	24,	30,	36,	42,		&c.	
3d or. diff's.	6,	6,	6,	6,	6,		&c.		

CUBE ROOT. The cube root of a quantity, is a quantity which being taken three times as a factor, will produce the given quantity: thus, 3 is the cube root of 27, because $3 \times 3 \times 3 = 27$.

Any number which can be resolved into three equal factors is a perfect cube, and its cube root may be found exactly. All other numbers are imperfect cubes, and their cube roots can only be found by approximation.

To find the cube root of a whole number.

Separate the number into periods of three figures each, beginning at the right hand; the left hand period will often contain less than three figures. Find the greatest perfect cube in the left hand period and place its cube root on the right, after the manner of a quotient in division. Subtract this cube from the left hand period, and to the remainder bring down the first figure of the next period and call this number the dividend.

Take three times the square of the root just found for a divisor, and see how many times it is contained in the dividend, and place the quotient for a second figure of the root required. Cube the number thus found, and if the result is less than the first two periods, the last figure is a figure of the root; if it is greater than the first two periods, it must be diminished successively by 1, till the cube of the root found is less than the first two periods; having found such a cube, subtract it from the first two periods, and bring down the first figure of the third period for a new dividend. Take three times the square of the root found, for a new divisor, and proceed as before, until all of the periods have been employed.

If the remainder is 0, the number is a *perfect cube*, and the root found is *exact*. If the remainder is not 0, the number is not a perfect cube, and the root found is true to within less than 1.

To find the cube root of a whole number to within less than a fractional unit $\frac{1}{n}$: *Multiply the number by* n^3, *and extract the cube root of the product to within less than* 1; *divide this result by* n, *and the quotient will be the root required.*

To extract the cube root of a vulgar fraction to within less than its fractional unit: *Multiply the numerator by the square of the denominator, and extract the cube root of the product to within less than* 1; *divide this result by the denominator, and the quotient will be the root required.*

To extract the cube root of a whole number, vulgar fraction, decimal, or mixed decimal, to any number of decimal places: *Place*

the given number under a decimal form, so that the number of decimal places shall be equal to three times the number required in the root; extract the cube root of this result to within less than 1, *and point off the required number of decimal places; the final result will be the root required.* It is to be remarked, in pointing off a mixed decimal into periods, that we begin at the decimal part and point off in both directions, the entire part to the left and the decimal part to the right; by annexing 0's, we can form as many periods of decimals as may be required.

To extract the cube root of a monomial: *Extract the cube root of the co-efficient for a new co-efficient; write after this each letter which enters the given expression, with an exponent equal to one-third of its exponent in that expression; the result is the root required.* If the co-efficient is not a perfect cube, or if any letter has an exponent which is not exactly divisible by 3, the expression is not a perfect cube.

To extract the cube root of a polynomial: Arrange the polynomial with reference to a particular letter, and extract the cube root of the first term; this will be the first term of the root required; divide the second term by three times the square of the term already found, and the quotient will be the second term of the root; cube the part of the root found, and subtract the result from the given polynomial, and divide the first term of the remainder by three times the square of the first term of the root; the quotient will be the third term of the root; cube the part of the root already found, and proceed as before, till a remainder is found equal to 0; the root found is that required. If no remainder is found equal to 0, or if any remainder is found whose first term is not exactly divisible by three times the square of the first term, the polynomial is not a perfect cube. We can often see, by inspection, that a polynomial is not a perfect cube; if any term of the polynomial which contains the highest or lowest power of any letter is not a perfect cube, the polynomial cannot be a perfect cube.

Besides the methods above explained for finding the cube roots of numbers, we may employ the binomial formula, which may be placed under the form

$$\sqrt[n]{x+a} = \sqrt[n]{x}\left(1 + \frac{1}{n}\cdot\frac{a}{x} - \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2}{x^2} + \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3}{x^3} - \text{\&c.}\right).$$

Find the nearest perfect cube to the given number, and substitute this in the formula for x; *subtract this cube from the given number, and substitute this difference, which will often be negative, in the formula for* a; *perform the operations indicated, and the result will be the required root.*

Thus, to extract the cube root of 31,

$$\sqrt[3]{31} = \sqrt[3]{27+4} = 3\left(1 + \frac{1}{3}\cdot\frac{4}{27} - \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{16}{729} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{5}{9}\cdot\frac{64}{19683}\ \text{\&c.}\right) = 3.14138.$$

There is still another method by means of continued fractions, which is entirely similar to that for extracting the square root of a number by continued fractions. See *Square Root, Logarithms.*

CU'BIC EQUATION. A cubic equation, containing but one unknown quantity, is one in which the highest exponent of the unknown quantity, in any term, is 3. Every cubic equation containing but one unknown quantity can be reduced to the general form

$$x^3 + px + q = 0,$$

in which the co-efficient of x^3 is 1, and the co-efficient of x^2 is 0.

Every cubic equation of the above form has three roots, all of which may be real, or one only may be real and the other two imaginary. It may be shown by the application of Sturm's rule for determining the number and places of the real roots of an equation, that the roots will all be real when p is essentially *negative*, and $\frac{p^3}{27} > \frac{q^2}{4}$, numerically. One of the roots only will be real and the other two imaginary, when p is essentially positive or when it is negative, and $\frac{p^3}{27} < \frac{q^2}{4}$, numerically. There is still another case in which p is essentially negative and $\frac{p^3}{27} = \frac{q^2}{4}$, numerically. In this case two of the roots are equal, and may be determined by the method of equal roots. See *Equal Roots.*

In the second case, that is, when only

one of the roots is real, the equation may be solved by the following formula:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$+ \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}};$$

this is Cardan's formula. If we place

$$P = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},$$

$$Q = \sqrt[3]{-\frac{p}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},$$

and $\quad a = \dfrac{-1 + \sqrt{-3}}{2},$

and regard only the numerical values of the cube roots, we shall have three formulas for the three roots:

$$\left.\begin{array}{ll} \text{1st root} & x = P + Q \\ \text{2d root} & x = aP + a^2Q \\ \text{3d root} & x = a^2P + aQ \end{array}\right\};$$

the second and third roots are imaginary.

1. Let it be required to find the three roots of the equation

$$x^3 - 6x - 9 = 0.$$

Here p is negative, being -6 and $\frac{p^3}{27} < \frac{q^2}{4}$, numerically; hence Cardan's formula is applicable. Substituting in that formula -6 for p and -9 for q, we find

$$x = \sqrt[3]{\frac{9}{2} + \sqrt{\frac{81}{4} - 8}} + \sqrt[3]{\frac{9}{2} - \sqrt{\frac{81}{4} - 8}}$$

$$= \sqrt[3]{\frac{9}{2} + \frac{7}{2}} + \sqrt[3]{\frac{9}{2} - \frac{7}{2}} = 3;$$

In this case $P = 2$ and $Q = 1$; hence, the imaginary roots, after reduction, are

$$x = \frac{-3 + \sqrt{-3}}{2} \quad \text{and} \quad x = \frac{-3 - \sqrt{-3}}{2}.$$

When the roots are all real, Cardan's formula fails to give their values; for in that case the two terms of the second member of the formula become imaginary, and although the imaginary parts must necessarily destroy each other, yet all attempts to put them under such a form as to get rid of them have proved unsuccessful. For this reason this is called the irreducible case.

The following method of solving cubic equations of all kinds is jointly due to BOMBELLI, VIETA, and GIRAUD. The formulas may be found demonstrated in Bonnecastle's Trigonometry.

If we assign to p and q their essential signs, the cubic equation may appear under one of the four following forms:

1st form	$x^3 + px - q = 0$;
2d form	$x^3 + px + q = 0$;
3d form	$x^3 - px - q = 0$;
4th form	$x^3 - px + q = 0$;

each of which will be considered in succession.

1. When $x^3 + px - q = 0$.

Assume

$$\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \tan z, \text{ and } \sqrt[3]{\tan(45^\circ - \tfrac{1}{2}z)} = \tan u;$$

then $\quad . \quad . \quad . \quad . \quad x = 2\sqrt{\frac{p}{3}} \times \cot 2u.$

Applying logarithms to these formulas, they become

$$\log \frac{1}{2}q + 10 - \frac{3}{2}\log\frac{p}{3} = \log(\tan z); \quad \text{and}$$

$$\frac{1}{3}\left\{\log\left(\tan(45^\circ - \frac{1}{3}z)\right) + 20\right\} = \log(\tan u)$$

Then

$$\log x = \frac{1}{2}\log\frac{4p}{3} + \log(\cot 2u) - 10.$$

2. When $x^3 + px + q = 0$.

Assume

$$\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \tan z, \text{ and } \sqrt[3]{\tan(45^\circ - \tfrac{1}{2}z)} = \tan u;$$

then $\quad x = -2\sqrt{\frac{p}{3}} \times \cot 2u.$

Applying logarithms to these formulas, they become

$$\log\frac{q}{2} + 10 - \frac{3}{2}\log\frac{p}{3} = \log(\tan z): \quad \text{and}$$

$$\frac{1}{3}\left\{\log\left(\tan(45^\circ - \frac{1}{2}z)\right) + 20\right\} = \log(\tan u).$$

Then $\cdots \log x = 10 - \frac{1}{2}\log\frac{4p}{3} - \log(\cot 2u).$

In both of these cases there is but one real root.

3. When $x^3 - px - q = 0$.

This form has two cases; 1st, when

$\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} < 1$; 2d, when $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} > 1$.

In the *first case*, assume

$\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} = \cos z$; and $\sqrt[3]{\tan(45^\circ - \frac{1}{2}z)} = \tan u$.

Then . . . . $x = 2\sqrt{\frac{p}{3}} \times \operatorname{cosec} 2u$.

By applying logarithms these formulas become

$10 + \frac{3}{2}\log\frac{p}{3} - \log\frac{q}{2} = \log(\cos z)$; and

$\frac{1}{3}\left\{\log\left(\tan(45^\circ - \frac{1}{2}z)\right) + 20\right\} = \log(\tan u)$.

Then

$$\log x = 10 + \log\frac{4p}{3} - \log(\sin 2u).$$

In this case there is but one real root.

In the *second case*, assume $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \cos z$, and x will have the three following values:

1. $x = 2\sqrt{\frac{p}{3}} \times \cos\frac{1}{3}z$,

2. $x = -2\sqrt{\frac{p}{3}} \times \cos(60^\circ - \frac{1}{3}z$,) and

3. $x = -2\sqrt{\frac{p}{3}} \times \cos(60^\circ + \frac{1}{3}z)$.

By applying logarithms, these formulas become

$\log\frac{q}{2} + 10 - \frac{3}{2}\log\frac{p}{3} = \log(\cos z)$; and

1. $\log x = \frac{1}{2}\log\frac{4p}{3} + \log(\cos\frac{1}{3}z) - 10$;

2. $\log x = \frac{1}{2}\log\frac{4p}{3} + \log\left(\cos(60^\circ - \frac{1}{3}z)\right) - 10$;

3. $\log x = \frac{1}{2}\log\frac{4p}{3} + \log\left(\cos(60^\circ + \frac{1}{3}z)\right) - 10$.

The value of x obtained from the first formula is to be taken with a positive sign; those from the second and third are to be taken with a negative sign.

In this case all the roots are real.

4. When $x^3 - px + q = 0$; this form has two cases: *first*, when

$\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} < 1$; *second*, when $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} > 1$.

In the *first case* assume

$\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} = \cos z$; and $\sqrt[3]{\tan(45^\circ - \frac{1}{2}z)} = \tan u$;

then, $x = -2\sqrt{\frac{p}{3}} \times \operatorname{cosec} 2u$.

By applying logarithms, these formulas become

$10 + \frac{3}{2}\log\frac{p}{3} - \log\frac{q}{2} = \log(\cos z)$; and

$\frac{1}{3}\left\{\log\left(\tan(45^\circ - \frac{1}{2}z)\right) + 20\right\} = \log(\tan u)$;

then, $-\log x = 10 + \log\frac{4p}{3} - \log(\sin 2u)$.

In this case, but one of the roots is real.

In the *second case*, assume

$\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \cos z$, and x will have the three following values,

1. $x = -2\sqrt{\frac{p}{3}} \times \cos\frac{1}{3}z$;

2. $x = 2\sqrt{\frac{p}{3}} \times \cos(60^\circ - \frac{1}{3}z)$;

3. $x = 2\sqrt{\frac{p}{3}} \times \cos(60^\circ + \frac{1}{3}z)$.

By applying logarithms, these formulas become

$\log\frac{q}{2} + 10 - \frac{3}{2}\log\frac{p}{3} = \log(\cos z)$; then

1. $\log x = \frac{1}{2}\log\frac{4p}{3} + \log(\cos\frac{1}{3}z) - 10$,

2. $\log x = \frac{1}{2}\log\frac{4p}{3} + \log\left(\cos(60^\circ - \frac{1}{3}z)\right) - 10$,

3. $\log x = \frac{1}{2}\log\frac{4p}{3} + \log\left(\cos(60^\circ + \frac{1}{3}z)\right) - 10$.

The value of x, found from the first formula, is to be taken with a negative sign; the values of x obtained from the second and third, are to be taken with positive signs. In this case, all the roots are real.

The last three cases in the third and fourth forms, come under the head of the *irreducible case* already considered.

To show the manner of solving an example of the irreducible case, let it be required to find the three roots of the equation,

$$x^3 - 3x - 1 = 0.$$

Here,

$$\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \frac{1}{2}\left(\frac{3}{3}\right)^{\frac{3}{2}} = \frac{1}{2} = .5 = \cos z,$$

$$\therefore\ z = 60^\circ.$$

1. $x = \quad 2\cos 20^\circ = \quad 1.8793852,$
2. $x = -2\cos 40^\circ = -1.5320888,$
3. $x = -2\cos 80^\circ = -0.3472964.$

Again, let it be required to find the three roots of the equation,

$$x^3 - 3x + 1 = 0.$$

Here, as before,

$$\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = .5 \text{ and } z = 60^\circ.$$

1. $x = -2\cos 20^\circ = -1.8793852,$
2. $x = \quad 2\cos 40^\circ = \quad 1.5320888,$
3. $x = \quad 2\cos 80^\circ = \quad 0.3472964.$

In the last example, the roots are equal to those of the first example, respectively, each being taken with a contrary sign.

Besides these methods, there is still another by means of series. In practice this will often be found more convenient than either of the others discussed.

The series employed depend on the form of the equation.

1. When the equation is of the form

$$x^3 + px - q = 0.$$

$$x = \frac{2q}{\sqrt[3]{2(27q^2 + 4p^3)}}\left\{1 + \frac{2\cdot 5}{6\cdot 9}\left(\frac{27q^2}{27q^2 + 4p^3}\right) + \frac{2\cdot 5\cdot 8\cdot 11}{6\cdot 9\cdot 12\cdot 15}\left(\frac{27q^2}{27q^2+4p^3}\right)^2 + \frac{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17}{6\cdot 9\cdot 12\cdot 15\cdot 18\cdot 21}\left(\frac{27q^2}{27q^2+4p^3}\right)^3 + \&c.\right\}$$

2. When the equation is of the form

$$x^3 - px \mp q = 0, \quad \text{and} \quad \frac{q^2}{4} > \frac{p^3}{27}.$$

The upper sign is to be used when q is negative, and the lower one when q is positive.

$$x = \pm\sqrt[3]{\frac{q}{2}}\left\{1 - \frac{2}{3\cdot 6}\left(\frac{27q^2 - 4p^3}{27q^2}\right) - \frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}\left(\frac{27q^2 - 4p^3}{27q^2}\right)^2 - \frac{2\cdot 5\cdot 8\cdot 11\cdot 14}{3\cdot 6\cdot 9\cdot 12\cdot 15\cdot 18}\left(\frac{27q^2-4p^3}{27q^2}\right)^3 - \&c.\right\}$$

3. When the equation is of the form

$$x^3 - px \mp q = 0 \quad \text{and} \quad \frac{q^2}{4} < \frac{p^3}{27}.$$

The signs to be used as before.

$$x = \pm 2\sqrt[3]{\frac{q}{2}}\left\{1 + \frac{2}{3\cdot 6}\left(\frac{4p^3 - 27q^2}{27q^2}\right) - \frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}\left(\frac{4p^3 - 27q^2}{27q^2}\right)^2 + \frac{2\cdot 5\cdot 8\cdot 11\cdot 14}{3\cdot 6\cdot 9\cdot 12\cdot 15\cdot 18}\left(\frac{4p^3 - 27q^2}{27q^2}\right)^3 - \&c.\right\}$$

This corresponds to the irreducible case, and the series gives one root; if we designate this root by r, the other two may be found by the following series:

$$x = \mp\frac{r}{2} \pm \frac{\sqrt{4p^3 - 27q^2}}{9\sqrt[3]{2q^2}}\left\{1 - \frac{2\cdot 5}{6\cdot 9}\left(\frac{4p^3 - 27q^2}{27q^2}\right) + \frac{2\cdot 5\cdot 8\cdot 11}{6\cdot 9\cdot 12\cdot 15}\left(\frac{4p^3 - 27q^2}{27q^2}\right)^2 - \frac{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17}{6\cdot 9\cdot 12\cdot 15\cdot 18\cdot 21}\left(\frac{4p^3-27q^2}{27q^2}\right)^3 + \&c.\right\}$$

The upper signs are to be used before r when q is negative, and the lower one when it is positive; the double sign before the series is to be used as in ordinary cases, that is, the plus sign corresponds to one root, and the minus sign to the other.

4. When the equation is of the form

$$x^3 - px \mp q = 0, \quad \text{and} \quad \frac{q^2}{4} < \frac{p^3}{27}$$

$$x = \pm 2\cdot\frac{2q}{\sqrt[3]{2(4p^3 - 27q^2)}}\left\{-\frac{2\cdot 5}{6\cdot 9}\left(\frac{27q^2}{3p^3 - 27q^2}\right) + \frac{2\cdot 5\cdot 8\cdot 11}{6\cdot 9\cdot 12\cdot 15}\left(\frac{27q^2}{4p^3 - 27q^2}\right)^2 - \frac{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17}{6\cdot 9\cdot 12\cdot 15\cdot 18\cdot 21}\left(\frac{27q^2}{4p^3\cdot 27q^2}\right)^3 + \&c.\right\}$$

This series also answers to the irreducible case. The rule for signs is the same as already explained.

If the root found by applying the series is denoted by r, the other two roots will be found by the following series:

$$x = \pm\frac{r}{2} \pm \sqrt[6]{\frac{4p^2 - 27q^2}{4}}\left\{1 + \frac{2}{3\cdot 6}\left(\frac{27q^2}{4p^3 - 27q^2}\right) - \frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}\left(\frac{27q^2}{4p^3 - 27q^2}\right)^2 + \frac{2\cdot 5\cdot 8\cdot 11\cdot 14}{3\cdot 6\cdot 9\cdot 12\cdot 15\cdot 18}\left(\frac{27q^2}{4p^3 - 27q^2}\right)^3 - \&c.\right\}$$

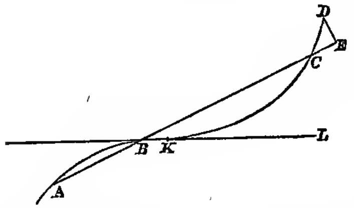
The signs are to be used as in the preceding case. With the aid of logarithms these

series present little or no difficulty in their application.

CUBIC PARABOLA. An Algebraic curve of the third order, whose general equation may be reduced to the form

$$y = ax^3 + bx^2 + cx + d.$$

If the origin of co-ordinates is taken at the vertex of the curve, its equation is $y = ax^3$. The curve consists of two branches KA and KD, infinite in extent, both of which have their convexities turned towards the line KL. Both are also tangent to the line KL at K; K is therefore a point of inflexion.



If a straight line ABC be drawn, cutting the curve in three points, A, B, and C, and if a straight line DE be drawn from any point of the curve perpendicular to it, then will DE be proportional to the parallelopipedon, whose edges are AB, AC, AE. This is a characteristic property of the cubic parabola, and it affords a method of constructing the roots of a cubic equation of the form

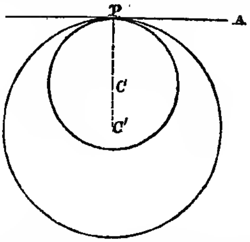
$$x^3 + a^2x = b^3.$$

The area of any portion of a cubic parabola is equal to three-fourths of its circumscribing parallelogram.

CUR'RENCY. [L. *currens*, from *curro*, to flow or run]. In Commerce, a term employed to express the aggregate amount of money, bills of exchange, and other substitutes for money, employed in buying, selling, and distributing the commodities of the various nations of the earth.

CUR'TATE CYCLOID. See *Cycloid*.

CURV'A-TURE. [L. *curvatura*, bowing, bending]. The curvature of a plane curve, at a point, is its tendency to depart from a tangent drawn to the curve at that point. In the circle, the tendency to depart from a tangent drawn at any point is always the same: hence, the curvature of the circle is constant throughout. If two circles C and C′ have a common tangent at a common point P, then will that circle which has the



least radius, have the greatest tendency to depart from the tangent, and consequently, the greatest curvature. We see, therefore, that the curvature of the circle varies inversely, as its radius. It is for this reason, that the reciprocal of the radius of a circle is assumed as the measure of its curvature.

In order to compare the curvature of different curves, or of the same curve at different points, we have simply to find the expression for the radii of the osculatory circles at the points, and then the greatest curvature will correspond to the least radius. The reason for this, is that the osculatory circle, from its nature, has a greater tendency to coincide with the curve at the point of osculation than any other circle; and so intimate is the relation between the curve and its osculatory circle, that for a small distance they may be regarded as absolutely coincident: hence, they have the same curvature, and we may take the measure of the curvature of the osculatory circle as the measure of the curvature of the curve.

The formula for the radius of the osculatory circle, or the radius of curvature, at any point of any curve, is

$$R = \pm \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

in which x and y are the co-ordinates of the point of osculation, which point may be anywhere on the given curve.

To apply this formula in any given case, differentiate the equation of the given curve twice; from the given equation and the two differential equations find expressions for the first and second differential co-efficients of

the ordinate, in terms of the abscissa; substitute these in the formula, and the resulting value of R will be the general expression for the radius of curvature at any point.

To find the value of R at any given point, substitute for x, in the general value of R, the abscissa of the given point, and the resulting value will be the required value of R for the particular point.

The general value of R, found as above explained, is a function of x, and its maximum or minimum value may be determined by the rules for finding the maxima or minima of functions of one variable.

To illustrate, let us consider the case of the conic sections, whose equations may be written under the general form

$$y^2 = 2px + r^2x^2,$$

the origin of co-ordinates being taken at the principal vertex.

By differentiation and combination, we find

$$\frac{dy^2}{dx^2} = \frac{(p + r^2x)^2}{y^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{p^2}{y^3}.$$

If we substitute these in the formula for R, taking the lower sign, it gives

$$R = \frac{[p^2 + (r^2+1)(2px + r^2x^2)]^{\frac{3}{2}}}{p^2}.$$

To find those points of the curve at which R is either a maximum or a minimum, assume

$$u = p^2 + (r^2+1)(2px + r^2x^2); \text{ whence,}$$

$$\frac{du}{dx} = (r^2+1)(2p + 2r^2x) = 0 \cdots (1);$$

$$\text{or,} \qquad x = -\frac{p}{r^2},$$

$$\left(\frac{d^2u}{dx^2}\right)_{x=-\frac{p}{r^2}} = 2(r^2+1)r^2 \cdot\cdot (2).$$

In the parabola $r^2 = 0$, whence $x = \infty$; that is, the point where R is a maximum, is at an infinite distance.

By substituting these values in R, we have

$$R = \infty, \quad \text{or} \quad \frac{1}{R} = 0;$$

that is, the parabola has no curvature at an infinite distance; or in other words, it coincides with a straight line.

In the ellipse,

$$r^2 = -\frac{b^2}{a^2}, \text{ in which } b^2 < a^2, \text{ and } p = \frac{b^2}{a}.$$

These values give $x = a$, and make

$$\frac{d^2u}{dx^2} = -\frac{2(a^2 - b^2)b^2}{a^4},$$

which is negative. Hence, at the points whose abscissas are a, the value of R is a maximum, and the curvature a minimum, but these points are the extremities of the conjugate axis.

In the hyperbola,

$$r^2 = +\frac{b^2}{a}, \text{ and } p = \frac{b^2}{a},$$

which give $x = -a$; but for $x = -a$, y is imaginary, which shows that there is no point of the curve at which the radius of curvature is a maximum.

The form of expression (1), does not at once indicate the conditions which render R a minimum; but we have, from the differential equation of the curve,

$$\frac{dx}{dy} = \frac{y}{p + r^2x} \cdots (3).$$

Now, if we multiply equations (1) and (3), member by member, we have

$$\frac{du}{dy} = 2(r^2+1)y = 0; \text{ whence } y = 0, \text{ and}$$

$$\left(\frac{d^2u}{dy^2}\right)_{y=0} = 2(r^2+1),$$

which is always positive: hence, in all of the conic sections, the radius of curvature at the principal vertex is a minimum.

From the foregoing discussion, it appears that the curvature of the ellipse is a minimum at the vertices of the conjugate axis, and that the curvature of any conic section is a maximum at the principal vertex.

It may be observed that the radius of curvature at the principal vertex is always equal to half the parameter. As the parameter decreases, the curvature at the principal vertex increases; and when the parameter becomes 0, the curvature is infinite. This last supposition corresponds to the case in which the conic sections become straight lines. See *Eccentricity*, and *Parameter*.

Curvature of Surfaces. The curvature of a surface at any point, is its tendency to depart from a tangent plane to the surface at that point.

If the tangent plane to a surface at the point at which the curvature is to be consi-

dered be taken as the plane XY, and the normal to the surface at the point be taken as the axis of Z; then every plane through the axis of Z will cut a section from the surface, and the curvature of these sections at the origin of co-ordinates will be different in each. If we denote by q, q'', and q', respectively, what

$$\frac{d^2z}{dx^2}dx^2, \quad \frac{d^2z}{dy^2}dy^2 \quad \text{and} \quad \frac{d^2z}{dxdy}dxdy,$$

become, when x, y, and z, are made equal to 0, and by ϕ the angle which the plane of the normal section makes with the plane XZ, we shall have for the radius of curvature of the normal section at the origin,

$$R = \frac{1}{q\cos^2\phi + 2q'\cos\phi\sin\phi + q''\sin^2\phi} \cdot (1);$$

whence,

$$\frac{1}{R} = q\cos^2\phi + 2q'\cos\phi\sin\phi + q''\sin^2\phi \cdot (2).$$

If we make $\phi' = 90° + \phi$, and denote what R becomes by R', we have

$$\frac{1}{R'} = q\sin^2\phi - 2q'\cos\phi\sin\phi + q''\cos^2\phi;$$

and by addition

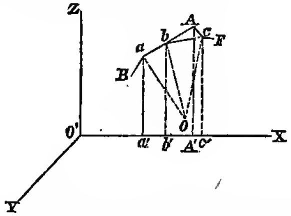
$$\frac{1}{R} + \frac{1}{R'} = q + q'';$$

that is, the sum of the reciprocals of the radii of curvature of two normal sections at right angles to each other, when taken at the point of contact, is *constant*.

We see from this result, that the section of least curvature is perpendicular to the section of greatest curvature. If the normal to the surface is the axis of revolution of the surface, the values of R become equal, and every curve of section will have the same curvature at the point of tangency, which will be, in that case, a vertex of the surface.

Curvature of Lines of Double Curvature. If two curved surfaces intersect each other, the line of intersection will, in general, be a curve of double curvature; and if at any point of it, a plane be passed so as to coincide with two consecutive elements of the curve, this plane is called an *osculating plane*. The radius of the osculatory circle, which coincides with these elements, is the radius of the curve at the assumed point.

To investigate an expression for this radius, let us take the length of the curve, denoted by s, as the independent variable; in which case, ds will be constant, and the elements of the curve equal to each other.



Let ab and bc, each equal to ds, be two consecutive elements of a curve of double curvature, and let O be the centre of a circle passing through the points a, b, and c, then will Oa be the radius of the osculatory circle or radius of curvature. Produce the element ab till the prolongation bA is equal to ab, and draw cA; then, since the angle Abc is equal to aOb, the triangles bAc and aOb will be similar, and we shall have

$$aO : ab :: bc : Ac, \quad \text{or}$$
$$R : ds : ds : Ac; \text{ whence,}$$
$$R = \frac{ds^2}{Ac}.$$

If we project the several lines ab, bc, bA and Ac upon the axis of X, by perpendiculars to that axis, we shall have

$$a'b' = b'A' = dx,$$

and $$b'c' = d(x + dx) = dx + d^2x,$$

which gives for A$'c'$, (the projection of Ac upon the axis of x),

$$A'c' = dx + d^2x - dx = d^2x.$$

In like manner, if we project Ac upon the axis of Y and Z, severally, we shall find the projections equal to d^2y and d^2z; hence,

$$Ac = \sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}; \text{ whence,}$$

$$R = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}}; \text{ or}$$

$$R = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}}.$$

If we do not regard s as the independent variable, we may replace

$$\frac{d^2x}{ds^2} \text{ by } \frac{d\left(\frac{dx}{ds}\right)}{ds}, \quad \frac{d^2y}{ds^2} \text{ by } \frac{d\left(\frac{dy}{ds}\right)}{ds},$$

and $$\frac{d^2z}{ds^2} \text{ by } \frac{d\left(\frac{dz}{ds}\right)}{ds};$$

whence, by performing the operations indicated,

$$\frac{d^2x}{ds^2} \text{ becomes } \frac{d^2x}{ds^2} - \frac{dx}{ds}\cdot\frac{d^2s}{ds^2};$$

$$\frac{d^2y}{ds^2} \text{ becomes } \frac{d^2y}{ds^2} - \frac{dy}{ds}\cdot\frac{d^2s}{ds^2};$$

$$\frac{d^2z}{ds^2} \text{ becomes } \frac{d^2z}{ds^2} - \frac{dz}{ds}\cdot\frac{d^2s}{ds^2}.$$

These, in the formula above deduced, give

$$R = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 - \left(\frac{d^2s}{ds^2}\right)^2}},$$

a formula much used in mechanics.

CURV'A-TURE, CHORD OF. See *Chord*.

CURVE. [L. *curvus*, bent; from *curvo*, to bend]. A curve is a line which changes its direction at every point; that is, no three consecutive points of which lie in the same straight line.

The portion of the line between two consecutive points, is an element of the curve. If we denote the length of the curve by s, the length of any element will be denoted by ds.

CURVED LINES. Curves may be either *plane curves*, or *curves of double curvature*. A plane curve is one all of whose points are in the same plane. A curve of double curvature is one in which no more than three consecutive points lie in the same plane.

The only curve considered as belonging to elementary geometry, is the circle. See *Circle*.

In the higher branches of mathematics, curves are classed according to the nature of their equations.

The first division of lines is into *algebraic* and *transcendental*. An *algebraic* line is one in which the relation between the co-ordinates of its points may be expressed by means of the ordinary operations of algebra; that is, *addition, subtraction, multiplication, division, raising to powers denoted by constant exponents*, and *extracting roots indicated by constant indices*.

A transcendental line is one in which the relation between the co-ordinates of its points cannot be thus expressed. It is to be observed, that in the higher mathematics, lines are always regarded as being defined by their equations. If we regard a line as being generated by a point moving according to some fixed law, the expression of that law, by means of the algebraic language, will be the equation of the line, and may be regarded as the analytical definition of the line.

If a point moves at random, the path which it describes is not regarded as a curve, but simply as a crooked line. Hence, we see the distinction between a curve and a crooked line; the former is generated in accordance with a mathematical law; the latter is generated without reference to any law. The distinction is analagous to that between *music* and unmeaning *noise*.

In *transcendental lines*, the relation between the co-ordinates of the points of the lines is expressed either by the aid of *exponential, logarithmic*, or *trigonometric* functions. In consequence of the intimate connection between these several classes of functions, it happens that we can generally refer all these relations to that existing between logarithmic quantities.

Amongst the transcendental lines may be mentioned the *logarithmic curve*, the *cycloid*, the *spirals*, the *catenary*, &c.

Algebraic lines are classed into orders depending upon the degree of their equations.

Lines, whose equations are of the first degree, with respect to the variables, are called lines of the *first order*. This order embraces only the straight line, which, for the purpose of classification, is often ranked as a curve.

Lines, whose equations are of the second degree, with respect to the variables which enter them, are called lines of the *second order*. This order of lines includes what have been called the conic sections; that is, the ellipse, parabola, and hyperbola, together with their particular cases. The order does not include any other lines.

In general, a line whose equation is of the n^{th} degree, with respect to the variables which enter it, is called a line of the n^{th} *order*.

The relation between the co-ordinates of any curve, may always be expressed by means of an equation, and if the curve lie wholly in a plane, that equation will contain two, and only two, variables.

Conversely, every equation between two variables is the equation of a plane curve, as may readily be shown: if we assume a value for one of the variables, substitute it for that variable in the equation, and deduce the cor-

responding value of the other variable, the assumed and deduced values will be the co-ordinates of a point, which may be constructed by known principles; in like manner, any number of points may be constructed, and the curve line drawn through them will be that represented by the equation.

This line is often called the locus of the equation; it is more properly the locus of the point, which moves in accordance with the law expressed by the equation.

In order to construct this locus: from what has preceded, we see that we must know the values of the constants which enter the equation. When the constants which enter an equation are known, in addition to the form of the equation, the curve is *completely* given. When the form of the equation only is given, the line is said to be given in *kind*. The form of the equation then determines the kind of line, and the constants which enter it serve to determine its extent and position with respect to the co-ordinate axes.

Since the equation of a straight line is of the first degree, it follows, that if we combine the equation of a straight line with the equation of a curve of any order, we shall in general get as many pairs of values for x and y, as there are units in the number which denotes the order of the curve. It may happen, however, that some of the sets of values are imaginary, but imaginary roots go in pairs: hence, if the curve is of an odd order, there will always be one real point of intersection.

Curved Surfaces. A curved surface is one in which, if a point be taken at pleasure, and any number of secant planes be passed through it, these planes will in general cut curved lines from the surface: thus, the surface of a sphere, cone, &c., are curved surfaces.

Curved surfaces are classified in geometry, into,

1. *Single Curved Surfaces*, which may be generated by a right line, moving in such a manner that any two consecutive positions shall be in the same plane. The conic surface and the cylindric surface belong to this class. Besides these, there is a numerous family of surfaces, which may be generated by a straight line, moving in such a manner as to continue tangent to a curve of double curvature.

The distinguishing characteristic of single curved surfaces is, that they may be developed, or rolled out upon a plane.

2. *Double Curved Surfaces*, are those which can only be generated by a curved line: thus, the surface of a sphere, ellipsoid, or paraboloid of revolution, are double curved surfaces.

3. *Warped Surfaces*, are those which may be generated by a right line, moving in such a manner that its consecutive positions shall not be in the same plane. The hyperbolic paraboloid, the conoid, and the hyperboloid of one nappe, are examples of warped surfaces.

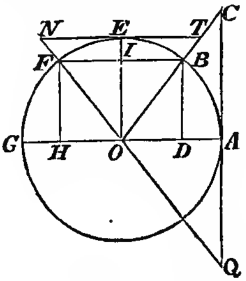
Neither double curved nor warped surfaces are capable of being developed.

In analysis, surfaces are classed in a manner entirely analogous to that employed for classing lines. Every equation between three variables represents a surface of some kind, and if it is of a degree superior to the first, it is a curved surface. Every equation of the first degree between three variables, is the equation of a plane. Every equation of the second degree between three variables, is the equation of a surface of the *second order*. Surfaces of the second order comprise the ellipsoid, the hyperboloids and the paraboloids, with their several varieties, and none others.

Generally, a surface is of the n^{th} order, when its equation is of the n^{th} degree between three variables.

Curves of Sines, Cosines, Versed Sines, Tangents, Cotangents, Secants, and Co-secants. Curves whose equations are respectively

$y = \sin x$, $y = \cos x$, $y = \text{ver-sin}\, x$, $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \cos x$.

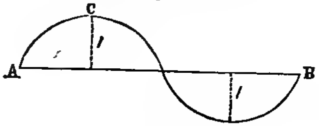


If we conceive a circle to roll upon a

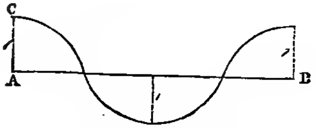
straight line, continuing in the same plane, and at the point of contact perpendiculars to be erected equal respectively to the sine, cosine, versed sine, &c., of the arc from the origin of the arcs to the point of contact, then will the loci of the extremities of these ordinates be the curves whose equations are given. After the generating circle has rolled once over, if it be again rolled over, a curve will be generated equal in all respects to that first generated; and so on, every time that the curve is rolled over, even to an infinite number of times.

Denote the circumference of the generating circle by π, and the radius by 1. Then lay off on a straight line the distance ab equal to 2π, to represent the development of the circumference of the circle. The curves may be constructed by points, from their equations, by means of a table of natural sines, and when constructed, they will present the appearances represented in the figures.

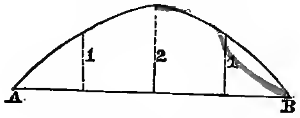
Curve of Sines.



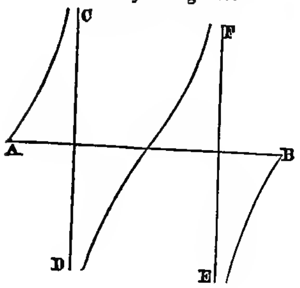
Curve of Cosines.



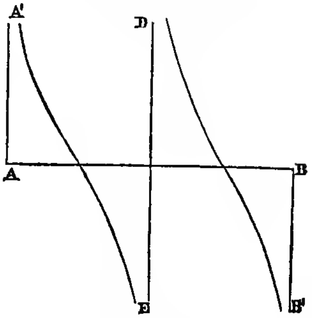
Curve of Versed Sines.



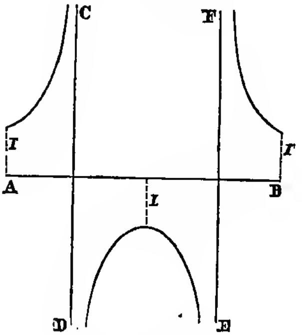
Curve of Tangents.



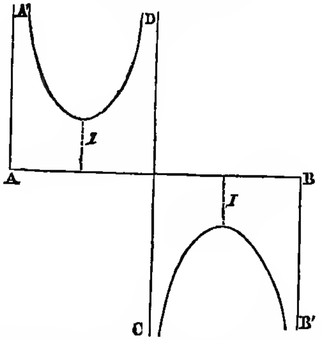
Curve of Cotangents.



Curve of Secants.



Curve of Cosecants.



In the figures, the lines AA′, BB′, CD and EF, are asymptotes to the infinite branches.

The curve of sines, sometimes called the sinusoid, is entirely similar to the curve of cosines, commencing to estimate both from the point C: this should be the case, since,

$$\sin x = \cos(90° - x).$$

By differentiating the equation of the curve of sines, we find

$$\frac{dy}{dx} = \cos x, \text{ and } \frac{d^2y}{dx^2} = -\sin x = -y;$$

from the first of these results, we deduce the equation of a tangent to the curve at a point whose co-ordinates are x'' and y'', as follows:

$$y - y'' = \cos x'' (x - x'').$$

If $x'' = 2n\pi$, n being any whole number, we have $\cos x'' = 1$; at these points the tangent makes an angle of 45° with the axis of X.

If $x'' = (2n + 1)\pi$, $\cos x'' = -1$, and for these points the tangent is inclined to the axis of X in an angle of 135°.

If $x'' = \frac{n}{2}\pi$, $\cos x'' = 0$, and at those points, the tangent is parallel to the axis of X, the ordinates at the alternate points are alternately maxima and minima.

From the second result, $\frac{d^2y}{dx^2} = -y$, we infer that the curve is always concave towards the axis of X.

To find an expression for the area of the sinusoid, we have the formula,

$$S = \int y dx = \int \frac{y dy}{\sqrt{1-y^2}} = -(1-y^2)^{\frac{1}{2}} + C.$$

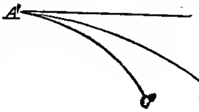
If we integrate between the limits $y = 0$ and $y = 1$, we have $A = 1$; hence the entire area included between one branch and the axis is equal to twice the square described upon the radius of the generating circle. Like discussions may be had with respect to the other curves above considered.

CUR-VI-LIN'E-AR. [L. *curvus*, bent]. Appertaining to curved lines; bounded by a curved line or lines; thus, a circle or an ellipse is a curvilinear figure, so also is a spherical triangle.

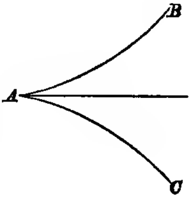
CUSP. [L. *cuspis*, a point]. A cusp point of a curve.

Cusp Point. A cusp point of a curve is a point at which two branches are tangent to each other, so that a point generating the curve suddenly stops at the cusp and returns for a time in the same general direction from which it arrived at the cusp point. A and A′ are cusp points. Cusp points are of two kinds.

1. When the two branches have their convexities turned in the same direction with respect to the common tangent at the cusp point.

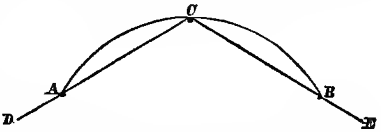


2. When they have their convexities turned in opposite directions with respect to the common tangent at the cusp point.



CYC'LO-GRAPH. [Gr. κυκλος, a circle, and γραφω, to describe]. An instrument employed to describe arcs of circles when the radii are greater than can be obtained with the dividers. It is also used in those cases in which the dividers cannot be employed.

The most simple form of this instrument is that employed by artificers in their work DC and CE are two arms turning about an



axis C, capable of being set to any given angle and clamped. At C is an arrangement for holding a pencil or a piece of chalk. A and B represent two nails or pins driven into a wall or board, to show the extremities of the arc to be described. If now the instrument be set at the proper angle and moved about so that the arms should constantly touch A and B, then will the pencil trace the required arc.

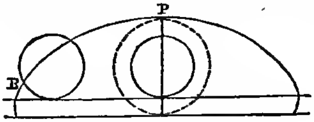
CYC'LOID or TROCHOID [Gr. κυκλος, a circle, and ειδος, form]. A curve which is generated by a point in the plane of a circle, when the circle is rolled along a straight line, always continuing in the same plane.

1. If the generating point is upon the circumference of the generating circle, the curve is called the *common cycloid*.

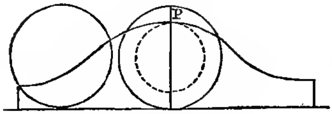
2. If the generating point lies without the circumference of the generating circle, the curve is called the *curtate cycloid*.

3. If the generating point lies within the

circumference of the generating circle, the curve is called the *prolate* or *inflected cycloid.*

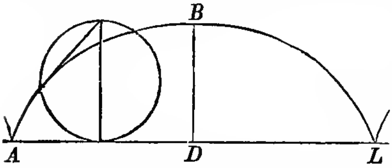


Curtate Cycloid.



Prolate Cycloid.

1. *Common Cycloid.* The rolling circle is called the *generating circle.* The point P which generates the curve is called the *generating point.* The line AL is the *base*, and BD the *axis*.



It is plain that each time the circle is rolled over, a portion of the curve, equal to ABL, will be described. Each of these portions are called *branches*. The number of branches is infinite.

If the origin of co-ordinates be taken at A, and the axis of x coinciding with the base AL, the equation of the curve is

$$x = \text{ver-sin}^{-1} y - \sqrt{2ry - y^2}.$$

This is the equation of a single branch, and it is unnecessary to regard more than one branch, because they are all equal to each other in every respect; therefore, if we deduce the properties of one branch, they will be common to all other branches.

The differential equation of the cycloid, which is more used than the equation of the curve, is

$$dx = \frac{ydy}{\sqrt{2ry - y^2}} = \frac{dy}{\sqrt{\frac{2r}{y} - 1}};$$

In this and the preceding equation, r is the radius of the generating circle, and ver-sin$^{-1}\, y$ is taken in that circle. It is also to be observed that as regards the double sign $\pm$, of the radical, in both equations, the upper sign is applicable to the portion of the branch on the left of the axis, and the lower one to the portion of the branch on the right of the axis.

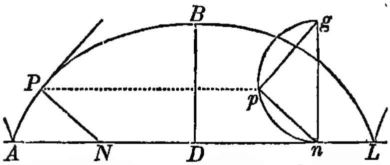
The following are some of the principal properties of the common cycloid:

1. The greatest ordinate is equal to $2r$ and the least one equal to 0; there are no negative ordinates.

2. The ordinate $2r$ coincides with the axis, and the tangent to the curve at its extremity is parallel to the base.

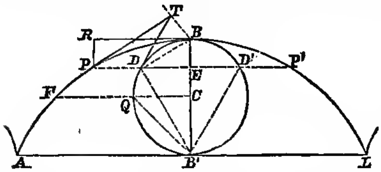
3. The tangent to the curve at the point whose ordinate is 0, is perpendicular to the base. This point is a cusp point of the first species, and it is also a point of concurrence with the base.

4. If gn represents a diameter of the generating circle in one of its positions, and is perpendicular to the base, and gpn a semicircle described upon it, then will the tangent to the curve, at any point P, be parallel to the corresponding chord gp, drawn to the upper extremity of the diameter, and the normal PN will be parallel to the supplementary chord pn, drawn to the lower extremity of the same diameter.



5. If two chords of the generating circle be drawn through the upper extremity of the diameter ng, and on opposite sides of it, making a given angle with each other, then will the locus of the point of intersection of the parallel tangents be a curtate cycloid. If a tangent be drawn to the curve at the vertex, the portion of it which is intercepted between any pair of these tangents will be equal in length to the corresponding arc of the generating circle; that is, to the part between the lower extremities of the chord parallel to the tangents. If the tangents are at right angles, the locus of their point of intersection will pass through the upper vertices of the rectangle described upon the base and axis of the curve.

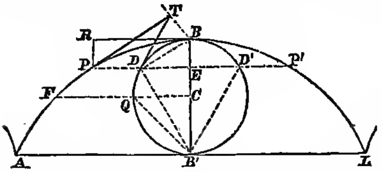
6. If a tangent DT be drawn to the circle described on the axis, and the corresponding tangent PT be drawn to the cycloid, the locus of their point of intersection is an involute



of the generating circle. The arc BP of the cycloid is equal to twice the chord BD of the generating circle, and the length of one branch is equal to four times the diameter of the generating circle.

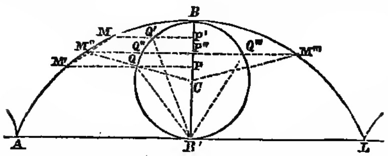
7. If the rectangle BEPR be completed, the area BPR is equal to the area of BDE, in the circle, and consequently, the area included between one branch of the curve and the base is equal to three times the area of the generating circle.

8. If a line PP′ be drawn parallel to the base and bisecting BC, then is the area PBP′ equal to the equilateral triangle B′DD′. If a



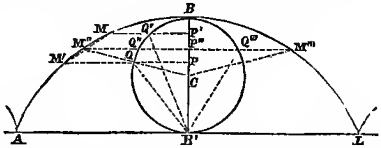
line CF be drawn through the centre of the generating circle parallel to the base, the area BCFPB is equal to the area of the triangle B′QC, or half of the square described upon the radius of the generating circle.

9. If two lines, PM and P′M′, be drawn parallel to the base, so that BP′ = CP, C being the centre of the generating circle,



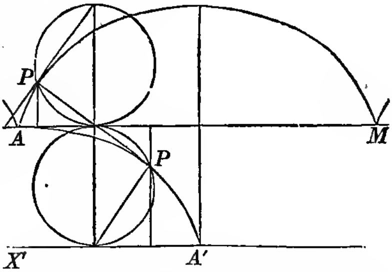
and the points M and M′ be joined, the area of PP′MM′ is equal to the difference between the two triangles B′PQ and B′P′Q′. If PM and P′M′ are on opposite sides of the axis, the area is equal to the sum of the two triangles.

10. If a line be drawn through P″, the middle point of CB, parallel to the base, and



C be joined with M″ and M‴, then will the cycloidal sector M″CM‴ be equivalent to the isosceles triangle BQ″Q‴.

11. The evolute AP′A′ of a semi-branch of the cycloid, is equal, in all respects, to the other semi-branch. If, whilst the generating circle of the cycloid rolls along the line AM,



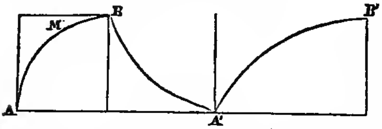
a second circle equal to it be rolled along X′A′, the two remaining tangent to each other on the line AM, then will the point P′ generate the evolute, and the tangent to the evolute at P′ will pass through the point of contact of the two generating circles, and be normal to the first cycloid.

12. The radius of curvature at any point of the cycloid is equal to twice the normal; at the cusp point it is 0, and a *minimum*: at the vertex of the axis it is twice the diameter of the generating circle, and a *maximum*.

13. The area of the surface generated by revolving one branch of the curve around its base, as an axis, is equal to $\frac{64}{3}$ of the area of the generating circle; and the volume of the corresponding solid of revolution is $\frac{5}{8}$ of the circumscribing cylinder.

14. If any curve A*m*B be taken, whose tangents at A and B are at right angles to the co-ordinate axes respectively, and its evolute BA′ be taken, beginning at B; then the evo-

lute of this last curve A′B′ be taken, beginning at A′, and so on, each succeeding evolute will approach in its nature a semi-cycloid, and will ultimately coincide with it.



The cycloid possesses some remarkable mechanical properties, the most important of which are the following:

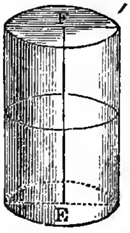
1. It is the curve of quickest descent from one point to another; that is, if two points lie one above the other, but not in the same vertical line, a heavy body will descend from the highest to the lowest along an arc of an inverted cycloid quicker than along any other curve passing through the two points, quicker even than along the straight line joining the two points. See *Brachystochrone*.

2. It is the tautochronous curve; that is, if a pendulum be made to vibrate on the arc of a cycloid, the time of vibration will always be the same, no matter what may be the length of the arc over which the vibration may take place. This result can only be approximately true in practice, since the theoretical considerations from which it was deduced can only be approximately fulfilled in any experimental operation.

CY-CLOID′AL, appertaining to a cycloid. A cycloidal segment is a segment included between an arc of a cycloid and its chord; a cycloidal sector is a portion of a cycloid bounded by an arc of the cycloid and two lines drawn from its extremities to the middle of its axis.

CY-CLOM′E-TRY. [Gr. κυκλος, a circle, and μετρεω, to measure]. The art of measuring circles.

CYL′IN-DER. [Gr. κυλινδρος, from κυλινδω, to roll. L. *cylindrus*]. In Plane Geometry, a cylinder is a solid which may be generated by revolving a rectangle AEFD about one of its sides EF.



This side is the *axis*. The opposite side generates a single curved surface, called the *convex* or *lateral surface* of the cylinder, and the two adjacent sides generate circles called *bases* of the cylinder.

If any plane be passed through the axis, it will cut from the cylinder a meridian section which will be a rectangle double the *generating* rectangle; any plane passed perpendicular to the axis, will cut from the cylinder a circle equal to either base. The distance between the bases is called the *altitude* of the cylinder, and is measured by the length of the axis. Cylinders are similar when generated by similar rectangles revolved about their homologous sides. If the same rectangle be successively revolved about two adjacent sides, the two cylinders generated are called *conjugate cylinders*. If the rectangle becomes a square, the conjugate cylinders are equal solids. The area of the convex or lateral surface of a cylinder is equal to the circumference of its base multiplied by its altitude; or, denoting the area required by A, the altitude of the cylinder by h, and the radius of the base by r, we have the formula

$$A = 2\pi \cdot r \cdot h.$$

If we include the areas of the bases, the formula becomes

$$A = 2\pi r\,(r + h).$$

By changing r into h, and h into r, we have the formula for the area of the surface of the conjugate cylinder,

$$A' = 2\pi h\,(r + h);$$

that is, the areas of the complete surfaces of two conjugate cylinders are to each other as their altitudes.

The formula for the volume of a cylinder, denoting the volume by V, is

$$V = \pi r^2 h,$$

and for the conjugate cylinder,

$$V' = \pi h^2 r;$$

whence we see that the volumes of conjugate cylinders are as their altitudes, or as the radii of their bases.

The volumes of similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.

CYLINDRICAL SURFACE or CYLINDER, in higher geometry, is a surface which may be generated by a straight line moving in such a manner as constantly to touch a given curve

and be continually parallel to its first position. The moving line is called the *generatrix;* the line along which it moves, is called the *directrix;* any position of the generatrix is called an *element* of the surface. Any section of the surface, by a plane, is called a *base.* If the base has a centre, the straight line drawn through it, and parallel to an element, is the *axis* of the cylinder.

If the plane of a base is perpendicular to the axis, or to an element, the cylinder is *right,* otherwise it is *oblique.*

If the vertex of a cone be moved to an infinite distance, the base remaining fast, the cone becomes a cylinder; hence, all the properties of a cylinder may be deduced as particular cases of the corresponding properties of the cone.

If the base of a cylinder be taken in the plane XY, its equation will be

$$f(x, y) = 0,$$

and the equation of the cylinder will be

$$f(x - az,\ y - bz) = 0.$$

In any particular case, having given the equation of the base or directrix, we substitute in that equation for x and y the expressions $x - az$, and $y - bz$; the resulting equation will be that of the surface, in which a and b are respectively the tangents of the angles which the projections of an element upon the planes XZ and YZ respectively make with the axis of Z. By attributing suitable values of a and b, the elements may be made to take any direction.

For further properties of the cylindrical surface, see *Cone* and *Conic Surface.*

CYL-IN'DRIC-AL. Having the properties of a cylinder, or resembling a cylinder in form.

CYL-IN'DRI-FORM. Having the form of a cylinder.

CYL'IN-DROID. A right cylinder with an elliptical base.

CYL-IN-DRO-MET'RIC. Belonging to a scale used in measuring cylinders.

CY'PHER. See *Cipher.*

D. The fourth letter of the English alphabet. In the Roman system of numeral notation, it stands for *five hundred;* with a dash on it thus, $\overline{\mathrm{D}}$, it denotes *five thousand.*

In the varying scale of English currency, d is used as a symbol to denote *penny* or *pence:* thus,

£	s.	d.
4	2	7,

is read, *four pounds, two shillings* and *seven pence.*

DA'TA. [L. *data,* plural of *datum,* given or known]. In mathematics, is a term employed to express all the given quantities and elements of a proposition.

In a problem, the *data* are the given parts, by means of which we are enabled to determine the unknown or required parts. If we have an equation expressing a relation between several unknown quantities, and assume values for all except one, these assumed values are *data* for finding the remaining one. This one is found from the data by the application of the principles of mathematics. In geometrical problems, the data are certain lines, surfaces, solids, or angles.

In the demonstration of a theorem, the data are the definitions, axioms and the conclusions of previous demonstrations; that is, such of them as may be necessary.

DA'TUM. The singular of data.

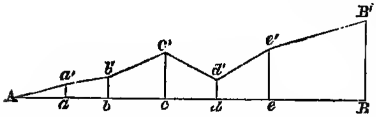
Datum Line. In *Surveying,* a line of true level, to which the points of a vertical section of the earth's surface are referred, for the purpose of determining its *slope* or *grade.*

This line is principally employed in making preliminary surveys for a line of railroad, canal or aqueduct. The datum line is usually taken through the lowest point of the section, or else through a point below the lowest point, so that all the vertical ordinates, by means of which the section is determined, may lie on the same side of the datum line. This arrangement is not absolutely necessary, but is usually made for the sake of convenience. To conceive the position of the datum line, let us take the case of a survey for a line of railroad, and suppose that the leveling is commenced at the lowest point. Through this point imagine a surface of true level, indefinite in extent, to be passed; next imagine a vertical cylinder to be passed through the proposed route, projecting it upon the level surface; this projection is the *datum line,* and if we conceive the projecting cylinder to be developed or rolled upon a

plane tangent to it along one of its elements, the datum line (neglecting the curvature of the earth) will be developed into a straight line, and the route along the surface of the earth will develop into a curve line, whose ordinates at different points are the distances of these points above the datum line.

If this curve be delineated upon paper, it forms what is called the vertical section of the route.

In the figure annexed, AB represents the



datum line developed upon a plane, and A a' b' c' d' e' B′ represents the vertical section of a route leading from A to B′.

In order to find the data for making a plot of this section, we measure, by means of a chain or tape, the horizontal distances Aa, ab, bc, &c., between the points Aa', $a'b'$, $b'c'$, &c., and by means of a level, determine the difference of level between the points A and a', A and b', A and c', &c., making the necessary corrections for curvature.

To plot the section: Draw the straight line AB to represent the datum line, and lay off on it from A the measured distance Aa; at a erect a perpendicular, and make it equal to the difference of level between A and a'; lay off on the datum line from a the measured distance ab; at b erect a perpendicular to it, and make it equal to the difference of level between A and b'; continue this operation till we reach the extreme point B′; then draw a curve line through the points A, a', b', c', &c., B′, and it will be the plot of the section required.

It is to be observed, that, as the horizontal distances Aa, ab, bc, &c., are very great in comparison with the vertical distances aa', bb', cc', &c., it is necessary to employ two scales in making the plot: one for the horizontal distances, and the other for the vertical distances. So that a line which represents a vertical distance, would represent several times that distance if laid off on the datum line.

DAY. A period of time which elapses between two consecutive transits of one of the heavenly bodies over the meridian. As the motion of the heavenly bodies are not all uniform, we distinguish several kinds of days, which differ but slightly from each other.

1. The ordinary *solar day*, is the time which elapses between two consecutive transits of the sun's centre over the meridian of a place. On account of the elliptical orbit of the earth, and of the inclination of the plane of the elliptic to that of the equator, the length of the solar day varies slightly, at different periods of the year. On account of this variability, the ordinary *solar day* has been found inconvenient, as a unit of time, and astronomers employ another unit.

2. Mean Solar Day. This, as its name indicates, is a period of time equal to the arithmetical mean of all the ordinary solar days in a year. The accumulated difference between the lengths of the ordinary and mean solar day, estimated from a common epoch, is what is called the *equation of time*. Ordinary clocks and watches are regulated to indicate *mean solar time*, and they are corrected, from time to time, by means of observations made upon the sun, and the equation of time.

3. Sidereal Day, is the period of the earth's rotation upon its axis once, and is measured by the interval of two consecutive transits of a fixed star over the meridian of a place.

On account of the motion of the sun amongst the stars, from west to east, the mean solar day is somewhat longer than the sidereal. Each day is divided into 24 equal parts, called hours; each hour into 60 equal parts, called minutes; and each minute into 60 equal parts, called seconds; and, according to the kind of day which is thus subdivided, we have *mean solar*, or *sidereal time:* A mean solar day of 24 hours is equal to $24^h.$ $03^m.$ $56.5554^s.$ of sidereal time; and a sidereal day of 24 hours, is equal to $23^h.$ $56^m.$ $04.0907^s.$ of mean solar time.

4. Civil Day. This is usually counted from one midnight to the next, though the epoch from which it is reckoned, is different in different countries.

5. Astronomical Day, commences at noon and continues till the next noon, and is reckoned through 24 hours; so that there is

oftentimes a difference of date for the same instant, when reckoned according to the one or the other of these methods.

For example, June 10^{th}, 10 A. M., by the civil reckoning, would be June 9^{th}. 22^h., by the astronomical reckoning; but June 10^{th}, 10, P. M., would be June 10^{th}. 10^h. astronomical reckoning. In order to convert a date in civil reckoning into the corresponding astronomical date, if it is before noon, subtract 1 from the number of the day, and add 12 to the number of hours; but if it is afternoon, the dates in both systems will be the same.

DEAD RECKONING. In Navigation, the estimation of a ship's place, arrived at without observation of any of the heavenly bodies. The place of the ship is estimated, or roughly computed from the courses run determined by the compass, the length of time run upon each course, the rate of sailing as determined by the log, due allowance being made for drift, leeway, &c.

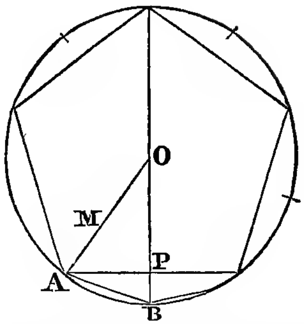
DEC'A-GON. [Gr. δεκα, ten, and γωνια, a corner]. A polygon of ten angles and ten sides. If the sides are all equal, and the angles also all equal, it is a *regular decagon*, and may be inscribed in a circle. If the radius of a circle be divided in *extreme and mean ratio*, that is, so that the greater segment shall be a mean proportional between the whole radius and the smaller segment, then will the greater segment be equal to one side of the regular inscribed decagon. If we denote the radius of the circle by r, and the side of the inscribed decagon by s, we have

$$r : s :: s : r - s, \quad \text{or,} \quad s = \tfrac{1}{2} r (\sqrt{5} - 1).$$

We have also, if we denote the area of the decagon by A,

$$A = s^2 \times 7.694209.$$

1. To inscribe a regular decagon in a given circle. Let O be the centre and OA the radius of the given circle. Divide the radius OA into extreme and mean ratios at the point M. Take OM, the greater segment, and lay it off from A to B; the chord AB will be the side of the regular decagon, and by applying it ten times to the circumference of the circle, the decagon will be inscribed in the circle.



To construct a regular decagon upon a given line as a side: construct, as just explained, a regular decagon inscribed in any given circle, then draw a line parallel to one side, and make it equal to the given line; through one extremity of this line draw a second line parallel to the adjacent side and make it also equal to the given line; through the three points, thus determined, pass a circle and apply the given line ten times as a chord, and the resulting polygon will be the required decagon.

DEC'AGRAMME. [Gr. δεκα, ten, and F. *gramme*, a unit of weight]. A French weight of ten grammes, each gramme being equivalent to 15.438 grains Troy.

DEC'A-LI-TRE. [Gr. δεκα, ten, and F. *litre*, measure]. A French measure of capacity, containing ten litres or 610·28 cubic inches.

DEC-AM'E-TRE. [Gr. δεκα, ten, and μετρον, a measure]. A French measure of length, containing ten metres, or 393.71 English inches.

A *decigram*, *decilitre*, and a *decimetre*, are respectively the tenth part of a gramme, litre and metre.

DEC'I-MAL. [L. *decimus*, tenth, from *decem*, ten]. Any number expressed in the scale of tens is a decimal. The system of arithmetic in which numbers are expressed decimally is called decimal arithmetic. But by the term decimal, a decimal fraction is generally understood.

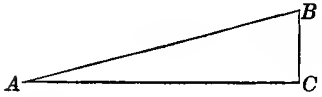
DECIMAL FRACTION. A fraction whose denominator is some power of ten, thus $\frac{2}{10}$, $\frac{3}{100}$, $\frac{5}{1000}$, are decimal fractions. For the sake of brevity it has been agreed, in writing decimal fractions, to omit the denominator,

and to place a point before the numerator in such a manner as to indicate the number of 0's in the denominator. The above examples would be written respectively .2, .03, .003; the number of places of figures which follows the decimal point indicates the number of 0's in the denominator. If there are not a sufficient number of places of figures in the numerator, 0's are prefixed until the whole number of places of figures in the numerator is just equal to the number of 0's in the denominator, and then the point is prefixed. This point is called the decimal point. See *Arithmetic* and *Arithmetical Scale*.

DE-CLI-NA'TION. [L. *declino*, to slope]. A declination circle in spherical projections is a great circle, whose plane passes through the axis of the sphere.

Declination of the Needle, in Surveying, is the same as the *variation of the needle*. See *Variation*.

DE-CLIV'I-TY. [L. *declivitas*, from *declivis*, sloping]. In Topographical Surveying, the slope or inclination of a surface downwards. The term is used in contrast with acclivity, which is the inclination or slope upwards. The measure of the declivity or acclivity of a line may be expressed in degrees, minutes, and seconds, but it is more often expressed by the ratio of the base to the altitude of a right angled triangle, constructed upon any part of the line. Thus the slope of the line



AB, (AC being horizontal), is measured by the ratio $\frac{BC}{AC}$. If BC is $\frac{1}{80}$ of AC, the slope is $\frac{1}{80}$. If through any point on the surface of a hill any number of vertical planes be passed, cutting out vertical sections, these will in general have different declivities. The line of greatest declivity, or the line sought by running water in flowing from one level to another, is the section cut out by that vertical plane which is perpendicular to the tangent plane to the surface at the point.

DE-CRĒASE'. [L. *decresco*; *de*, from, and *cresco*, to grow]. To diminish. When a less number is taken from a greater, the latter is said to be decreased by the former.

Decreasing Function. In Analysis, one quantity is a decreasing function of another, when it decreases as the other increases. Thus, in the equation

$$y = \frac{M}{x},$$

y is a decreasing function of x, because as x is increased y is diminished, and the reverse. A quantity may be a decreasing function of another between certain limits, and an increasing function between other limits. Thus, in the equation

$$y = a\,(b - x)^{2n},$$

y is a decreasing function of x between the limits $x = -\infty$ and $x = b$, but an increasing function between the limits $x = b$ and $x = +\infty$.

The differential co-efficient of a *decreasing function* is always *negative*, whilst that of an *increasing function* is *positive*.

Decreasing Series. A series is decreasing when each term is less than the preceding one. Thus, a geometrical progression is decreasing when the ratio is less than 1. In any series whatever, if the quotient obtained by dividing any term by the preceding is numerically less than 1, the series is decreasing.

DEC'RE-MENT. [L. *decrementum*, from *decresco*, to decrease]. In Calculus, the name given to the quantity which is subtracted from the variable in order to find a preceding state of any given function.

DEC'U-PLE. [L. *decupulus*. Gr. δεκαπλους]. Tenfold, containing ten times as many.

DE-DŪCE'. [L. *deduco*; *de*, from, and *duco*, to lead or draw]. To infer something from what precedes. To draw a conclusion from given premises. The conclusion thus drawn is called a *deduction*, and the method of reasoning is called *deductive*.

DE-DUCT'. [L. *deduco*, to draw out from]. To take from; to subtract.

DE-FECT'ĪVE. [L. *defectivus*, imperfect]. Wanting a marked characteristic.

Defective Hyperbola. A curve having two infinite branches and but one rectilinear asymptote.

Its general equation is

$$xy^2 + ey = - ax^3 + bx^2 + cx + d.$$

DEF'ER-ENT. In the Ptolemaic system of the world the planets are supposed to move in circular orbits, the centres of which are at the same time revolving in other circular orbits. These secondary circles are *deferents* of the first orbits, which are called *epicycles*. The system of deferents and epicycles was invented to explain certain observed phenomena, such as the eccentricity, perigee, apogee, &c. See *Epicycle*.

DE-FĬ''CIENT. [L. *deficio*, to fail]. Wanting; incomplete.

Deficient Numbers, in Arithmetic, are those, the sum of whose aliquot parts is less than the number. Thus, 8 is a deficient number, for $4 + 2 + 1 = 7 < 8$; also, 16 is a deficient number, for $8 + 4 + 2 + 1 = 15 < 16$.

DEF-IN-Ĭ''TION. [L. *definitio*; *de*, from, and *finio*, to limit]. A definition is such an enumeration and description of the attributes of an object, as serve to explain its nature and character, and also to distinguish it from every other thing. Definitions are of two kinds:

First. Those in which it is only intended to explain the meaning of a word, or of a term employed.

Second. Those which, besides explaining the meaning of the word or term employed, imply also that there exists or may exist a thing corresponding to the word.

Definitions which do not imply the existence of corresponding things, that is, definitions of names, are those usually found in the dictionary of a language. They explain the meaning of a term or word by giving some equivalent expression which may happen to be better known. Definitions which also imply the existence of things, do more than this. For example: "A plane triangle is a polygon of three sides." This definition does two things.

1st. It explains the meaning of the word *triangle*.

2d. It implies that there exists, or may exist, a polygon having three sides.

To define a word, when the definition is to imply the existence of a thing, is to enumerate its most obvious and characteristic properties; this requires a thorough knowledge of these properties, in order that we may seize upon those best fitted for the purpose of definition.

In mathematics, and indeed in all exact sciences, names imply the existence of things which they designate, and the definition of those names express one or more attributes of the things named. No correct mathematical definition can be framed which shall not express certain of these attributes of the thing named. Furthermore, every definition in mathematics is a tacit assumption of some proposition which is expressed in the definition, and it is this circumstance which renders these definitions of so much importance.

All reasoning in mathematics which appears to rest ultimately upon definitions, does, in fact, rest upon the intuitive inference that things corresponding to the words defined have a conceivable existence as subjects of thought, and do, or may have, *proximately*, an *actual* existence.

The following rules afford the means of testing a definition and determining upon its value:

1. The definition must be *adequate*, that is, neither too much extended nor too narrow for the word defined.

2. The definition must in itself be *plainer* than the word defined, else it would not define it.

3. The definition should be expressed in a *convenient number of appropriate words*.

4. When the definition implies the existence of a thing the certainty of that existence must be intuitive.

DE-GREE'. [Fr. *degré*, a steep, a grade]. In Algebra the degree of a term is the number of literal factors which enter it, and is denoted by the sum of the exponents of all the literal factors of the term. Thus a^2b^3 is of the 5th degree.

The degree of a power is the number of equal factors which are taken to form the power.

The degree of a radical is the number of times which the radical must be taken as a factor to produce the quantity under the radi

cal sign. The degree of a radical is indicated by its index.

The degree of an equation containing but one unknown quantity, is the greatest number of times which the unknown quantity enters any term as a factor; thus,

$$ax^4 + bx^2 + cx = d,$$

is an equation of the *fourth* degree. The degree is always indicated by the highest exponent of the unknown quantity in any term.

The degree of an equation containing more than one unknown quantity, is denoted by the greatest sum of the exponents of the unknown quantities in any term; thus,

$$mx^2y + nyx = q,$$

is an equation of the *third* degree.

In trigonometry, a degree is the 360th part of the circumference of a circle. The degree serves as a unit of comparison for angles; for, on account of the uniform curvature of the circumference of a circle, the same length of arc in equal circles corresponds to equal angles at the centre; and in different circles, similar arcs correspond to equal angles at the centre. If with the vertex of an angle as a centre, and with any radius whatever, a circle be described, and if the entire circumference be divided into 360 equal parts, beginning at one side of the angle, the number of parts which fall between the sides will be the measure of the angle. This number will be entirely independent of the length of the radius employed.

In the French decimal system of measures, it was proposed to divide the entire circumference of the circle into 400 equal parts, to be called degrees, each of which was to be divided into 100 equal parts or minutes, and hese again were to be subdivided into 100 parts or seconds. In this method, each second of arc would be equal to the millionth part of a quadrant.

This arrangement was adopted by Laplace in his Mécanique Céleste, and by some other distinguished writers at the time it was proposed, but it seems to be nearly abandoned.

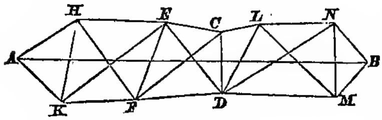
If we call the radius of the circle 1, the length of a degree of the circumference, or the 360th part of the circumference, is 0.00872665. In the French system it is 0.0078535.

Degree of Latitude. On the surface of the earth is the length of a portion of a meridian between two points, whose latitudes differ from each other by one degree. In consequence of the spheroidal figure of the earth, it happens that the length of a degree of latitude is different at different distances from the equator. By measuring the lengths of a degree of latitude at different points on the earth's surface, a knowledge of the true form and real dimensions of our globe may be determined.

The operation is one of great nicety, and its successful execution has drawn largely upon the resources of science. A general idea, however, of the method of proceeding is by no means difficult to conceive. In the first place, in order to obviate the effects of superficial irregularities, we refer all measurements to the level of the sea; that is, we conceive the surface of the ocean to be extended beneath the continents, and to this surface we conceive every operation to be reduced. This being understood, let two stations be selected on the same meridian, or as nearly so as possible, and let the distance between them be measured with the greatest accuracy. This distance should be as great as possible, because any error in the measurements would be less felt in a long line than in a short one: the latitudes of the two stations are next determined, astronomically, with the utmost precision. The difference of the latitudes in degrees, together with the measured distance in yards, will make known the average length of a degree of latitude between the assumed stations. A long series of observations is necessary to find the true latitude of a point, and, since an error of a single second in arc corresponds to about 100 feet, we see the necessity of making the distance between the assumed stations as great as possible. The direct measurement of the distance between two points, might be effected by means of the base apparatus in a country favorable to its use, but it has generally been found more convenient to proceed by means of a system of triangulation.

Let A and B represent the selected points between which the distance is to be determined. A level space CD is selected, upon which the length of a base line CD is measured by the base apparatus used in geodetical surveys. Suitable points, E, F, G, H, &c.,

are selected and marked by signals, so that



lines joining them may form a system of triangles as nearly equilateral as may be; these are taken in such a manner that the vertices of the extreme triangles shall fall at the points A and B. The angles of these triangles are next carefully measured by a theodolite, and, after the proper reductions are made, these measurements give the necessary data for determining the length of the line AB. The very operations which are requisite in determining the length of AB, require a knowledge of the form and dimensions of the earth; but each succeeding operation gives us these elements with greater accuracy, and the measurements already made give them with sufficient accuracy for making the necessary reductions.

The first measurement of a degree of latitude, undertaken on correct principles, was undertaken by Eratosthenes, who lived during the third century before Christ. He determined the distance between Alexandria in Lower Egypt, and Syene in Upper Egypt. The result of this measurement gave for the length of a degree of latitude, 694½ stadia; but from our want of knowledge of the exact length of the ancient stadium, no very precise idea can be formed as to the accuracy of the result.

About the middle of the 16th century, Fernel made a rough measurement between Paris and Amiens, and deduced the length of a degree of latitude 364,960 English feet. In 1635, Norwood measured the distance between London and York, and found the length of a degree to be 367,176 feet. In 1735, the Academy of Sciences at Paris, in order to decide upon the true figure of the earth, resolved to have two arcs of the meridian measured with all the accuracy of modern science. The one of these arcs was to be taken as near the equator, and the other as near the pole, as possible. The site of the former arc was chosen in Peru, and its measurement was committed to the charge of Boguer, Godin, and Condamine. After many difficulties and hardships, the operations were completed at the end of about ten years, and the result of the measurement gave for the length of a degree of latitude at the equator 362,912 English feet. The other arc was located in Lapland, and its measurement was undertaken by Maupertius, Clairaut, and Lemonnier. This party was more fortunate, and accomplished their mission in about sixteen months, and as a result gave for the length of a degree of latitude at the parallel of 66° 20′ 11″, 365,697 English feet.

Since these expeditions, several arcs of meridians have been measured, varying in length from a single degree up to nearly sixteen degrees, and all the results go to show that the figure of the earth is that of an oblate spheroid, whose shorter axis coincides with the axis of rotation, which agrees with the form pointed out by theory. If the material of the earth had been originally fluid, analysis shows that under the action of gravity and the centrifugal force, the surface would differ only in a small degree from that which repeated measurements show to be its actual shape.

Besides the measurements already pointed out, there are some others which seem worthy of a particular mention.

Lacaille, in 1751, measured an arc at the Cape of Good Hope, and in the same year Maire and Boscovich measured one in the Roman States. In 1762, Leisganig measured one in Hungary, and in 1764 one was measured in the United States by Mason and Dixon.

In 1762, the measurement of an arc, extending through the whole of France, from Dunkirk to Barcelona, was undertaken by Mechain and Delambre, for the purpose of determining a basis for a decimal system of weights and measures. This operation was completed with every precaution that science could suggest, and, in point of accuracy, its results stand unrivalled by those of any similar undertaking.

In 1790, the measurement of an English arc was commenced under the direction of the Board of Ordinance; with the advantage of most excellent instruments, and the aid of refined theory, it was pushed from Dunnore in the Isle of Wight to Burleigh Moor in Yorkshire, nearly four degrees.

Two arcs of the meridian have been measured under English superintendence in India. The first extended only about one degree and a half. The second extends about 16 degrees; it was commenced, and about 10 degrees finished by Col. Lambton; the remaining 6 degrees were completed by Capt. Everest.

Besides these, several minor arcs have been measured at subsequent periods in different portions of the world.

For convenience of reference, the following table is inserted, showing the results obtained from some of the most important measurements, and giving the latitude of the middle point of each arc measured, as well as the authority on which the measurement depends. These results are taken principally from an article in Brande's Encyclopedia.

TABLE OF MEASURED ARCS.

No.	Where measured.	Authority.	Length of arc in deg.	Lat. of middle points.	L'gth of deg. in Eng. feet.
			° ′ ″	° ′ ″	
1	Peru, S. America	Bouguer, Godin, and Condamine	3 7 3	1 31 0 S.	362,809
2	India	Colonel Lambton	1 34 56	12 32 21 N.	362,988
3	India	Lambton and Everest	15 57 39	16 8 22 N.	363,040
4	France	Mechain and Delambre	12 22 12	44 51 0 N.	364,644
5	England	Board of Ordnance	3 57 13	52 35 45 N.	365,032
6	Hanover	Gauss	2 0 57	52 32 17 N.	365,301
7	Lithuania	Struve	3 35 5	58 17 37 N.	365,377
8	Sweden	Svanberg	1 37 20	66 20 11 N.	365,697

In the second Indian arc above tabulated, latitudes were determined at six different points of the arc; in the French arc, at seven points; in the English arc, at four points; and in the Lithuanian, at three points; so that the measurement of the Indian arc affords *five* separate determinations of the length of a degree, the French *six*, the English *three*, the Lithuanian *two*. In the remaining arcs only the latitudes of the extremities of the arc were determined. The above table embraces therefore the results of *twenty* distinct measurements of arcs of the meridian.

From these results, by well-known mathematical processes, the following elements of the earth's figure have been deduced.

It is shown that the meridian section approximates very closely to an ellipse, whose conjugate axis coincides with the axis of the earth, the elements of which are as follows:

	English feet.	*English miles.*
Equatorial or transverse axis,	41,843,330,	7924.87.
Polar or conjugate axis,	41,704,788,	7898.63.
Difference of axes	138,542,	26.24.

The difference between the greater and lesser axis divided by the greater, is called the ellipticity of the meridian. If we denote this ellipticity by e, we shall have, from the above results,

$$e = \frac{1}{302.026}.$$

If we designate the length of the equatorial diameter by a, that of the polar diameter by b, the latitude of any place by l, and the length of a degree of latitude at that place by d, we shall have the following formula:

$$d = a(1 - e + 3e\sin^2 l)\,3600 \sin 1'';$$

from which the length of a degree of latitude, anywhere upon the earth's surface, may be computed.

It is to be observed, that the elements as above given, do not exactly agree with those adopted as most correct, by astronomers and men of science.

The terrestrial elements, as given by Bessel, after a complete discussion of all the data, and which have been adopted in this country by the Coast Survey, and by the corps of Topographical Engineers, are as follows:

		English feet.
Equatorial axis	(a)	41,847,194.
Polar axis	(b)	41,707,308.
Ellipticity	$\left(\frac{a-b}{a}\right)$	$\frac{1}{299.66}$.

In connection with these elements, the fol-

lowing formula is used for finding the length of a degree of latitude, at any point of the earth's surface, viz.:

$$D = \overset{\text{ft.}}{364{,}575.579} - \overset{\text{ft.}}{1{,}831.008}\cos 2l$$
$$+ \overset{\text{ft.}}{3.906}\cos 4l + \overset{\text{ft.}}{0.006}\cos 6l;$$

in which D denotes the length of a degree, and l the latitude of the middle point of the degree.

The following table exhibits the length of a degree of latitude at various points on the surface of the earth:

Lat. of middle point.	Length in nautical miles.	Length in statute miles.
20°	59.669	68.779
25	59.706	68.822
30	59.749	68.871
35	59.796	68.925
40	59.847	68.984
45	59.899	69.044
50	59.951	69.104

The length of a degree of longitude at the equator, is 69.160 statute miles.

Degree of Longitude. The 360th part of any circle of latitude. As the circles of latitude vary in length from the equator to the pole, it follows that the length of a degree of longitude will be very different under different parallels of latitude; but as the earth is a surface of revolution, the length of a degree of longitude, on the same parallel of latitude will always be the same.

If we could determine, by measurement, the lengths of several degrees of longitude, in different parallels of latitude, we should have all the data necessary to determine the terrestrial elements already considered. This measurement may be made in a manner entirely similar to that employed in determining the length of an arc of the meridian; but on account of the great difficulty of determining the longitudes of the extreme points of a measured arc, the results of such measurements have generally been unsatisfactory.

If we assume that the figure of the earth is an oblate spheroid, and take Bessel's terrestrial elements, as given in the preceding article, the length of a degree of longitude, in any latitude, may be computed by the following formula, viz.:

$$D' = \overset{\text{ft.}}{365{,}491.098}\cos l - \overset{\text{ft.}}{305.823}\cos 3l$$
$$+ \overset{\text{ft.}}{0.384}\cos 5l;$$

in which D' denotes the length of a degree of longitude, and l the latitude of the parallel in which it is taken.

The following table gives the lengths of a degree of longitude, in statute miles, for every degree of latitude from 20° to 50°, inclusive.

Deg. of parallel.	Length of deg. in stat. miles.	Deg. of parallel.	Length of deg. in stat. miles.
20°	65.015	36°	56.013
21	64.594	37	55.300
22	64.154	38	54.568
23	63.695	39	53.819
24	63.216	40	53.053
25	62.718	41	52.271
26	62.200	42	51 473
27	61.664	43	50.659
28	61.109	44	49.830
29	60.536	45	48.986
30	59.944	46	48.126
31	59.334	47	47.251
32	58.706	48	46.362
33	58.060	49	45.460
34	57.396	50	44.542
35	56.715		

DE MOIVRE'S FORMULA. A name given to the formula

$$(\cos x + \sqrt{-1}\sin x)^m = \cos mx + \sqrt{-1}\sin mx,$$

because first deduced and published by a mathematician named De Moivre. This formula is of so much interest, in a scientific and logical point of view, that we annex the course of reasoning employed in its deduction.

Let $y = \sin x$, and $v = \cos x$; whence

$$dy = v dx, \quad \text{and} \quad dv = -y dx.$$

If both members of the first of these differential equations be multiplied by $\sqrt{-1}$, and the resulting equation be added to the second equation, member to member, there results the equation

$$dv + \sqrt{-1}\,dy = [-y + v\sqrt{-1}]dx \cdot\cdot (1),$$

whence,

$$dv + \sqrt{-1}dy = (y\sqrt{-1} + v)\,dx\sqrt{-1}, \text{ and}$$

$$\frac{d(v + \sqrt{-1}y)}{v + \sqrt{-1}y} = dx\sqrt{-1} \cdot\cdot (2).$$

By integration,

$$l(v + \sqrt{-1}y) = x\sqrt{-1};\ \text{whence},$$

$$v + \sqrt{-1}y = e^{x\sqrt{-1}} \cdots (3).$$

The constant to be added is 0, because when $x = 0$, $y = 0$.

Substituting for v and y their values, we have

$$\cos x + \sqrt{-1}\sin x = e^{x\sqrt{-1}} \cdots (4);$$

or, since the formula is general, for x write mx.

$$\cos mx + \sqrt{-1}\sin mx = e^{mx\sqrt{-1}} \cdots (5).$$

Now, if both members of equation (4) be raised to the m^{th} power, we have

$$(\cos x + \sqrt{-1}\sin x)^m = e^{mx\sqrt{-1}} \cdots (6).$$

Equating the first members of equations (5) and (6), we have the formula sought, viz:

$$(\cos x + \sqrt{-1}\sin x)^m = \cos mx + \sqrt{-1}\sin mx \cdots\cdots\cdots (7).$$

This formula may be made the basis of a system of analytical trigonometry.

Assume the equations,

$$\left.\begin{array}{l}\cos x + \sqrt{-1}\sin x = e^{x\sqrt{-1}}\\ \cos y + \sqrt{-1}\sin y = e^{y\sqrt{-1}}\end{array}\right\}\cdots (8).$$

Multiplying these, member by member, we find

$$\cos x\cos y - \sin x\sin y + (\sin x\cos y + \sin y\cos x)\sqrt{-1} = e^{(x+y)\sqrt{-1}} \cdot (9).$$

But, from Demoivre's formula,

$$\cos(x+y) + \sin(x+y)\sqrt{-1} = e^{(x+y)\sqrt{-1}} \cdot (10),$$

and, since the second members of (9) and (10) are the same, the first members must be equal; hence,

$$\cos x\cos y - \sin x\sin y + (\sin x\cos y + \sin y\cos x)\sqrt{-1} = \cos(x+y) + \sin(x+y)\sqrt{-1} \cdot (11).$$

In order that equation (11) may be satisfied, the real terms and the imaginary terms in the two members must be separately equal to each other; hence,

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \cdot (12),$$
$$\sin(x+y) = \sin x\cos y + \sin y\cos x \cdot (13).$$

From these formulas, all of the formulas of trigonometry may be deduced analytically.

This is but a single application of the formula; many others might be given. It may not be inappropriate to call the attention of the reader to the fact, that formulas (12) and (13) have been deduced from reasonings based wholly on imaginary expressions. The resulting formulas are rigorously true, as may be shown by direct reasoning upon the lines themselves. The results being true, the question arises as to the logic of the demonstration made use of. The only answer that can be given to that question is, that imaginary expressions do represent quantities, that they do admit of logical interpretation, and that the name *imaginary* as applied to them is erroneous, if understood to mean that they have no existence, and is calculated to deceive the reader with respect to their true logical interpretation.

DEM-ON-STRĀ'TION. [L. *de*, from, and *monstro*, to show]. A demonstration is a course of reasoning brought to a conclusion. The object of a demonstration is always to show that a certain result is the necessary consequences of assumed premises.

In every mathematical demonstration, the assumed premises are *definitions*, *axioms*, and *previously established propositions*; besides these, hypotheses are often made, which aid in the reasonings employed. The arguments are the links which connect the premises logically with the conclusion or ultimate truth to be proved.

The nature of a demonstration is the same in all branches of mathematics. Two kinds of demonstration are, however, distinguished, which differ as to the method of reaching the conclusion—the *direct* and the *indirect*—the latter of which involves what is usually styled the *reductio ad absurdum*. These are also sometimes called *positive* and *negative* demonstrations.

In the *direct* method of demonstration, the premises are definitions, axioms, and previously established propositions. In this method, by a process of logical argumentation, the quantities, of which something is to be proved, are shown to have the marks of that something; that is, they are shown to fall under some definition, axiom, or proposition, already proved.

The indirect method consists in assuming an hypothesis of such a nature that either it or its opposite must be true. The assumed hypothesis is then made a premise and compared, by a process of logical reasoning, with definitions, axioms, and established proposi-

tions, and the reasoning continued until a conclusion is arrived at, which either agrees or disagrees with some known truth. Now, if the conclusion agrees with a known truth, the hypothesis is said to be established or proved, but if it disagrees with a known truth, the hypothesis is disproved, and its contrary is necessarily established. Many of the demonstrations of geometry, and a large share of those of algebra, belong to the latter class.

The demonstrations in mathematics afford examples of the most perfect application of the principles of pure reason to the development of truth. The ideas employed are clearly defined by a fixed, certain, and definite language; the only axioms assumed, are those which are universally true, and to which the mind necessarily assents from its very constitution; in the course of the reasoning, every link in the chain of argument is clearly connected with some well established truth, by the infallible rules of logic, and the conclusions deduced are irresistible.

We shall give an illustration of the two methods of demonstration, chosen from the simplest propositions of elementary geometry.

1st. As an example of the direct method, let us take the first proposition in Legendre's Geometry:

" *If one straight line meet another straight line, the sum of the two adjacent angles will be equal to two right angles.*"

Let the straight line DC (next figure) meet the straight line AB in the point C; then will the sum of the adjacent angles, DCA and DCB, be equal to two right angles.

To prove this proposition, we assume the definition of a right angle, viz:

" *If a straight line meets another straight line, making the adjacent angles equal to each other, each angle is called a right angle, and the first line is perpendicular to the second.*"

We also assume the following axioms:

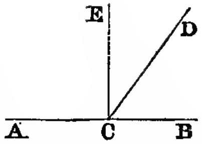
1. *Things which are equal to the same or to equal things, are equal to each other.*

2. *A whole is equal to the sum of all its parts.*

3. *If the same or equal things be added to equals, the sums will be equal.*

We also assume the postulate,

That a perpendicular can always be drawn to a given straight line at a given point.



Let a perpendicular CE be drawn to the line AB at the point C; then from the definition both ECA and ECB are right angles. From the second axiom ACD is equal to ACE plus ECD; and from the third axiom, ACD plus DCB is equal to ACE plus ECD plus DCB; but ECD plus DCB is, from the second axiom, equal to ECB; and from the first axiom we finally have ACD plus DCB equal to ACE plus ECB; that is, equal to two right angles, which agrees with the enunciation of the proposition.

In this demonstration the premises are axioms, a definition and a postulate; the final conclusion is reached by a course of direct reasoning.

As an example of the indirect method of demonstration, we shall select the second proposition of Legendre.

" *Two straight lines which have two points in common, coincide throughout their whole extent, and form one and the same straight line.*"

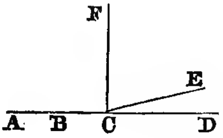
To prove this proposition, we shall assume in addition to the axioms, definition, and postulate already employed, the result of the preceding demonstration and the additional axioms.

4. *Between two given points only one straight line can be drawn.*

5. *If the same or equal things be subtracted from equals, the remainders will be equal.*

Let A and B be common points of the two given straight lines.

In the first place, from axiom fourth, the two lines must coincide between the given points A and B.



Let us suppose that beyond B they begin to separate at some point, as C, and that the first line takes the direction CD and the second one the direction CE. At C let CF

be drawn perpendicular to AC. Since ACE is a straight line, and since the line FC meets it at C, from the preceding proposition we have ACF + FCE, equal to two right angles.

Since ACD is a straight line, and the line FC meets it at C, we have from the preceding proposition,

$$ACF + FCD,$$

equal to two right angles. From the first axiom,

$$ACF + FCE = ACF + FCD.$$

Now from the fifth axiom we have

$$FCE = FCD,$$

which is manifestly absurd, since a part can never be equal to the whole. Hence, the hypothesis made that the lines begin to separate at a point, is untrue; but if they do not begin to separate at a point they must coincide throughout their whole extent, which proves the proposition. In this method of demonstration we introduced the hypothesis that the lines did actually begin to separate at a point, and this led to a conclusion which was absurd, because contrary to a known truth. This is called the *reductio ad absurdum*.

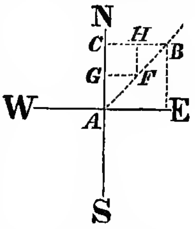
DEN'A-RY SCALE. A uniform scale whose ratio is ten. See *Arithmetical Scale*.

DE-NOM'IN-ĀTE. [*de*, from, and *nomino*, to name]. That which may be named or specified. A denominate quantity is one whose unit of measure is a concrete quantity, as 7 feet, 8 pounds, &c. Or it is one in which the value of the unit of the quantity is named. The term is opposed to *abstract quantity*. Thus 7 lbs. and 10 feet are denominate numbers; n feet or m pounds are denominate quantities; whilst 7 and 10, n and m are, respectively, abstract numbers or quantities.

DE-NOM'IN-Ā-TOR, in Arithmetic and Algebra, is that term of the fraction which indicates the value of the fractional unit. In the fraction $\frac{3}{7}$, 7 is the denominator, and indicates that the fractional unit is $\frac{1}{7}$; in all cases 1 divided by the denominator is the unit of the fraction. In its primitive signification, the denominator is what we have defined it above, but in generalizing the language of algebra the term *denominator* is applied to that part of any expression under a fractional form, which lies below the horizontal line, signifying division. In this sense the denominator is not necessarily a number, but may be any expression, either positive or negative, real or imaginary.

The denominator of a decimal fraction is generally suppressed, and its value is indicated by means of the decimal point, and is always equal to 1 followed by as many 0's as there are places of figures immediately following the decimal point; thus, the denominator of the decimal fraction .0076 is 10,000. See *Decimal*.

DE-PÄRT'URE. [L. *de*, from, and *partio*, to separate]. In Surveying, the departure of a course is the distance between two meridians drawn through its extremities. Thus, if AB represent a course, NS and BE meridians drawn through its extremities, then is AE equal to the departure. In plane surveying, on account of the shortness of the courses in comparison with the radius of the earth, we may regard the two meridians NS and BE as parallel. If we designate the bearing NAB of the course AB, by ϕ, the length AB by l, and the departure by d, we shall have from the right angled triangle ABE, the formula

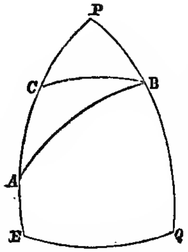


$$d = l \sin \phi.$$

This formula may be used in computing a table of latitudes and departures. See *Traverse Table*. The departure of a course is always equal to the double meridian distance of the middle point of the course when referred to the meridian through the extremity. If the course makes *Easting*, the departure is regarded as *positive*; if *Westing*, it is *negative*.

DEPARTURE OF A COURSE, in Navigation, is the distance between the meridians through the extremities of the course, expressed in degrees. Let P represent the Pole of the earth, EQ an arc of the Equator, AB a course supposed to be an arc of a great circle, PB and PE meridians through its extremity; then is EQ the departure expressed in degrees. In this case the departure is equal to the difference of longitude of the extreme points

of the course. In the spherical triangle BPA, it is plain that from the latitudes of the points A and B, and their difference of longitude, we may compute the length AB of the course. If we know their latitudes and the length of the course, we may compute their difference of longitudes. For a more complete discussion of this subject, see *Navigation*.



DE-PRES'SION. [L. *de*, from, and *premo*, to press]. Depression of equations in algebra is the operation of reducing the degree of an equation. This is effected by dividing both members by a divisor that will divide them without a remainder, and the operation depends upon the principle that if a is a root of an equation, the second member of which is 0, then will the first member be divisible by the unknown quantity minus a. If we know one or more roots of an equation, its degree may be diminished by as many units as there are known roots, by continually dividing both members by the corresponding binomial factors.

An equation may always be depressed to one of a lower degree in either of the following cases:

1st. When the equation contains equal roots; for the manner of effecting the operation in this case, see *Equal Roots*.

2. When two of the roots are numerically equal, with contrary signs, as $+a$ and $-a$.

Let us suppose that we know that the equation

$$x^5 - 3x^4 - 17x^3 + 27x^2 + 52x - 60 = 0 \quad (1)$$

has two roots equal with contrary signs. Then, we may, without destroying the equality, change $+x$ into $-x$, which gives the equation

$$-x^5 - 3x^4 + 17x^3 + 27x^2 - 52x - 60 = 0;$$

or,

$$x^5 + 3x^4 - 17x^3 - 27x^2 + 52x + 60 = 0 \quad (2).$$

If we apply the process for finding the greatest common divisor to the first members of equations (1) and (2), we shall find that they have one, $x^2 - 4$, which, placed equal to 0, gives $x = +2$, and $x = -2$; if we divide both members of the given equation by $x^2 - 4$, we shall find for the depressed equation

$$x^3 - 3x^2 - 3x + 15 = 0.$$

In like manner, any equation of this kind may be depressed.

3. A *reciprocal equation*, that is, one in which one root is the reciprocal of another, may be depressed.

Every reciprocal equation may be reduced to the form

$$x^m + Px^{m-1} + Qx^{m-2} + \cdots + Qx^2 + Px + 1 = 0,$$

in which the co-efficients of terms equally distant from the extremes of the first member, are numerically equal. Conversely, every equation of this form, or which can be reduced to this form, is a reciprocal equation.

In the first place, if m is odd and the last term is $+1$ or -1, it may be reduced to a reciprocal equation of an even degree by dividing both members by $x + 1$ or $x - 1$, as the case may be.

If m is even and equal to $2n$, we first divide both members by x^n, and then substitute for $x + \frac{1}{x}$, a new unknown quantity y, the resulting equation in y will be of the n^{th} degree only.

For example, let the reciprocal equation be

$$x^5 - 6x^4 + 5x^3 + 5x^2 - 6x + 1 = 0.$$

Dividing both members of the equation by $x + 1$, we have the reciprocal equation

$$x^4 - 7x^3 + 12x^2 - 7x + 1 = 0.$$

Dividing both members of the last equation by x^2, and arranging,

$$x^2 + \frac{1}{x^2} - 7\left(x + \frac{1}{x}\right) + 12 = 0,$$

in which, if we substitute y for $x + \frac{1}{x}$, and reduce, we shall find

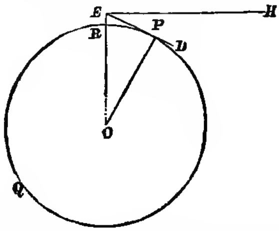
$$y^2 - 7y + 10 = 0,$$

which may be solved; therefore, all the roots of the given equation of the fifth degree may be determined.

Depression of the Visible Horizon: or, Dip of the Horizon. The angle of depression included between a horizontal line and a line drawn through the eye, tangent to the surface of the earth.

Let PQ represent a vertical section of the earth's surface, E the position of the eye,

EH a horizontal line, and ED a line through the eye, tangent to PQ; then is the angle of depression, DEH, called the *depression* or *dip*



of the horizon. This angle will evidently depend for its value upon the radius OP of the earth, and the height ER of the eye above the earth's surface. In the right-angled triangle EOP, since EO is perpendicular to EH, and OP perpendicular to ED, the angle EOP is equal to the dip.

Designating the height of the eye above the surface of the earth by h, and the radius by R, we have the hypothenuse OE equal to $R + h$, and from the principles for solving right-angled triangles, we have the following formula:

$$\cos \text{EOP} = \frac{R}{R + h},$$

from which, by substituting for R its constant value, and for h different values from 1 foot upwards, a table may be computed. The following table exhibits the dip for different heights of the eye from 1 to 100 feet.

Height of eye in feet	Dip	Height of eye in feet	Dip	Height of eye in feet	Dip
1	0′ 58″	13	3′ 27″	26	4′ 52″
2	1 21	14	3 36	28	5 5
3	1 40	15	3 42	30	5 15
4	1 56	16	3 50	35	5 39
5	2 9	17	3 57	40	6 4
6	2 21	18	4 4	45	6 27
7	2 33	19	4 11	50	6 46
8	2 44	20	4 17	60	7 25
9	2 53	21	4 23	70	8 1
10	3 2	22	4 30	80	8 34
11	3 10	23	4 36	90	9 6
12	3 19	24	4 42	100	9 35

The above table is principally of use in making observations at sea. In consequence of the depression of the visible horizon, due to the elevation of the eye above the surface of the sea, it follows that all angles of elevation, referred to the sea horizon, will be too great by the dip; hence, every such measured angle must be diminished by the value of the dip taken from the above table.

The following practical rule affords a very good approximate value for the dip: "*Take the square root of the number of feet which the eye is above the surface, and multiply the result by* 1.063; *the product will express the number of minutes in the dip.*"

The actual dip observed is less than that given by the formula by about $\frac{1}{14}$ of the theoretical dip; this is again affected by the temperature of the sea; when the sea is warmer than the air, the dip is greater than the theoretical value, and when it is colder the reverse obtains.

DER-I-VĀ′TION. [L. *derivatio*, derivation]. The operation of deducing one function from another according to some fixed law, called the law of derivation.

The operations of differentiation and integration are examples of derivation. The function operated upon is called the *primitive* function, and the resulting function is called the *derivative* or *derived* function.

The general method of derivations has for its object the discovery of the law of relation between primitive and derived functions. Arbogast has investigated the general subject of derivations as a separate branch of analysis, and has treated it at some length under the name of *Calculus of Derivations*.

DERIVATIONS, CALCULUS OF. A name given by Arbogast to a method of developing functions into a series, by the aid of certain general formulas deduced from the principles of the Calculus of operations.

The principle which lies at the bottom of the Calculus of derivations is, that if any operation be performed upon an expression, the *form* of the result will be entirely independent of the nature of the expression to be operated upon.

The binomial formula, as an illustration of this principle, is,

$$(x + a)^m = x^m + max^{m-1} + m\frac{m-1}{2}a^2x^{m-2} + \&c. \cdots$$

in which each term is derived from the pre-

ceding, in accordance with a fixed law, that remains the same whatever may be the nature of the expressions a, x and m.

Another principle is, that all symbols of operation are distributive; that is, when an expression is to be operated upon by several different processes, it is entirely immaterial in what order the operations are to be performed. As a familiar example of this principle, we may instance the fact that if a given expression x is to be multiplied by m, and the result divided by n, it is entirely immaterial whether we multiply the expression x by m, and divide the result by n, or we divide the expression by n and multiply the result by m, or finally conceive both operations to be performed together.

In the Calculus many instances of this distributive nature of symbols of operation occur.

Thus, if an expression has to be differentiated, and the integral of the result taken, the same effect may be produced by integrating the expression and then taking the differential of the result.

Also, if an expression is to be differentiated, first with respect to one variable, and that result differentiated with respect to a second variable, we may, without affecting the final result, differentiate first with respect to the second variable, and then differentiate the result with respect to the first variable; thus,

$$\frac{d\left(\frac{du}{dy}\right)}{dx} = \frac{d\left(\frac{du}{dx}\right)}{dy}$$

Instances of this principle occur in every branch of mathematics; indeed, every combination of symbols of pure operation must necessarily be distributive.

One great advantage of Arbogast's method of development, is the system of notation employed. In this respect he has extended and generalized the methods before employed in particular cases. As an example of his system of notation, we may instance his indicated method of developing the expression $(a + x)^m$. Now whatever operation may be indicated by m, that is, whether m is positive, negative, entire, fractional, real or imaginary, the form of the resulting development will be the same. The terms of the series will involve the successive integral powers of x; commencing at the 0 power, the first term of the series is always a^m, and the second term is always of the form max^{m-1}; the co-efficient of the second term is found by multiplying the first term by the exponent of a in that term, and then diminishing the exponent of a by 1 in the product. Let D be assumed to denote this operation; then will Da^m denote ma^{m-1}, Da^{m-1} will denote $(m-1)a^{m-1}$, Da^{-n} will denote $-na^{-n-1}$, and so on. D^2a^m denotes that the operation is to be performed twice in succession upon a^m, or

$$D^2a^m = m(m-1)a^{m-2},$$

and in like manner

$$D^3a^m = m(m-1)(m-2)a^{m-3},$$

and generally,

$$D^na^m = m(m-1)(m-2)\cdots(m-n+1)a^{m-n}.$$

The formula for development will then become

$$(a+x)^m = a^m + Da^mx + \frac{D^2am}{1.2}x^2 + \frac{D^3a^m}{1.2.3}x^3 + \cdots \frac{D^na^m}{1.2.3.n} + \&c.$$

By means of this system of notation M. Arbogast deduces a general formula for the development of

$$f(a + bx + cx^2 + \cdots + sx^n + \cdots),$$

in which f denotes any function whatever. When the indicated operations are actually to be performed, he shows how to deduce the co-efficients in a simple manner, in terms of the constants which enter the given functions.

He next deduces the general form of the development of any functions of two or more polynomials arranged with reference to the ascending powers of a single variable, and consequently deduces a method of finding the form of the product of two or more series arranged with reference to the same variable.

He finally shows the form of the development of any function of two or more polynomials arranged with reference to the ascending powers of two or more variables.

The application of these principles serves to simplify the investigations to be made in discussing the theory of equations, the form and nature of series, the doctrine of chance, of multiple arcs, the reversion of series, &c.

DE-RĪVED′ POLYNOMIAL. In Alge

bra, a polynomial which is derived from a given polynomial which is a function of one unknown quantity, as x, by *multiplying each term by the exponent of the unknown quantity in that term, diminishing the exponent of the unknown quantity in the product by* 1, *and taking the algebraic sum of the results.*

Thus, the derived polynomial of

$$x^5 + 4x^4 + 3x^2 + 2x + 1 \quad \text{is}$$

$$5x^4 + 16x^3 + 6x + 2 \cdot \cdot$$

It is plain that derived polynomials are nothing else than differential co-efficients, as explained in Calculus. In Algebra, the only polynomials considered, are those which are entire functions of x; that is, those in which the exponent of the unknown quantity in each term is a whole number.

The second derived polynomial of a given polynomial, is the derived polynomial of the first derived polynomial.

The third derived polynomial, is the derived polynomial of the second derived polynomial, and so on.

The derived polynomials, taken in their order, are called *successive derived polynomials.* Their number is equal to the highest exponent of the unknown quantity in any one term.

In the case already considered, we have the successive derived polynomials as follows:

$x^5 + 4x^4 + 3x^2 + 2x + 1$ . .	given polynomial.		
$5x^4 + 16x^3 + 6x + 3$	1st derived polynomial.		
$20x^3 + 48x^2 + 6$ . . .	2d	"	"
$60x^2 + 96x$	3d	"	"
$120x + 96$	4th	"	"
120	5th	"	"

When the co-efficient of the highest power of the unknown quantity is 1, the last derived polynomial consists of but a single term, and is equal to the continued product of the natural numbers from 1 up to this exponent inclusively. In the case taken, we have $120 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, as is evident.

If the given polynomial is placed equal to 0, and then resolved into its binomial factors of the first degree with respect to the unknown quantity, then will the first derived polynomial be equal to the algebraic sum of the quotients obtained by dividing the given polynomial by each of these factors: that is, if we designate the given polynomial by X, its binomial factors, supposed to be m in number, by $x - a$, $x - b$, $x - c \ldots x - l$, and the successive derived polynomials by Y, Z, W, V, &c., we shall have

$$Y = \frac{X}{x-a} + \frac{X}{x-b} + \frac{X}{x-c} + \cdot\cdot + \frac{X}{x-l}.$$

The second derived polynomial divided by $1 \cdot 2$, is equal to the sum of the quotients obtained by dividing the given polynomial by all the different products of the binomial factors, taken in sets of 2; that is,

$$\frac{Z}{1 \cdot 2} = \frac{X}{(x-a)(x-b)} + \frac{X}{(x-a)(x-c)} + \frac{X}{(x-b)(x-c)} + \ldots + \frac{X}{(x-k)(x-l)}.$$

In like manner, we shall have the equations

$$\frac{W}{1 \cdot 2 \cdot 3} = \frac{X}{(x-a)(x-b)(x-c)} + \frac{X}{(x-a)(x-b)(x-d)} + \cdot + \frac{X}{(x-h)(x-k)(x-l)};$$

and so on for the remaining successive derived polynomials.

DE-SCEND′ING SERIES. [L. *de*, from, and *scando*, to climb]. A series is descending, when each term is numerically less than the preceding one: thus, the progression

$$8 : 4 \quad 2 : 1 : \&c.,$$

is a descending series.

DE-SCRĪBE′. [L. *de*, from, and *scribo*, to write]. To delineate or mark the form of a figure In Geometry, the term describe is used as nearly synonymous with construct: thus, to describe a circle, is the same as to construct a circle, and so on.

DE-SCRĪB′ENT. In Geometry, is the same as the generatrix. In case of a line, the describent is a point; and of a surface, it is a line. See *Generatrix.*

DE-SCRIP′TĪVE GEOMETRY. That branch of geometry which has for its object the graphic solution of all problems involving three dimensions, by means of projections upon auxiliary planes. Two auxiliary planes are usually employed, called *planes of projection.* The one is taken horizontal, and is called the *horizontal plane of projection;* the other is for convenience taken vertical, and called the *vertical plane of projection;* the

intersection of these planes is called the *ground line*. By their intersection, the planes of projection form four diedral angles, which are numbered as follows: the first lies above the horizontal, and in front of the vertical plane; the second lies above the horizontal, and behind the vertical plane; the third lies directly below the second; and the fourth directly below the first.

In making a drawing upon a sheet of paper, according to the method of descriptive geometry, we take the plane of the paper to coincide with the horizontal plane of projection, and conceive that the vertical plane has been revolved about the ground line until that part which lies above the horizontal plane shall have coincided with the part of the horizontal plane beyond the ground line. Then the projections of all points upon the vertical plane which fall above the horizontal plane will, in the revolved position, be found above the ground line, and the reverse.

In this system, points, lines and surfaces are given by their projections, two of which are, in general, sufficient to fix the position of these elements in space.

The projection of a point upon a plane, is the foot of the perpendicular drawn from the point to the plane; the perpendicular is called the *projecting line* of the point. If the projection is made upon the vertical plane, it is called the *vertical projection;* if upon the horizontal plane, the *horizontal projection* of the point, the projecting lines receive corresponding names. To show that the two projections of a point upon the planes of projection determine the position of the point with respect to the planes of projection: Let a perpendicular to the horizontal plane be erected through the horizontal projection; it will contain the point from the definition of a projection; again let a perpendicular to the vertical plane be erected through the vertical projection of the point; this will also, for a like reason, contain the point, and since the two straight lines both contain the point, it must be found at their point of intersection; and since they can intersect in but one point, this point will be completely determined with respect to the two planes. If now we pass a plane through the two projecting lines, it will be perpendicular to both planes of projection, and to their common intersection, the ground line. Its intersection with the horizontal plane will also be perpendicular to the ground line, and when the vertical plane is revolved about this line, to coincide with the horizontal plane, the vertical projection of the point will continue in this plane; and after the revolution, the vertical and horizontal projections of the point will both be found in the same straight line perpendicular to the ground line.

The projections of a line are made up of the projections of all its points, and since the projecting lines of these points, with respect to each plane, are parallel, they make up the surface of a plane when the given line is straight, and of a cylinder when it is curved. Hence, the projection of a line upon any plane, is the intersection of the plane with a cylindrical surface passed through the line, and having its elements perpendicular to the plane. If the projection is made upon the horizontal plane, it is called the *horizontal*, if upon the vertical plane, it is called the *vertical projection* of the line.

Since the projections of a point determine the position of the point, it follows that the projections of a line determine its position with respect to the planes of projection.

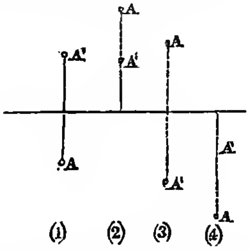
A surface may be represented in projection in two ways—either by the projection of some of its principal elements, or by passing a cylinder tangent to or enveloping the surface, and whose elements shall be perpendicular to the plane of projection; in this case, the intersection of the cylindrical surface with the plane of projection gives the contour of the projection upon the plane: the enclosed area is the projection, and is named from the plane upon which it is made.

By the aid of these simple conventional principles, a great variety of problems may be solved, which are immediately applicable in architecture, sculpture, painting, civil and military engineering, fortification, &c.

With reference to the projections of a point, it is to be observed, that when the point occupies different positions in space, with respect to the planes of projection, its projections will also have different positions with respect to the ground line.

If the point is in the first angle, its horizontal projection will be in front, and its vertical projection behind the ground line; as

(A, A′), A denoting the horizontal, and A′ the vertical projection of the point (1).



If the point is in the second angle, its projections will be situated as in (2). If the point is in the third or fourth angle, its projections will be situated as represented in (3) and (4). If it is in the horizontal plane, it will be its own horizontal projection, and its vertical projection will be in the ground line. If it is in the vertical plane, it will be its own vertical projection, and its horizontal projection will be in the ground line. If it is in the ground line, both projections coincide with the point itself.

With respect to lines, it is to be remarked:

1st. Both projections of a straight line are straight lines: if a given straight line be parallel to either plane of projection, its projection on that plane will be parallel to the line itself, and its projection in the other plane will be parallel to the ground line. If a given straight line is perpendicular to either plane of projection, its projection on that plane will be a point, and its projection on the other plane will be perpendicular to the ground line. If a straight line is parallel to the ground line, both projections will be parallel to the ground line: the projection of a limited straight line upon either plane, will be equal to the line itself when the line is parallel to the plane of projection; in all other cases, it will be less than the line.

2d. A curve line will be projected into an equal curve when it is in a plane parallel to the plane of projection, and the other projection of the line will be parallel to the ground line. The projection of a plane curve will be a straight line when the plane of the curve is perpendicular to the plane of projection, and both projections of a plane curve will be straight lines when the plane of the curve is perpendicular to the ground line; in that case, the projections of the line do not determine its position, since all curves in such a plane are projected into the same straight line perpendicular to the ground line.

The art of descriptive geometry, having for its object nothing more than the application of the principles of mathematics to the solution of a particular class of practical problems, a sufficient degree of accuracy may be attained by regarding curved lines as polygons having a great many sides, which are very small, these polygons being inscribed in the curves considered. The greater the number of points taken, the nearer will these polygons approach to a coincidence with the curves. If the number of points be greater than any assignable number, the polygons will differ from the curves in no sensible degree. In like manner, curved surfaces are regarded as inscribed polyhedral surfaces, which approximate more or less closely to the surface in question, as the number of faces is increased. If the number of faces be regarded as infinite, the polyhedral surfaces will differ from the one considered in no sensible degree.

In this point of view, we regard the prolongation of one of the sides of the polygon as a tangent to the line, and the prolongation of one of the plane faces of the polyhedral surface as a tangent plane to the surface.

For the purposes of definition, we regard the points in the former case as very near to each other, and in the latter case we consider the polyhedral faces very small.

Hence the following definitions: *A tangent line to a curve is a straight line, which passes through two consecutive points of the curve. A tangent plane to a surface is a plane which has at least one point in common with the surface, through which, if any secant planes be passed, the straight line cut from the plane will be tangent to the curve cut from the surface.*

If a curve is in a plane, the tangent at any point of it will lie wholly in that plane. If two curves have two consecutive points in common, they are tangent to each other at the first point; for, a straight line through these consecutive points is tangent to both at the common point. If two lines are tangent, their projections are tangent; for, the projections of the common consecutive points will be consecutive, and will be found at the

same time in the projections of both curves. In descriptive geometry, lines are divided into three classes.

1st. Straight lines, which do not change their directions between any two of their points.

2d. Curves of single curvature, or curves all of whose points lie in the same plane; and,

3d. Curves of double curvature, or curves all of whose points do not lie in the same plane. In a curve of double curvature, a plane may be passed through any three points, whether consecutive or not, but four consecutive points cannot lie in the same plane.

Curves are regarded as being generated by points, moving according to some fixed law, or as resulting from the intersection of surfaces which are generated by a line, either straight or curved, moving according to some fixed law.

In generating a surface, by moving a line, several cases may occur; this gives rise to the classification of surfaces as follows:

1st. Plane Surfaces, which may be generated by a straight line, moving in such a manner as to touch a given straight line, and continue parallel to its first position.

2d. Single Curved Surfaces, which may be generated by a straight line, moving in such a manner that its consecutive positions shall lie in the same plane.

3d. Double Curved Surfaces, which can only be generated by a curved line; and,

4th. Warped Surfaces, which may be generated by a straight line, moving in such a manner that its consecutive positions shall not lie in the same plane.

In the generation of surfaces, the moving line is called the *generatrix*, the lines which it constantly touches are called directrices, and any position of the moving line is called an element of the surface.

The problems usually referred to descriptive geometry, may be included under one of the following heads:

1st. To pass a plane tangent to a surface, or to draw a tangent to a curve.

2d. To find the line of intersection of two surfaces; and,

3d. To develop a surface upon a plane.

The general problem of passing a plane tangent to a given surface, at a given point, may be solved as follows:

Through the given point pass two planes, which shall cut from the surface two lines intersecting each other at the given point; draw straight lines tangent to the curves of intersection at the given point, and through them pass a plane: it will be the tangent plane required. The tangent lines may be constructed by any of the methods described in practical geometry.

To find the line of intersection of two surfaces, we pass auxiliary surfaces intersecting the given surfaces; each auxiliary surface will cut the given surfaces in lines, and the points in which these lines intersect, will be points of the required curve. The auxiliary surfaces are to be chosen in such a manner, that the lines cut from the given surfaces shall be the simplest elements possible. The right line is the simplest element, then the circle, and after these the conic sections.

If it is required to draw a tangent line to the curve of intersection of two surfaces, at any point, it may be done as follows:

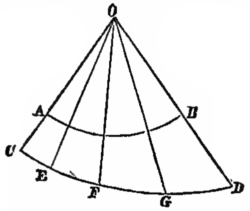
Pass two planes, one tangent to each surface at the common point; their intersection will be tangent to the line of intersection at the given point.

To develop any surface on a plane: No surface can be developed on a plane, unless it belong to the class of single curved surfaces; for, if it is a warped surface, we know, by definition, that no two consecutive elements can be made to lie in the same plane: therefore, it cannot be developed or rolled out upon a plane surface. If a double curved surface be laid upon a plane, it will only touch it in a point, as in the case of a sphere; and if it be rolled along the successive points of contact, will trace out a line upon the plane. In the case of a single curved surface, however, if it be laid upon a plane, two consecutive elements will lie in the plane; and if the surface be rolled along as each preceding element is lifted out from the plane, a succeeding one is brought into it, and so on. Finally, after every element of the surface to be developed has touched the plane, the portion of the plane touched is equal in area to the surface, and is called the development of the surface.

The surfaces which can be developed in this way are conical surfaces, cylindrical surfaces, and a third class of single curved sur-

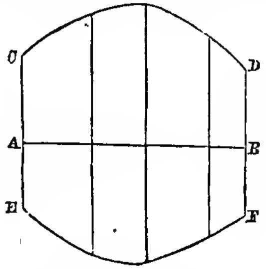
faces, which may be generated by moving a straight line in such a manner as to remain constantly tangent to a curve of double curvature.

To develop a conic surface, we first suppose a sphere to be described, whose centre is at the vertex of the cone: then, if the cone be laid upon a plane, the intersection of the sphere and cone will develop into a circle whose radius is equal to that of the sphere: the points in which any element of the cone intersects this in development, may be easily found, and therefore, the developed position of that element drawn; having its position, its length is next found and laid off; the line



uniting the extremities A, E, F, G, D, of these lines is the development of the base, and the area OCD is the development of the surface.

In order to develop the surface of a cylinder, we conceive it to be intersected by a plane perpendicular to its elements: then, if the cylinder be laid upon a plane, and rolled over, the intersection of the cutting plane and cylinder will develop into a straight line AB, and the point in which it cuts any ele-



ment may be found. Through this point draw a straight line, and on it measure a distance, each way equal to the distance from the point to each base: the line thus constructed will be the developed position of one element. Having found a sufficient number of developed elements, draw the lines CD and EF through their extremities, and these, together with the extreme elements, will limit the development of the surface. The development of the third class of single curved surfaces is more difficult and of less practical importance than those which we have considered. See *Development of Surfaces.*

The principles of Descriptive Geometry are of immediate application in the subjects of Shades, Shadows and Perspective, Spherical Projections, and Stone-cutting,—which see.

DE-SĪGN′. [L. *de*, from, and *signo*, to seal or stamp]. A term which is sometimes, though improperly, used as synonymous with drawing. Design is the mental conception of the artist of any particular subject, and an expression of the conception may take place through the medium of a drawing, of a piece of statuary, or in any other work of art.

The arts which are generally called arts of design, are painting, sculpture, and architecture.

DE-TERM′IN-ATE. [L. *determinatus*, limited]. That which has limits. In opposition to that which is without limits.

Determinate Equation. One which admits of a finite number of solutions. Every equation which contains but one unknown quantity, and which is not identical, is determinate.

If a group of equations be independent of each other, and equal in number to the number of unknown quantities which they contain, the group is determinate, and there will be but a finite number of sets of values for the unknown quantities.

Determinate Geometry. That branch of geometry which has for its object the solution of determinate problems. The solution is usually effected by means of algebraical analysis. See *Analytical Geometry.*

Determinate Problems. Those which admit of a finite number of solutions. In every determinate problem, the given conditions determine the number, and afford the means of finding the required parts.

Determinate problems may usually be solved analytically by the following rule:

Conceive the problem solved, and draw a figure whose parts shall respectively represent the given and required parts of the problem. Draw such other lines as may be required to establish the relations between the known and required parts; denote the known parts by the leading letters of the alphabet, and the required parts by the final letters; consider the relations between the known and unknown parts, and express these relations by means of equations, of which there must be as many as there are required parts; combine the equations, and find the values of the unknown quantities; construct these values, and the problem is solved. See *Application of Algebra to Geometry*.

DETERMINATE QUANTITY. One which admits of but a finite number of values. Thus, in an equation containing but one unknown quantity, that quantity is said to be determinate.

DE-VEL'OP. [Fr. *developper*, to unfold]. In algebraic language, to develop an expression is to change its form by the execution of certain indicated operations, without changing the value of the expression. Thus, in the equation,

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3;$$

the first member is the indicated cube of $x + a$, and the second member is its development.

The term development often implies that the equivalent expression is a series having an infinite number of terms; in this case a finite number of terms can only approximate to the true development.

DEVELOPMENT OF AN EXPRESSION. An equivalent expression, in which certain indicated operations have been performed.

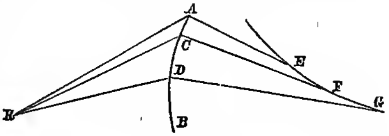
The form of the development will depend in a great measure upon the nature of the indicated operation, and it may be finite, or it may be infinite in extent. By far the greater part of algebraical developments have an infinite number of terms, in which case a few of the leading terms, together with the law of the development, are sufficient to determine the series to any desired extent.

A great many developments may be made by means of Taylor's and McLaurin's formulas, which see.

DEVELOPMENT OF A SURFACE. If a single curved surface be rolled upon a plane till every element comes in contact with the plane, that portion of it which is touched is called the development of the curved surface. See *Descriptive Geometry*.

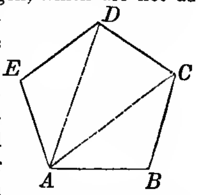
DĪ-A-CAUS'TIC CURVE. [Gr. διακαιω, to burn or inflame].

If AB represent a section of the surface of a refracting medium, R the radiant point, RA, RC, RD, &c., rays of light incident upon the



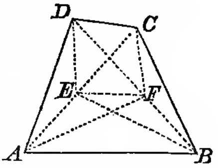
surface, and AE, CF, DG, &c., refracted rays, then is the curve which is tangent to all the refracted rays a diacaustic curve. The section is supposed to be taken through the radiant and the centre of the deviating surface.

DĪ-AG'O-NAL. [Gr. διαγωνιος, from δια, and γωνια, a corner]. In Geometry, a diagonal is a straight line joining the vertices of two angles of a polygon, which are not adjacent. In the polygon ABCDE, the lines AC and AD are diagonals. If two sides which are not adjacent be prolonged till they meet, and their point of intersection be joined with a vertex not adjacent, this line is often called a diagonal, for it is found to possess all the mathematical properties of a diagonal.



Diagonals of quadrilaterals:

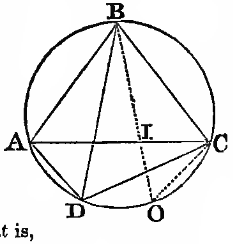
1. In any quadrilateral the sum of the squares of the two diagonals is equivalent to the sum of the squares of the four sides diminished by four times the square of the line joining the middle points of the diagonals; that is,



$$AC^2 + BD^2 \asymp AB^2 + BC^2 + CD^2 + DA^2 - 4EF^2.$$

If the quadrilateral is a parallelogram, the last term is equal to zero.

2. In any quadrilateral inscribed in a circle, the rectangle of the two diagonals is equivalent to the sum of the rectangle of the opposite sides taken two and two; that is,

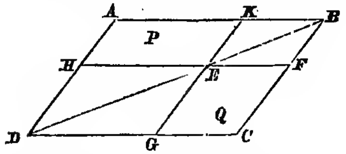


$$AC \times BD \Leftrightarrow AB \times DC + AD \times BC.$$

If the quadrilateral is a parallelogram, it must necessarily be a rectangle; the diagonals will be equal, and we shall have the square of either diagonal equal to the sum of the squares of either two adjacent sides.

3. In any parallelogram, either diagonal divides the figure into two equal triangles, and the two diagonals mutually bisect each other.

4. In any parallelogram, a straight line through the middle point of either diagonal, divides the parallelogram into two equal parts.



5. If through any point E of the diagonal of a parallelogram AC, parallels be drawn to the sides of the parallelogram, forming the two parallelograms P and Q, these are called *complementary parallelograms* about the diagonal, and are always equivalent to each other.

6. The diagonal of a square is incommensurable with its side.

7. In any trapezoid the sum of the squares of the two diagonals is twice the sum of the squares of the lines which bisect the opposite sides, taken two and two.

A diagonal of a polyhedron is any straight line joining the vertices of any two polyhedral angles, which do not lie in the same face.

In perspective, a diagonal is any horizontal line which makes an angle of 45° with the perspective plane. Two diagonals may be drawn through any point in space, and, on account of the ease with which their perspectives may be constructed, they are of much use in finding the perspective of points. See *Perspective*.

DIAGONAL SCALE. See *Scale*.

DĪA-GRAM. [Gr. διαγραμμα]. A drawing or pictorial delineation, made for the purpose of demonstrating or illustrating some property of a geometrical figure.

DĪ'AL. [L. *dies*, a day]. An instrument for determining the hour of the day, by means of a shadow cast by the sun.

In the construction of a sun-dial, the object is to find the angular distance of the sun from the meridian of the place at any instant, and thence to determine the hour.

In the construction of dials, the sun's apparent motion is supposed to be uniform throughout the day, and to take place in a circle whose plane is parallel to the equator: neither of these suppositions is strictly correct; but for all the purposes of dialing, they are both sufficiently so, as they afford results sufficiently accurate for ordinary purposes.

The surfaces upon which dials are constructed, may vary infinitely in shape and position, giving rise to a great number of constructions, but the same geometrical principle runs through them all.

If a dial is constructed upon a horizontal plane, it is called a *horizontal dial*. Such a dial shows the hours from sunrise to sunset.

If it is constructed upon the plane of the prime vertical, that is, on a vertical plane perpendicular to the meridian, it is called an *erect vertical dial*, and shows the hours from 6 o'clock in the morning to 6 o'clock in the evening.

If it is constructed upon any other vertical plane, it is called a *vertical declined dial*, and the limits between which it shows the hours will depend upon its inclination to the meridian. If it coincides with the plane of the meridian and faces the east, it is called an *east dial*, and shows the hours from sunrise to 12 o'clock. If it faces the west, it is called a *west dial*, and shows the hours from 12 o'clock till sunset.

If the dial is constructed upon a plane which is perpendicular to the plane of the meridian, but not vertical, it is called an *inclined dial*.

If it is constructed upon a plane which is neither perpendicular to the meridian nor to the horizon, it is said to be *deinclined.* Such dials are rare, and of very little importance.

Amongst inclined dials may be distinguished *the polar dial*, whose plane is parallel to the axis.

We shall only indicate some of the most useful constructions, referring the reader to more extended treatises for a complete account of the art of dialing.

If we conceive twelve planes to be passed through the axis of the earth, making equal angles with each other, they will divide the surface into twenty-four equal lunes. The curves of intersection of these planes with the surface of the earth, are called *hour circles*, for the sun occupies just an hour in passing over the space between each two. We shall consider one of these circles as passing through the place where the dial is to be constructed, and suppose the meridians or hour circles starting from this one, and going around towards the west, and numbered 1, 2, 3, &c., up to 12, which will correspond to the lower meridian of the place; commencing again at this meridian, and numbering as before, 1, 2, 3, &c., to 12, we shall arrive at the meridian from which we set out.

If now we conceive a plane to be passed through the centre of the earth parallel to the sensible horizon of the place, it will intersect the meridian planes in 24 straight lines concurring at the centre, each of which must bear the same number as the corresponding hour circle. If now the upper half of the sphere be removed, the axis, supposed a material line, only remaining, it is evident that the shadow of this axis will be successively projected upon the lines marked 1, 2, 3, &c., at the hours of the day corresponding to the respective numbers. This is, then, a sun-dial placed at the centre of the earth; the axis, in this case, is called the *style*, and the lines are *hour-lines*.

Now, the dimensions of the earth are so small in comparison with the distance of the sun from it, that they may be disregarded. If, therefore, through the place on the earth's surface, we erect a style parallel to the axis, and draw lines parallel to the hour-lines already considered, we shall have a *horizontal dial.*

It is evident from the same considerations, that a sun-dial may be transported to any point of the same meridian, provided all its parts are disposed in positions parallel to their primitive positions.

In this case, the dial will, for every other point of the meridian, be an *inclined dial.* We may also revolve the whole dial about the axis of the earth as an axis without change; hence, all dials of equal inclination are the same for every point in the same parallel of latitude. It is to be observed that the construction explained will fail at the equator. In that position the style is horizontal, and the hour-lines are parallel to it. At the poles, the style of a horizontal dial is vertical, and the hour-lines make equal angles with each other.

If it is desirable to construct a dial upon any oblique plane, we conceive a parallel plane through the centre of the earth, cutting the meridian planes instead of the horizontal plane first passed, and proceed as before. If we wish to construct a dial upon a curved surface, we conceive a curved surface passing through the centre of the earth concentric with the given surface; the auxiliary hour-lines will be curved, and the corresponding hour-lines upon the dial must be drawn concentric with them.

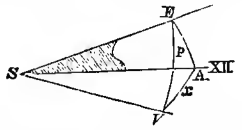
It is sufficient to remark, that, if we wish the dial to indicate half-hours, quarter-hours, &c., the number of meridian planes considered must be doubled, quadrupled, &c. Hour-planes are distant from each other 15°; hence, half-hour planes are at a distance from each other of 7° 30′, and quarter-hour planes at a distance of 3° 45′, &c.

The style which is employed may be a slender metallic rod, or it may consist of a thin plate of metal, generally of a triangular shape, and terminating at one edge in a smooth and sharply defined line. If a dial is to be constructed on a horizontal plane, the line indicating 12 o'clock must coincide exactly with the meridian. If the dial is to be constructed upon a vertical wall, the 12 o'clock line will be vertical.

To investigate a formula which shall indicate the method of constructing a *horizontal dial.*

Let SE represent the style, SA the 12 o'clock line, ASV the plane of the dial, and

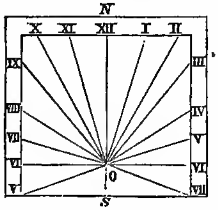
SV the shadow cast upon the dial-plate by the style any number of hours before or after twelve o'clock. If S be assumed as the cen-



tre of a sphere whose radius is 1, the several planes ASE, ASV, and SEV, will cut from the surface a spherical triangle EVA, in which we know the side EA equal to the latitude of the place; the angle VEA equal to the hour-angle from 12 o'clock, expressed in degrees at the rate of 15° to an hour; and the right angle EAV. If now we denote the latitude by l, the hour-angle by p and the required arc AV, which measures the angle ASV, by x, we shall have from the rules for solving spherical triangles

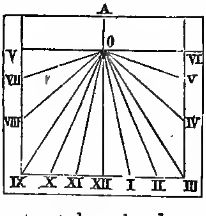
$$\tan x = \sin l \tan p \cdots\cdot (1).$$

To find the angle x corresponding to 1 o'clock or 11 o'clock, make $p = 15°$ and e equal to the latitude of the place. If now we successively attribute to p different values corresponding to different hours from 0 up to 7, 8, or any other limit, the corresponding angles may be computed by the formula, and being laid off and numbered as in the diagram, the dial-plate will be constructed. This is to be placed perfectly horizontal, and so that the line NS may be exactly in the meridian; the style is to be attached so as to pass through the point O, and making with the line NS an angle equal to the latitude of the place, being at the same time in a vertical plane through NS.



If it be required to construct a dial upon a vertical wall, perpendicular to the meridian, formula (1) will reduce to

$\tan x = \tan p \cos l$, and the several hour-lines may be constructed as already ex-

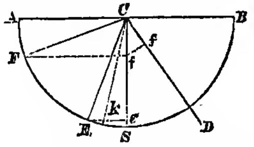


plained. In this case, the 12 o'clock line is perpendicular to the horizon, and the style is to be placed in a vertical plane through AB, passing through the point O, and making with the vertical wall an angle equal to the complement of the latitude.

If the style is a metallic plate, then two parallel lines must be drawn at a distance from each other, equal to the thickness of the plate between which the shadows will fall at 12 o'clock. The angles on each side must be equal, and laid off from the corresponding parallel.

The hour-lines upon the plate of an erect *vertical* or *horizontal dial*, in the cases already considered, admit of an elegant geometrical construction.

For the hour-lines of the horizontal dial:



With C as a centre, and a radius equal to 1, describe an arc of a circle ASB, and through C draw AB and CS at right angles to each other, and also the line CD, making the angle DCB equal to the latitude l of the place. From S and A lay off the arcs SE and AF, each equal to p, the hour angle, and draw Ff and Ee perpendicular to CS, and also ff' perpendicular to CD; then will $\frac{Cf'}{Ce} = \tan x$. For,

$Cf = \sin p$, $Ce = \cos p$, and $Cf' = \sin p \sin l$; whence,

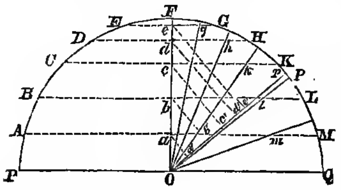
$$\frac{Cf'}{Ce} = \tan p \sin l = \tan x.$$

If, therefore, the distance ek be laid off equal to Cf', and the line Ck drawn, Ck will be the hour-line corresponding to the angle p.

We may, therefore, construct the hour-lines of a horizontal sun-dial as follows:

Describe a semi-circle PFQ, and divide it into arcs of 15° each; draw the line FO perpendicular to PQ, and draw chords EG, DH, CK, &c., through the points of division, at equal distances from P and Q, cutting FO in e, d, c, b, &c. Draw the line OP, making the angle POQ equal to the latitude of the

place. Through e, d, c, &c., draw the lines ee', dd', &c., perpendicular to OP. Lay off



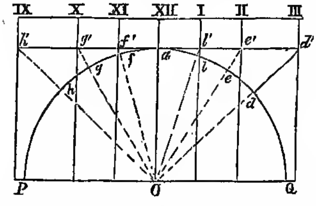
the distances $eg = \mathrm{O}a'$, $dh = \mathrm{O}b'$, $ck = \mathrm{O}c'$, $bl = \mathrm{O}d'$ and $am = \mathrm{O}e'$; through O draw Og, Oh, Ok, Ol, and Om; on the other side of OF draw symmetrical lines, and they will be the hour-lines of a horizontal dial.

If it had been required to construct half hour or quarter hour lines, we should have divided the arc PO into parts of $7\frac{1}{2}°$ or $3\frac{3}{4}°$, and the construction would have been the same.

In order to construct the hour-lines upon a vertical dial, whose plane is perpendicular to the meridian, we have only to change the latitude into its complement, and the construction will be the same as that already described.

It has been observed that at the equator the hour-lines upon a horizontal dial will be parallel to each other; this, with the exception of the horizontal dial at the pole, presents the simplest case of construction. The style being parallel to the dial plate, the hour-lines are at distances from the 12 o'clock line, which are proportional to the tangents of the hour angles.

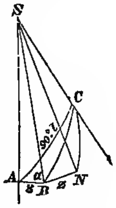
Draw two straight lines at right angles to each other, intersecting at the point O where the style is to pierce the dial-plate. With O as a centre, and any radius OQ, describe a



semicircle PaQ, and from the point a in which it cuts the perpendicular Oa, lay off arcs ab, ac, ad, &c., equal, respectively, to 15°, 30°, 45°, &c.; through the point a draw a tangent to the circle, and through the points of division and the centre draw lines till they intersect the tangent in the points b', c', d', &c.; through the last points draw straight lines parallel to Oa, and they will be the hour-lines. The hour-lines gradually become more and more distant from each other, till finally the hour-line of 6 o'clock will be at an infinite distance from Oa; this dial, therefore, can only be used for three or four hours before and after noon.

The horizontal dial at the pole presents no difficulty. The style is vertical, and the hour-lines make angles of 15° with each other.

To deduce formulas for constructing the hour-lines upon a *vertical declined dial*:



Let SC represent the style parallel to the axis of the earth, piercing the dial plane at S; SA the intersection of the dial plane and the meridian plane, which will be a vertical line; SB the projection of the style upon the dial plane, and called the *substyle*; SN the shadow of the style upon the dial plane at any hour corresponding to the hour angle p. Conceive a sphere with its centre at S, and a radius 1, to be described; its intersections with the several planes CSA, CSB, CSN, and ASN, will give the right angled spherical triangles CAB and CBN. Denote the side AB by s; the side BN by z; the side CA by $90° - l$; the side BC by θ; the angle CAB by a; the angle ACB by u; and the angle ACN by p. The rules for solving spherical triangles will give from the triangle ACB,

$$\tan s = \cot l \cos a \quad (1), \quad \sin\theta = \cos l \sin a \ (2).$$

Since l is the latitude of the place, and a the angle which the plane of the meridian makes with the dial plane, both of which are supposed known, equation (1) will determine the angle which the substyle makes with the vertical SA, and equation (2) will, with equation (1), fix the position of the style in space. When the plane of the dial inclines towards the east, the *substyle* will be on the left of the vertical SA, and when it inclines to the west it will be on the right, the vertical declined dial being supposed to face the south; if it faces the north the reverse will obtain.

To show the method of tracing the hour lines, we have, from the triangle ABC

$$\cot u = \sin l \tan a \quad \cdot \quad \cdot \quad (3)$$

In the triangle BCN, right angled at B, we have, since the angle at C is equal to $p - u$,

$$\tan z = \sin \theta \tan (p - u) \quad \cdot \cdot \cdot (4)$$

Then, to find the hour-line corresponding to the hour angle p, we first find the constant angle u, from equation (3), and by substituting in (4), we get the value of z, which determines the hour-line.

We have supposed SN situated on the right of the *substyle;* if this line falls in the angle ASB, we shall have $u > p$, and z will be negative. Finally, if SN falls to the left of SA, u will be negative, and equation (4) takes the form

$$\tan z = \sin \theta \tan (p + u) \quad \cdot \cdot \cdot (5)$$

It is to be observed that if $p = u$, z will be equal to 0, and the shadow will fall upon SA.

By giving suitable values to a, l and p, all the hour-lines may be found from the above equations.

The following method of tracing the hour-lines of a *vertical declined dial,* is given by Delambre. In the spherical triangle ACN, (last figure), let the side AC $= 90° - l$, the angle A $= a$, and ACN $= p$; denote, also, AN by x, CN by y; whence,

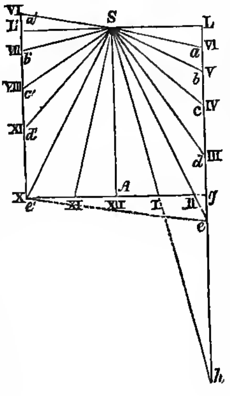
$$\cos x = \sin l \cos y + \cos l \sin y \cos p\,;$$
$$\cos y = \sin l \cos x + \cos l \sin x \cos a\,;$$

from which we deduce, by eliminating $\cos y$,

$$\left.\begin{aligned}\cos x\,(1 - \sin^2 l) &= \sin l \cos l \sin x \cos a \\ &+ \cos l \sin y \cos p\end{aligned}\right\} (6)$$

We have, also,

$$\sin y \sin p = \sin a \sin x \quad \cdot \cdot \quad (7)$$



Eliminating $\sin y$, between equations (6) and (7), and dividing both members of the resulting equation by $\sin x \cos^2 l$, we have

$$\cot x = \tan l \cos a + \frac{\sin a \cot p}{\cos l} \cdot \cdot (8)$$

From equation (8) we deduce the following construction:

Draw a horizontal line LL′, and through S, its middle point, draw a vertical line SA; SA will lie in the meridian plane. Denote the parts SL and SL′ each by m, and draw through L and L′ the lines Lh and L′h', parallel to SA; also draw any horizontal line $e'g$ below LL′.

Let Sb be any hour-line; then the angle $ASb = x$; the triangle SLB gives

$$Lb = SL \cot x\,; \qquad \text{whence,}$$

$$Lb = m \tan l \cos a + \frac{m \sin a \cot p}{\cos l} \cdot \cdot \cdot \cdot (9)$$

Now, for the hour-line of 6 o'clock, $p = 90°$, and

$$La = a = n \tan l \cos a.$$

The line Sa is the hour-line of 6 o'clock in the evening, and its prolongation Sa' is the hour-line of 6 o'clock in the morning. Since the first term of Lb, is a; the second term will express the lengths of ab, ac, ad, &c., or the distances from the point a to the points in which the hour-lines cut the vertical Lh. Denoting the general value of these distances by ϕ, we shall have

$$\phi = \frac{m \sin a \cot p}{\cos l} \cdot \cdot \cdot \cdot \cdot (10).$$

By means of formula (10), the different values of ϕ are computed and laid off from a on Lh; then, since for the corresponding hours in the morning the value of p will be equal to its value in the evening, we shall have $ab = a'b'$, $ac = a'c'$, &c., from which the hour lines may be constructed.

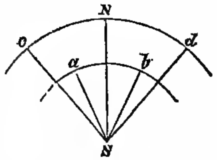
By a proper combination of the preceding principles, every variety of dial upon a plane surface may easily be constructed.

Sometimes, instead of a style, a disk of metal is employed, pierced by a hole, through which the light falls upon the dial plane. In this case, we conceive a straight line to be drawn through the centre of the hole parallel to the axis of the earth; the point in which it pierces the dial plane is the centre of the dial, and the hour lines are to be constructed about this point as though this imaginary line were the style.

Sometimes a narrow slit is made in a plate, so that the light shining through it may fall upon the dial plane; in this case, the slit is to be placed parallel to the axis of the earth, and regarded as a style in all respects.

To construct the meridian line upon any

horizontal plane. Assume one point through which it is to pass, and with that point as a centre, describe several concentric circles, and erect a vertical pin of such length that the shadow of its top may at noon fall within some of the concentric circles, and at evening fall without them. In the morning, mark the points c, a, in which the end of the shadow crosses each circle, and in the afternoon mark the points b, d, in which the end of the shadow crosses the same circles. Join these points with the centre, and bisect the angles formed by joining the points in which the shadow crosses the semi-circle and the centre; the bisecting lines of all these angles ought to coincide with each other and the meridian.



By repeating the operation from day to day, great accuracy may be reached.

It is to be observed, that the time given by a sun-dial is *true*, or *solar* time, and only agrees with mean time at four different days of the year. At any other period, time indicated by a dial must be reduced to clock, or *mean time*, by applying a small correction called the *equation of time*. See *Equation of Time.*

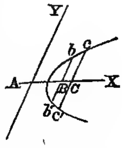
DIAL-PLATE. The plate on which the hour lines of a dial are drawn.

DĪ'AL-ING SCALE. A scale constructed according to the principles explained under the head of *dial*, and of some use in constructing dials. See *Dial.*

DĪ-AM'E-TER. [Gr. διαμετρος, δια, through, and μετρον, measure] A straight line which bisects a system of parallel chords drawn in a curve.

If a plane curve, given by its equation, has a diameter, the co-ordinate axes may be so taken that one of them, the axis of X, for example, shall coincide with this diameter, and the other be parallel to the chords which it bisects. For this position of the co-ordinate axes, the equation must be of such a form, that for any assumed value of x there will be two corresponding values of y, equal with contrary signs. The two values, taken together, when real, correspond to one of the bisected chords.

Suppose that AC is a diameter of the curve $c'b'\ bc$, that the axis of X coincides with it, and that the axs of Y is parallel to the chords bb', cc', which it bisects: for the assumed value of $x =$ AB, the two corresponding values of y must represent, respectively, the distances Bb and Bb', which together make up the chord bb'; and in like manner, for any other assumed value of x, which corresponds to points of the curve.



We have, then, the following rule for ascertaining whether any plane curve, given by its equation, has a diameter.

Substitute for x and y, in the given equation, their values taken from the formulas for passing from the given system to any rectilineal system in the same plane, and see if such values can be given to the arbitrary constants which enter the new equation, as to place it under such a form, that for every value of x, the corresponding values of y shall be equal with contrary signs. If any such values exist, the curve will have a diameter, and the number of sets of such values will determine the number of diameters.

To illustrate, let it be required to ascertain whether the curves of the second order, that is, the conic sections, have any diameters, and if so, how many.

The general equation of the conic sections is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \quad \cdots (1).$$

in which the axes may be considered at right angles with each other. The formulas for passing from a rectangular system of co-ordinates to any rectilineal system in the same plane, are

$$x = a'' + x' \cos a + y' \cos a' \quad \cdots (2), \text{ and}$$

$$y = b'' + x' \sin a + y' \sin a' \quad \cdots (3).$$

Substituting these values for x and y, in equation (1), and dropping the accents, we find an equation of the form

$$a'y^2 + b'xy + c'x^2 + d'y + e'x + f' = 0 \quad \cdots (4).$$

in which

$$b' = [2a \tan a \tan a' + b(\tan a + \tan a') + 2c] \cos a \cos a'.$$

$d' = [(2ab'' + ba'' + d) \tan a' +$
$(2ca'' + bb'' + e)] \cos a'$.

Equation (4) will be of the required form, if

$$b' = 0, \quad \text{and} \quad d' = 0, \text{ or}$$

$$2a \tan a \tan a' + b (\tan a + \tan a') + 2c = 0 \quad (5).$$

$$(2ab'' + ba'' + d) \tan a' + (2ca'' + bb'' + e) = 0 \cdots\cdots\cdots (6).$$

Equations (5) and (6) are equations of condition that must be satisfied, in order that the conic sections may admit of a diameter. In these equations, a'', b'', a and a', are arbitrary constants. These equations can, in general, be satisfied for an infinite number of sets of values of a and a'.

From equations (5) and (6), we deduce

$$\tan a' = -\frac{2ab'' + ba'' + d}{2ca'' + bb'' + e} \cdots\cdot (7),$$

$$\tan a = -\frac{2c + b \tan a'}{2a \tan a' + b} \cdots\cdots\cdot (8).$$

Any assumed values of a'' and b'', will give, in equation (7), a value for tan a', and this in equation (8), will give a corresponding value for tan a; and since there are an infinite number of sets of values that may be assumed for a'' and b'', it follows that there are an infinite number of positions of the co-ordinate axes, such that the axis of X shall be a diameter of a curve, and the axis of Y parallel to the chords which it bisects. The conic sections have, therefore, an infinite number of diameters.

If we suppose

$$b^2 - 4ac = 0 \quad \text{or} \quad 2c = \frac{b^2}{2a},$$

equation (8) becomes

$$\tan a = -\frac{\frac{b^2}{2a} + b \tan a'}{2a \tan a + b} =$$

$$-\frac{b}{2a}\left\{\frac{b + 2a \tan a'}{b + 2a \tan a'}\right\} = -\frac{b}{2a};$$

which, since tan a is constant, that is, *entirely independent of* a', shows that all diameters in the parabola are parallel to each other.

If in equation (6), we assume any value for tan a', there will be an infinite number of sets of values for a'' and b'', which will satisfy it; if we then regard a'' and b'' as variables, equation (6) must be the equation of a straight line, and the origin may be anywhere on this line, but the origin is necessarily upon the diameter, which is assumed as the axis of X; hence, equation (6) is the equation of the diameter which coincides with the axis of X.

Now, equation (6) will be satisfied, if we make a'' and b'' equal to the co-ordinates of the centre; hence, every diameter of a conic section passes through the centre.

If we suppose the origin of co-ordinates to be placed at the centre, which may always be done, except in the case already considered, in which $b^2 - 4ac = 0$, we shall have

$$a'' = \frac{2ae - bd}{b^2 - 4ac}, \quad \text{and} \quad b'' = \frac{2cd - be}{b^2 - 4ac}$$

for the co-ordinates of the centre. These values of a'' and b'' being substituted in equation (7), reduce it to

$$\tan a' = \frac{0}{0} \cdots\cdot (9);$$

which shows that there is an infinite number of diameters, all passing through the centre, and also that any straight line through the centre is a diameter.

Furthermore, from the form of (5), from which equation (7) was deduced, it is evident that if we assume any value of tan a', and deduce the corresponding value of tan a, and then make tan a' equal to this value, and again find the corresponding value of tan a, this last value will be the same as the value originally assumed for tan a'. This shows that, if one of the co-ordinate axes is a diameter, and the other axis parallel to the chords which it bisects,—the second co-ordinate axis will also be a diameter, and the first one parallel to the chords which it bisects. Such diameters are called *conjugate diameters*; and since tan a' may have any value, it follows: 1st, that there is an infinite number of pairs of conjugate diameters in the ellipse and hyperbola; and 2d, that *every diameter* has a conjugate.

Equation (5), or (7), is the *equation of condition for conjugate diameters*, for the ellipse and hyperbola. The preceding considerations show that the parabola has no conjugate diameters.

In like manner, we may discuss the equation of any given curve with respect to its diameters.

If a diameter is perpendicular to the chords which it bisects, it is called an *axis*. The

parabola has one axis, and each of the other conic sections two axes. The circle has an infinite number of axes, every diameter being an axis.

DĪAM′E-TRAL CURVE. A curved line which bisects a system of parallel chords drawn in any given curve.

If a given curve has a diametral curve, the co-ordinate axes may be taken in such a manner that the axis of y shall be parallel to the bisected chords, and then, if the equation be solved with respect to y, it will be of the form

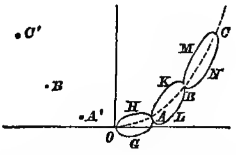
$$y = f(x) \pm \sqrt{f'(x)} \cdots\cdots (1);$$

in which, $y = f(x), (2)$, is the equation of the diametral curve. This is plain; for if we assume any value of x, the corresponding values of y in equation (1) will be equal to the corresponding value of y in equation (2), increased and diminished by the corresponding value of the radical. If then we construct the curve whose equation is $y = f(x)$, it will bisect all the chords of the given curve, which are parallel to the axis of y.

For example, the curve whose equation is

$$y = ax^2 \pm \sqrt{x \sin bx},$$

has a diametral curve whose equation is $y = ax^2$; that is, the diametral curve is a



parabola OAC. On the right of the origin, there is a succession of closed branches, OA, AB, BC, the consecutive ones having common tangents parallel to the axis of y. On the left of the origin, the diametral curve exists; but the oval branches are reduced to a system of conjugate or isolated points, A′, B′, C′, &c., corresponding to the values of x, which make $\sin bx = 0$.

To ascertain whether any given curve has a diametral curve, we have to see whether the co-ordinate axes can be so placed that the resulting equation will be of the form of equation (1). If they cannot, the curve has no diametral curve.

It may be observed that the diameters of conic sections, discussed in the last article, are only particular cases of diametral curves.

Diametral Plane. A plane which bisects a system of parallel chords drawn in a surface. By a discussion entirely analogous to that relating to diameters, it may be shown that every surface of the second order has an infinite number of diametral planes. If a diametral plane is perpendicular to the chords which it bisects, it is called a *principal plane* of the surface: every surface of the second order has at least one principal plane, and may have three.

Three diametral planes are said to be conjugate, when each one bisects a system of chords parallel to the intersection of the other two. Whenever a surface of the second order has a centre, there is always an infinite number of sets of conjugate planes. Every diametral plane passes through the centre, when the surface has a centre; and conversely, every plane through the centre is a diametral plane.

In a surface of the second order, we may find the equation of a diametral plane, which bisects a system of chords parallel to the axis of z. The method of proceeding indicated in the discussion of diameters may, with some modification, be applied to diametral planes.

Diametral Surface is a curved surface, which bisects a system of parallel chords drawn in the surface, a particular case of which is the diametral plane.

Whenever the equation of the surface can be placed under the form

$$z = ax^m + by^n + c \pm \sqrt{f(x)},$$

it admits of a diametral surface whose equation is

$$z = ax^m + by^n + c.$$

The reason is evident. The discussion is entirely analogous to that for diametral curves.

DĪ-E′DRAL ANGLE. The angular space included between two planes which meet each other in a common straight line; the planes are called *faces*, and the straight line is the edge of the angle. The measure of a diedral angle is the angle included between two straight lines, one in each face, and both perpendicular to the edge at the same point.

DIF′FER-ENCE. [L. *dis*, and *fero*, to bear or move apart]. The result obtained by

subtracting one quantity from another. When it is not specified which quantity is to be taken from the other, it is generally understood that the less is to be taken from the greater, so that the difference is positive. As far as the numerical value of the difference is concerned, it makes no difference which is taken for the subtrahend or which for the minuend.

Differences, method of. The name given to a method of finding an expression for the sum of any number of terms of a series.

Let a, b, c, d, &c., represent the successive terms of a series formed according to any law; then, if each term be subtracted from the succeeding one, the remainders will form a second series, called the *first order of differences* If we again subtract each term of this series from the succeeding one, we shall form another series called the *second order of differences*, and so on, as exhibited in the annexed table:

SERIES OF DIFFERENCES.

$$\begin{array}{lr}
a \;,\; b \;,\; c \;,\; d \;,\; e & \\
b-a \;,\; c-b \;,\; d-c \;,\; e-d \;, & \text{1st.} \\
c-2b+a \,,\, d-2c+b \,,\, e-2d+c & \text{2d.} \\
d-3c+3b-a \;,\; e-3d+3c-b & \text{3d.} \\
e-4d+6c-4b+a \,, & \text{4th.} \\
\text{\&c.} \qquad \text{\&c.} &
\end{array}$$

If we designate the first terms of the 1st, 2d, 3d, &c., order of differences by d_1, d_2, d_3, &c., we shall have

$d_1 = b - a$; whence,

$b = a + d_1$.

$d_2 = c - 2b + a$; whence,

$c = a + 2d_1 + d$.

$d^3 = d - 3c + 3b - a$; whence,

$d = a + 3d_1 + 3d_2 + d_3$.

$d^4 = e - 4d + 6c - 4b + a$; whence,

$e = a + 4d_1 + 6d_2 + 4d_3 + d_4$.

&c. &c. &c. &c.

And if we designate the term of the series which has n terms before it, by T, we shall find by continuing the above process,

$$T = a + nd_1 + \frac{n(n-1)}{1 \cdot 2} d_2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_3 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} d_4 + \text{\&c.} \quad \cdot \cdot \quad (1)$$

from which we may find any term of a series when we know the number of preceding terms, and the first terms of the successive orders of differences.

To deduce a formula for the sum of n terms of the series a, b, c, &c., assume the auxiliary series,

$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,$ &c.

The first order of differences is evidently a, b, c, d, &c., in the given series.

Now it is obvious that the sum of n terms of the given series is equal to the $(n+1)^{th}$ term of the auxiliary series. But the $(n+1)^{th}$ term of the auxiliary series may be deduced from formula (1) if we observe that the first term of the first order of differences is a, the first term of the second order of differences d_1, the first term of the third order of differences d_2, and so on.

Hence, making these changes in formula (1), and denoting the sum of n terms of the series by S, we have the formula

$$S = na + \frac{n \cdot (n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} d_3 + \text{\&c.} \cdot \cdot (2)$$

When all of the terms of any order of differences become equal, the terms of all the succeeding orders of differences are 0, and formulas (1) and (2) give exact results. When there are no orders of differences whose terms are all equal, the formulas do not give exact results, but approximations more or less accurate, according to the number of terms of the formulas employed.

Formula (1) will be referred to hereafter, in connection with the subjects of *Interpolation* and *Summation*.

To show the application of formula (2), let it be required to determine the number of cannon balls in a pile having a square base, and terminating in an apex of 1 ball.

Commencing at the top, the first layer contains 1 ball, the second layer 2×2 balls, the third layer 3×3 balls, and so on, the n^{th} layer containing $n \times n$ balls. It is therefore re-

quired to find the sum of n terms of the series

1^2,	2^2,	3^2,	4^2,	$\cdots n^2$	or
1	4	9	16	$\cdots$ &c. $\cdots$	series.
3	5	7	9	$\cdots$	1st order of differences.
	2	2	2	$\cdots$	2d order of differences.
	0	0	0	$\cdots$	3d order of differences.

Hence, $d_1 = 3$, $d_2 = 2$, $d_3 = 0$, $d_4 = 0$, &c., and $a = 1$.

Substituting these values in formula (2), we get

$$S = n + \frac{n \cdot (n-1)}{1 \cdot 2} \cdot 3 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot 2;$$

or,

$$S = \frac{n\,(n+1)\,(n+2)}{1 \cdot 2 \cdot 3} \quad \cdots .$$

Differences, Calculus of Finite. See *Calculus of Finite Differences.*

DIF-FER-EN'TIAL. A difference or part of a difference between two different states of a variable quantity. By some the differential is considered infinitely small, in which case the differential of a function is the same as the difference between two consecutive states of the function; by others it is considered as a finite quantity.

The differential of the *independent* variable is always constant, and equal to the difference between two consecutive states of that variable.

Those who regard differentials as finite quantities, define the differential of a function as follows:

If the variable be increased by a constant increment, called the differential of the variable, and the new state of the function be diminished by the primitive state, and this difference be developed according to the ascending powers of the increment, then is the term which is of the first degree with respect to the increment, called the *differential of the function.*

The co-efficient of the differential of the variable in the expression for the differential of the function, is the *differential co-efficient* of the function.

It is the differential co-efficient of a function which characterizes the function, and as this is independent of the differential, of the independent variable, it is entirely immaterial what value we suppose that to have.

In a function of two or more variables, the result obtained by differentiating with respect to one of them, is called *a partial differential taken with respect to that variable.* There will be as many partial differentials as there are independent variables, and their sum forms what is called the *total differential of the function.* Since, from their nature, the variables are entirely independent of each other, we may always find the partial differential of a function by simply operating upon it as though that were the only variable, by the simple rules for differentiating a function of one variable. To find the total differential of a function of two or more variables of the second order, we differentiate the total differential of the first order with respect to each of the independent variables, and take the sum of the results. In like manner, we find the total differentials of the third and higher orders. The co-efficient of the differential of the differential of either variable, in the expression for the partial differential of the functions, is called *the partial differential co-efficient of the function taken with respect to that variable.*

The following notation for partial differentials has been adopted. If

$$u = f(x, y, z, \text{\&c.,}) \text{ we have}$$

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz + \text{\&c.}$$

The simple expression du designates the total differential, whilst $\frac{du}{dx}dx$, $\frac{du}{dy}dy$, &c., designate partial differentials, in which the denominator in each case shows with reference to which variable the differential is taken.

This method of notation, though sometimes apparently cumbersome, is nevertheless sufficiently clear and explicit. The notation employed for differentials of the higher orders, is analogous; thus, if $u = f(x, y)$,

$$d^2u = \frac{d^2u}{dx^2}dx^2 + \frac{d^2u}{dxdy}dxdy + \frac{d^2u}{dydx}dydx + \frac{d^2u}{dy^2}dy^2. \quad (1.)$$

The simple symbol d^2u designates the total differential of the second order; the symbol $\frac{d^2u}{dx^2}$ indicates the result obtained by differentiating the function twice in succession, and both times with respect x; the symbol

$\frac{d^2u}{dxdy}$, stands for the result obtained by differentiating the function first with respect to x, and then that result with respect to y; the symbol $\frac{d^2u}{dydx}$ is the symbol which indicates that the given function has been differentiated first with respect to y, and the result with respect to x; and, finally, $\frac{d^2u}{dy^2}$ shows that both differentiations have been performed with respect to y. If a function of x and y be differentiated first with respect to y, and then with respect to x, or, if it be first differentiated with respect to x, and then with respect to y, the final result will be the same; that is, the symbol $\frac{d^2u}{dxdy}dxdy$ is equivalent to the symbol $\frac{d^2u}{dydx}dydx$: this agrees completely with what was said in regard to the distributive character of all pure symbols of operation. This principle also enables us to place formula (1) under the form,

$$d^2u = \frac{d^2u}{dx^2}dx^2 + 2\frac{d^2u}{dxdy}dxdy + \frac{d^2u}{dy^2}dy^2:$$

we have also the formulas

$$d^3u = \frac{d^3u}{dx^3}dx^3 + 3\frac{d^3u}{dx^2dy}dx^2dy + 3\frac{d^3u}{dxdy^2}dxdy^2 + \frac{d^3u}{dy^3}dy^3$$

$$d^4u = \frac{d^4u}{dx^4}dx^4 + 4\frac{d^4u}{dx^3dy}dx^3dy + 6\frac{d^4u}{dx^2dy^2}dx^2dy^2 + 4\frac{d^4u}{dxdy^3}dxdy^3 + \frac{d^4u}{dy^3}dy^3;$$

and so on, for the differentials of a higher order.

Similar results might be obtained, though more complicated, by considering the higher order of differentials of functions of three or more variables.

The formulas for differentiating all functions of a single variable, have been given under the head of CALCULUS DIFFERENTIAL, and their application to functions of two or more variables, presents no difficulty.

DIFFERENTIAL CALCULUS. See *Calculus Differential.*

DIFFERENTIAL CO-EFFICIENT. The differential co-efficient of a function of one variable is a function whose form depends upon that of the given function, and which may be derived from it by a fixed law called the law of differentiation. It is often called a derived function, and may always be obtained as follows:

Give to the variable a variable increment, and find the new state of the function; from this subtract the primitive state, and divide the remainder by the increment; pass to the limit of this ratio by making the increment equal to 0, *and the result will be the differential co-efficient required.*

In most cases, the differential co-efficient may be more easily obtained, but the above law, which is applicable in all cases, possesses the advantage of showing the nature of the relation which exists between the function and its differential co-efficient. We also see that the differential co-efficient is entirely independent of every consideration with respect to the method of arriving at it; so that, whether we regard the differential calculus as establishing relations between infinitely small elements, as did Leibnitz; or whether we adopt with Newton the theory of fluents and fluxions, or the method of limits; or, finally, whether we adopt Lagrange's theory of derived functions, the fundamental conception is always the same. The processes may differ in form, but the final results must be rigorously identical. With respect to the mathematical character of the differential co-efficient, it is the characteristic mark of a class of functions which only differ from each other by a constant quantity having the sign *plus* or *minus*. It is found in the applications of mathematics to science, that results may be more easily arrived at by operating upon these marks of the functions than upon the functions themselves; and it is this circumstance which has contributed so largely to the development and progress of the calculus. As in the practical operations upon numbers, we use their *marks* or logarithms for the purpose of facilitating arithmetical processes, and as we are able to pass at any instant to the numbers whose characteristics we are making use of, so in the analytical processes of science we may reason upon the relations existing between the differential co-efficients of functions, being able at any time to return to the

functions themselves by the aid of the integral calculus. The analogy between the use of logarithms and differentials is not perfect, but it serves to give a slight idea of the use of the calculus.

There is, however, a great advantage that the calculus possesses, which is, that we are often enabled to find the relations between the *differentials* of functions, and thence to find relations between the functions themselves that could have been discovered in no other way. For example, the relation between the differential of the length of a curve and the differentials of the co-ordinates of its points, can always be deduced, and from it the length of an arc may easily be found in terms of these co-ordinates, by the aid of the calculus, whilst in many cases the attempt to deduce this last relation by any other process, would prove futile. The principal applications of the calculus are to the higher geometry, the theory of equations and the investigations of philosophy, mechanics, machines, engineering, &c.

DIFFERENTIAL EQUATIONS. Equations which express the relations between variables and their differentials.

In every single equation involving two or more variables, values may be assumed at pleasure for all the variables, except one, and the corresponding value of that one will be made known by the equation; whence it appears that in every such equation we may regard all the variables, except one, as independent variables, and that one as a function of them all.

If, therefore, we differentiate both members of an equation, regarding one variable as a function of the remaining ones, and then place the two results equal to each other, the resulting equation will be a differential equation of the first order, and is called an *immediate differential equation.* An immediate differential equation is one which is obtained, without transformation, from a given integral equation; all other differential equations are called *mediate differential equations.*

If we have a group of simultaneous equations containing more variables than there are equations, we may, by combination and elimination, reduce them to a single equation whose differential equation may be found as just explained. Or, we may regard some one variable as the function, and after differentiating each equation separately, we may combine the resulting equations and the given equations in accordance with the rules for treating the differentials of implicit functions, and thus deduce a *mediate* differential equation which will be the same as that obtained by the former method. For example, if we have the equations,

$$y^2 = 2pz \ldots (1), \quad z^3 = 3ax \ldots (2),$$

we might deduce from (1) and (2) the equation

$$y^6 = 24ap^3x,$$

from which, by the rule, we should obtain

$$6y^5dy = 24ap^3dx, \quad \text{or} \quad \frac{dy}{dx} = \frac{4ap^3}{y^5};$$

or we might, from (1) and (2), derive, by differentiation, the equations,

$$2ydy = 2pdz, \text{ and } 3z^2dz = 3adx, \text{ or}$$

$$\frac{dy}{dz} = \frac{p}{y} \quad \text{and} \quad \frac{dz}{dx} = \frac{a}{z^2};$$

but

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{ap}{yz^2} = \frac{4ap^3}{y^5}.$$

A mediate differential equation may also be derived from a single equation, by forming its differential equation, and then combining this with the given equation and eliminating some one quantity, usually a constant.

If a differential equation be differentiated, and its differential equation found, this is called a differential equation of the *second* order. The differential equation of a differential equation of the second order, is one of the third order, and so on.

A mediate differential equation can always be found from any equation which shall be entirely independent of the constants that enter the given equation, as follows: *Differentiate the given equation as many times as there are constants, and combine the resulting equations with the given equation, eliminating all the constants; the resulting equation will be a mediate differential equation, and will express a relation between the variables and their differentials of the different orders.*

A *partial differential equation*, is one which expresses the relation between the variables and their partial differentials. Such equations are entirely independent of the form of the function. As an illustration, let us take the equation

$$\phi = F(x + at) + f(x - at),$$

in which F and f denote any arbitrary functions.

If we differentiate the equation twice with respect to x, and twice with respect to t, we shall have

$$\frac{d\phi}{dx} = F'(x + at) + f'(x - at),$$

$$\frac{d\phi}{dt} = aF'(x + at) + af'(x - at),$$

$$\frac{d^2\phi}{dx^2} = F''(x + at) + f''(x - at),$$

$$\frac{d^2\phi}{dt^2} = a^2 F''(x + at) + a^2 f''(x - at),$$

whence, by combining the last two,

$$a^2 \frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dt^2};$$

which expresses a relation between the partial differential co-efficients of ϕ, taken with respect to the two variables x and t, and is entirely independent of the forms indicated by f and F.

DIG'ITS. [L. *digitus*, a finger]. In Arithmetic, the ten characters

0, 1, 2, 3, 4, 5, 6, 7, 8, 9,

by the aid of which all numbers may be expressed.

DI-MEN'SION. [L. *dimensio*, *dis*, from, and *metior*, to mete]. In Geometry, extension in one direction. Every body is extended in three directions at right angles to each other, or has three dimensions, length, breadth and height, or thickness. A line is extended in one direction. A surface is extended in two directions, that is, it has length and breadth, but no thickness. A line has but one dimension, length, without breadth or thickness.

In Algebra, a literal factor of a product or term, is called a dimension: thus, the expression a^2b has three dimensions.

The use of the term in Algebra, is due to the fact, that we usually employ a symbol of the first degree to represent a line or magnitude of one dimension, a symbol of the second degree to denote a surface or magnitude of two dimensions, and one of the third degree to denote a volume or magnitude of three dimensions. Although a magnitude can have but three dimensions, we have come to speak of terms having four or more dimensions, so that the word, as adopted in Algebra, has widely departed from its original signification—a circumstance of frequent occurrence in mathematical terms.

DI-O-PHAN'TINE ANALYSIS. A branch of Algebra which treats of the method of solving certain kinds of indeterminate problems, relating principally to square and cube numbers, and rational right-angled triangles. The name is derived from DIOPHANTUS, a Mathematician of Alexandria, who first wrote on the subject, about the third century of the christian era. He solved a great number of curious problems, and it is to his treatise that we are at the present day indebted for most of our knowledge on the subject. The following problem will serve to show the nature of the Diophantine analysis.

Let it be required to find a right-angled triangle whose sides shall be commensurable with each other.

If we denote the lengths of the three sides by x, y and z, respectively, z denoting the hypothenuse, we shall have

$$z^2 = x^2 + y^2 \quad \ldots\ldots \quad (1).$$

It is required to find rational numbers, which, when substituted for x, y and z, will satisfy equation (1). We first substitute for z, $x + u$, which gives

$$x^2 + 2xu + u^2 = x^2 + y^2 \quad \text{or} \quad y^2 = 2xu + u^2,$$

whence $$x = \frac{y^2 - u^2}{2u} \quad \ldots\ldots \quad (2).$$

Any rational numbers assumed for y and u, will give a rational value for x and z. If it is required that the sides of the triangles shall be expressed in whole numbers, we have simply to assume such values for y and u, in whole numbers, as will make x a whole number. To find what numbers will satisfy this condition, we first find expressions for the sides of the triangle in terms of y and u, which are, when cleared of fractions,

$$2yu, \; y^2 - u^2 \text{ and } y^2 + u^2,$$

in which u cannot be equal to y and correspond to any triangle. First, assume $u = 1$, and $y = 2$; the three sides will be 4, 3 and 5. If we make $u = 1$, and $y = 3$, we have the sides 6, 8 and 10. If we make $u = 2$, and $y = 3$, the sides are 12, 5, 13, and so on. The problem evidently admits of an infinite number of solutions.

In this branch of analysis, but few rules can be laid down, the successful solution of each case requiring some device, chiefly dependent upon the nature of the particular problem.

A few examples will further illustrate the general method of treating Diophantine problems.

I. To separate a given square number into two parts, both of which shall be square numbers.

Denote the given square number by a^2, and the required square numbers by x^2 and and y^2, respectively; then we have only to satisfy the equation

$$x^2 + y^2 = a^2 \quad \text{or} \quad x^2 = a^2 - y^2.$$

Assume

$$a + y = \frac{px}{q}, \quad \text{and} \quad a - y = \frac{qx}{p},$$

from which we readily deduce

$$2a = \frac{(p^2 + q^2)x}{pq}, \quad \text{and} \quad 2y = \frac{(p^2 - q^2)x}{pq},$$

and from these equations, finally

$$x = \frac{2pqa}{p^2 + q^2}, \quad \text{and} \quad y = \frac{(p^2 - q^2)a}{p^2 + q^2} \cdots (1),$$

in which the values for p and q may be assumed at pleasure.

If a is equal to the sum of two squares, p and q may be assumed so that $p^2 + q^2 = a$, in which case the expressions for x and y will be entire; they will also be entire if $p^2 + q^2$ is equal to any factor of a, and as many different integer values for them may be found, as there are ways of resolving a, or any of its factors. into the sum of two squares.

Resolve $(65)^2$ into two squares.

Here $\quad 65 = 8^2 + 1^2 = 7^2 + 4^2$,

also 13 and 5, which are factors of 65, give

$$13 = 3^2 + 2^2, \quad \text{and} \quad 5 = 2^2 + 1^2.$$

In the first place, assume $p=8$, and $q=1$; these will give

$x = 16$, and $y = 63$; and $16^2 + 63^2 = 65^2$.

In the second place, assume $p = 7$ and $q = 4$; these give

$x = 56$, and $y = 33$; and $56^2 + 33^2 = 65^2$.

In the third place, assume $p = 3$, and $q = 2$; these give

$x = 60$, and $y = 25$; and $60^2 + 25^2 = 65^2$.

In the fourth place, assume $p = 2$, and $q = 1$; these give

$x = 52$, and $y = 39$; and $52^2 + 39^2 = 65^2$.

There are an infinite number of fractional solutions.

II. To resolve a number which is equal to the sum of two given squares, into two parts which shall be perfect squares.

Denote the given squares by a^2 and b^2, and the required ones by x^2 and y^2, respectively.

From the conditions of the problem, we have the equation

$$x^2 + y^2 = a^2 + b^2, \quad \text{or} \quad a^2 - x^2 = y^2 - b^2.$$

Assume

$$a + x = \frac{p(y + b)}{q}, \quad \text{and} \quad a - x = \frac{q(y - b)}{p};$$

whence, by combination, we deduce the values of x and y;

$$x = \frac{ap^2 + 2bpq - aq^2}{p^2 + q^2}, \quad \text{and}$$
$$y = \frac{bq^2 + 2apq - bp^2}{p^2 + q^2},$$

in which values may be assumed at pleasure, for p and q. If the given sum $a^2 + b^2$ can be resolved into factors which are squares of whole numbers, it will be better to resolve it into its factors which, in that case, will also be sums of two perfect squares; then their product will give the required squares: thus, if

$$a^2 + b^2 = (m^2 + n^2)(m'^2 + n'^2), \quad \text{then will}$$
$$a^2 + b^2 = (mm' \pm nn')^2 + (mn' \pm m'n)^2;$$

and, therefore,

$$x = mm' \pm nn', \quad \text{and} \quad y = mn' \pm m'n.$$

1. Resolve $(9^2 + 2^2)$ into two parts which shall be perfect squares, and also whole numbers.

Here $\quad (9^2 + 2^2) = (2^2 + 1^2)(4^2 + 1^2)$;

whence,

$$m = 2, \quad n = 1, \quad m' = 4, \quad \text{and} \quad n' = 1;$$

these give

$x = 9$, and $x = 7$; also $y = 2$, and $y = 6$,

hence, $\quad 85 = 9^2 + 2^2 = 7^2 + 6^2$.

The first pair are the given squares, and the second pair the required ones.

If the given number cannot be resolved into factors of the proposed form, the problem cannot be solved in whole numbers, but there will be an infinite number of fractional solutions.

2. Resolve $2^2 + 1^2$ into two parts which will be perfect squares. Here $a = 2$ and $b = 1$; which in the formulas first deduced, give for the assumed values

$p = 2$ and $q = 3$, $x = \frac{29}{13}$ and $y = \frac{2}{13}$, whence

$$2^2 + 1^2 = \left(\frac{29}{13}\right)^2 + \left(\frac{2}{13}\right)^2.$$

III. To find three square numbers which are in arithmetical progression.

Denote the required squares by x^2, y^2 and z^2; then from the conditions of the problem,

$$x^2 + z^2 = 2y^2.$$

Assume $x = m + n$ and $z = m - n$, whence

$$y^2 = m^2 + n^2.$$

Again assume $m = p^2 + q^2$ and $n = 2pq$; we have,

$$x = p^2 - q^2 + 2pq, \quad z = p^2 - q^2 - 2pq \text{ and } y = p^2 + q^2;$$

in which p and q may be assumed at pleasure. If

$p = 2$ and $q = 1$; $x^2 = 7^2$, $z^2 = 1^2$, $y^2 = 5^2$, and the progression is 49, 25, 1.

IV. To resolve the sum of three squares which are in arithmetical progression into three parts, which shall be perfect squares, and also in arithmetical progression.

Denote the given squares by a^2, b^2, and c^2, and the required squares by x^2, y^2 and z^2, respectively, and take $s = a^2 + b^2 + c^2$. From the conditions of the problem, we have

$$a^2 + b^2 + c^2 = x^2 + y^2 + z^2,$$
$$x^2 + z^2 = 2y^2, \text{ and evidently,}$$
$$y^2 = b^2; \text{ whence}$$
$$x^2 + z^2 = 2b^2.$$

We may, as in the second problem, deduce values of x and z in terms of the indeterminate quantities p and q; or, we may in the formulas there deduced, make $a = b$ and change y into z, whence

$$x = \frac{(p^2 + 2pq - q^2)b}{p^2 + q^2}, \quad y = b,$$

$$\text{and } z = \frac{(q^2 + 2pq - p^2)b}{p^2 + q^2},$$

in which values for p and q may be assumed at pleasure.

1. Find three squares in arithmetical progression, whose sums shall be equal to

$$1^2 + 5^2 + 7^2 = 75.$$

Here, $a = 1$, $b = 5$ and $c = 7$.

1st. If $p = 3$, and $q = 2$, then $x^2 = \left(\frac{85}{13}\right)^2$,

$$y^2 = 5^2, \text{ and } z^2 = \left(\frac{35}{13}\right)^2;$$

The required progression is

$$\left(\frac{85}{13}\right)^2, \quad 5^2, \quad \left(\frac{35}{13}\right)^2$$

If $p = 4$ and $q = 3$, the progression is

$$\left(\frac{31}{5}\right)^2, \quad 5^2, \quad \left(\frac{17}{5}\right)^2,$$

If $p = 5$ and $q = 4$, the progression is

$$\left(\frac{245}{41}\right)^2, \quad 5^2, \quad \left(\frac{155}{41}\right)^2,$$

&c., &c., &c., &c.

If it were required to find the values of x, y, and z, so that they would be whole numbers, it would be necessary to assume the given progression such that b^2 would be equal to the sum of two integral squares. This will always be the case when b is equal to the product of any number of prime factors, each of which is of the form $4n + 1$, n being a whole number.

Thus, if $b = (4 + 1)(12 + 1) = 65$, we have the following progressions,

$$13^2 + 65^2 + 91^2 = 12675,$$
$$23^2 + 65^2 + 89^2 = 12675,$$
$$35^2 + 65^2 + 85^2 = 12675,$$
$$47^2 + 65^2 + 79^2 = 12675.$$

DI-RECT′. [L. *directus*, from *dirigo*, to make straight].

Direct Demonstration, in contradistinction to the *indirect* demonstration, or the *reductio ad absurdum*. In the direct demonstration the premises employed in each step of the reasoning, are either axioms, definitions, or truths, previously demonstrated. In the indirect demonstration, the premises in some of the steps may depend upon one or more hypotheses. See *Demonstration*.

DI-RECT′ER PLANE. In the first class of warped surfaces, the plane to which all of the right lined elements are parallel, is called the *plane directer* of the surfaces. Since an infinite number of planes can be passed which shall be parallel to any plane directer, either one may be assumed as the plane directer of the surface; in fact, when the plane directer

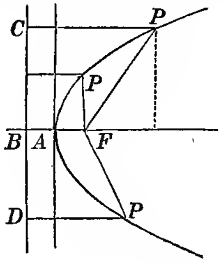
is pointed out, its direction only is intended. See *Warped Surfaces.*

DI-RECT'LY PROPORTIONAL. A term used in contradistinction to the term *inversely proportional.* Two quantities are directly proportional when they both increase or decrease together, and in such a manner that their ratio shall be constant.

DI-RECT'RIX of a conic section, is a straight line, such that the ratio obtained by dividing the distance from any point of the curve to it, by the distance from the same point to the focus, shall be constant. If PA represent a conic section, whose directrix is CD, and focus F, then will

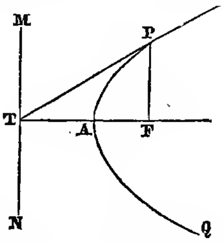
$$\frac{\mathrm{PC}}{\mathrm{PF}} = c, \text{ a constant quantity.}$$

If PC > PF, the curve will be an ellipse; if



PC = PF, it will be a parabola; and if PC < PF, the curve will be a hyperbola, In the ellipse and hyperbola, there are two directrices, each of which corresponds to one-half of the curve; in the parabola, there is but a single directrix. The directrix is always perpendicular to the principal axis, and if the curve is given it may be constructed as follows:

Let PAQ represent the curve, TF its axis,



and F its focus. Through F draw FP perpendicular to the axis till it meets the curve in P; at P draw the tangent PT, cutting the axis at T; through T draw a straight line MN perpendicular to the axis, and it will be the directrix required. If the curve is an ellipse or hyperbola, the construction will give a directrix corresponding to each focus lying on different sides of the centre and equally distant from it.

To find the analytical expression for the distance from the vertex of the curve A, to the point in which the directrix cuts the axis, take the equation of the tangent to the ellipse, which is

$$a^2yy'' + b^2xx'' = a^2b^2,$$

and make in it

$$x'' = \sqrt{a^2 - b^2} \quad \text{and} \quad y'' = \frac{b^2}{a};$$

it will become after striking out the common factor b^2,

$$ay + \sqrt{a^2 - b^2}x = a^2,$$

which is the equation of the focal tangent PT; making $y = o$, and finding the corresponding value of x, we have

$$x = \frac{a^2}{\sqrt{a^2 - b^2}} = \frac{a}{e};$$

e denoting the eccentricity of the curve, and x the distance of the point T from the centre. Subtracting from this, a which denotes the semi-transverse axis, we have

$$\mathrm{AT} = \frac{a(1 - e)}{e},$$

which since the expression is entirely independent of the conjugate axis, except in the value of e, will be true for either the ellipse or hyperbola. Making $e = 0$, which corresponds to the circle, we have $\mathrm{AT} = \infty$. Making $e = 1$ which corresponds to the case in which the ellipse becomes a limited straight line, we find $\mathrm{AT} = 0$; and for all other cases of the ellipse the value of AT will be found between the limits. Since the values of AT are always positive in the ellipse, it follows that the directrix cuts the axis at a greater distance from the centre than the principal vertex. For the hyperbola, $e > 1$, and the values of AT are negative; the directrix will therefore cut the axis between the centre and principal vertex. For $c = 1$ which corresponds to the case in which the hyperbola is a straight

line limited towards its centre AT $= 0$, or the directrix passes through the principal vertex. When $c = \infty$, which corresponds to the case in which the hyperbola becomes two parallel straight lines perpendicular to the transverse axis AT $= -a$, or the directrix passes through the centre of the curve. For all other values the directrix will lie between these extreme positions.

In the parabola, as we have already seen, the distance from the focus to the vertex of the curve is always equal to the distance from the vertex to the point in which the directrix cuts the axis. If $a = 0$, which corresponds to the case in which the ellipse becomes a point, and the hyperbola two straight lines which intersect each other, we shall have for both lines AT $= 0$, which shows that the directrix passes through the centre.

In Descriptive Geometry, a directrix is a line along which the generatrix moves in generating a warped or single curved surface.

DIS-CON-TIN′U-OUS. Broken off, interrupted, gaping.

Discontinuous Function is a function which does not vary continuously as the variable increases uniformly.

The function, $\frac{b}{a}\sqrt{x^2 - a^2}$, is a discontinuous function; for, if we suppose x to increase uniformly from $-\infty$, the function will decrease to the value $x = a$, when it becomes 0. From $x = -a$ to $x = +a$, the corresponding values of the function are imaginary; and from $x = a$ to $x = \infty$, the function increases.

The function, $\tan x$, is always an increasing function; but it changes its sign by passing from $+\infty$ to $-\infty$, and the reverse for $x = 90°$, and $x = 270°$.

DIS′COUNT. An allowance made by the creditor for the payment of money before it is due. The actual amount to be paid is called the *present value* of the bill or note, and the difference between the amount specified in the bill and the present value, is the *discount*.

If A gives a note to B for 106 dollars payable in one year, the *present value* of the note at 6 per cent is 100 dollars; because, if 100 dollars be placed at interest for one year, at 6 per cent, the amount at the end of the year will be just 106 dollars.

In any case, the true present value of a note, payable at some future day, is the sum which, being placed at interest for the given period, and at the given rate per cent, will at the end of the time amount to the sum specified in the note, or what is called *the face of the note*. The following rule will serve to determine the present value of a note.

Add to 1 *dollar its interest for the given time at the given rate, and divide the face of the note by this sum; the quotient will be the present value. Subtract this from the face of the note, and the remainder will be the discount.*

When the payments are to be made in installments at different times, find, by the above rule, the discount on each payment, and the sum of these will be the total discount.

Bank Discount. Bankers adopt a different rule for reckoning discount, and one which is much more favorable to them than the one just explained. This kind of discount is called *Bank Discount*, is always paid in advance, and is estimated as follows:

Compute the interest on the face of the note, for the given time, at the given rate per cent; this is the bank discount; and being taken from the face of the note, leaves what is paid to the holder.

The bank discount on a note of 106 dollars for one year, at 6 per cent, is \$6,36, and the present value is therefore \$99,64, instead of \$100, as given by the first rule.

The discount on notes is usually taken from tables, prepared for the purpose, from which the discount may easily be found for any sum, at any rate per cent, and for any given period of time.

DIS-CRĒTE′. [L. *discretus*, separate, distinct].

Discrete Proportion is one in which the ratio of the first term to the second is equal to that of the third to the fourth, but not equal to that of the second to the third: thus, 3 : 6 :: 8 : 16. The proportion 3 : 6 :: 12 : 24 is not a discrete but a continued proportion, or a geometrical progression. A *discrete* quantity is one which is discontinuous in its parts.

DIS-CUSS′. [L. *discutio*, to debate, to examine].

DIS-CUS′SION of a problem or of an equation, is the operation of assigning every reasonable value to the arbitrary quantities which enter the equation, and interpreting the results. An example of the discussion of an equation may be seen under the head of *Eccentricity*, where the equation expressing the eccentricity of a conic section is fully discussed.

DIS′TANCE. [L. *distantia*, *dis*, from, and *sto*, to stand apart]. The distance between two points is the length of a line joining the two points, expressed in terms of some line which is assumed as the unit of length. The numerical value of the distance is the ratio of the unit of measure to the distance to be measured or expressed. When not otherwise specified, the distance between two points is understood to mean the shortest distance, or the distance estimated on a straight line joining the points.

In Surveying, distances are distinguished as *vertical distances* or *heights*; *horizontal distances*, or those estimated in a horizontal plane; and *oblique distances*, which are neither horizontal nor vertical.

We shall only consider horizontal distances; these may be classed into *accessible* and *inaccessible*.

1. Accessible distances are those which may be measured by the direct application of some linear unit of measure.

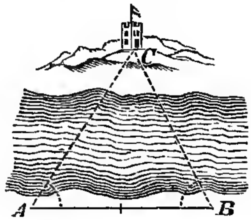
This class of distances requires no further consideration than the mere statement, that they are ascertained by the repeated application of some convenient scale, which may be a chain, tape, or measuring-rod. In field-surveying, the unit of measure is Gunter's chain, which is 66 feet in length, and is subdivided into hundredths, called *links*. Tapes and measuring-rods, when employed, may be of any convenient length, and may be subdivided into links, or into feet and parts of a foot.

In all cases, when a horizontal distance is to be measured, care should be taken to keep the chain or rod truly horizontal. The accuracy of the measurement will depend upon the care exercised, and upon the nature of the apparatus employed. When great accuracy is necessary, the base apparatus, already spoken of, is most to be relied upon.

2. Inaccessible distances are those which either cannot be reached, or which are inconvenient to reach, so as to apply to them the linear unit. Such distances are determined by measuring an auxiliary distance and certain auxiliary angles. From these data the required distances are computed by the rules of Trigonometry. They may also be determined, approximately, by various methods, employing the simple principles of Elementary Geometry. There is still a third method, by means of the known velocity of sound through the atmosphere. We shall consider these methods separately.

I. *By Triangulation*. There are several cases:

1. To determine the distance from a given point to an inaccessible point.



Let C be the inaccessible point, A the given point, and let it be required to determine the distance AC. Select some point B, from which both A and C are distinctly visible, and measure the distance AB, the angle CAB, and the angle CBA. Then, in the triangle CAB, there will be known a sufficient number of parts to determine the remaining ones. From the known angles A and B, the third angle C may be found, and then we shall have the proportion

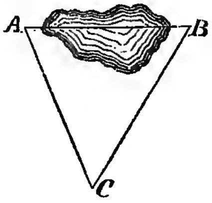
$$\sin C \quad \sin B :: AB : AC,$$

$$\therefore \quad AC = AB \frac{\sin B}{\sin C}.$$

2. To find the distance between two objects, when there is an intervening obstacle

Let A and B (next Figure) represent the objects between which the distance is required. Select some point C, from which both A and B are visible, and measure the distances between AC, BC, and the angle C.

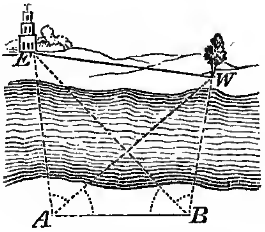
We have, to determine the two angles at A and B, the following proportion,



$$AC + BC : AC - BC :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B),$$

from which the angles may be found, and the side AB of the triangle ABC, may then be found by the method given in the first case.

3. To determine the horizontal distance between two inaccessible objects.



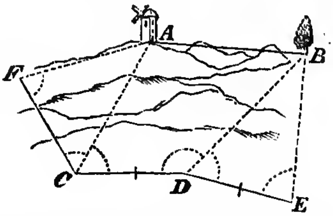
Let E and W be the objects. Select two stations, A and B, from which both objects may be distinctly seen, and which are visible from each other.

Measure the distance AB, the angles EBW, EBA, EAW, and WAB.

From the given parts of the triangle EAB, the side EB may be determined, as in the first example. Also, from the triangle ABW, the side BW may be determined by the same rule; then in the triangle EBW there will be known the sides EB, BW, and their included angle, from which EW may be found, as in the second case.

4. To determine the distance between two points which are inaccessible, and such that no station can be found from which both are visible at the same time.

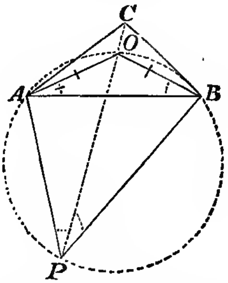
Let A and B represent the objects; select two points, C and D, which can be seen from each other, and such that the object B is visible from D, and the object A from C; select also a station E from which B and D are visible, and a station F from which A and C are



visible. Measure the distances FC, CD, DE, and the angles CFA, ACF, ACD, BDC, BDE and BED. In the figure draw an auxiliary line CB.

From the two triangles AFC and BDE, the sides AC and BD may be computed, as already explained in problem first. In the triangle BCD we shall then know two sides, and their included angle; we may, as in problem second, determine the side BC and the angle BCD; subtracting the angle BCD from ACD gives ACB, and in the triangle ACB we shall therefore know two sides and the included angle from which the third side AB may be found as before.

5. To find the distance of an object from either of three points whose relative positions are known. Let P denote the object and let A, B, and C denote the fixed points, and suppose that we know the distances AB, BC, and CA, also the angles ABC, ACB, and BAC. From P measure the angles APC and BPC.



In the figure draw AO, making the angle OAB equal to OPB and BO, and making the angle ABO equal to the angle APO. In the auxiliary triangle AOB, we know the side AB, and the angles at A and B, and we compute the side AO and the angle OAB; the angle OAB taken from CAB gives OAC, and

we shall then know the angle OAC, and the sides OA and CA, from which the angle ACP may be computed; then in the triangle ACP we shall have given the side AC, the angles ACP and APC, and consequently the third angle PAC; the sides AP and PC may then be computed. In like manner the side BP may be found. For a more complete discussion of this problem in its different cases, see *Problem of Three Points.*

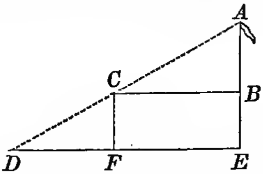
A judicious combination of the various cases just discussed, will be amply sufficient to determine any inaccessible distance.

II. To determine the distance between two points without instruments for measuring angles, by the aid of the principles of Elementary Geometry.

1. To determine the horizontal distance from a given point to an inaccessible object.

First method.

Let A be the inaccessible object, and E the point from which the distance EA is to

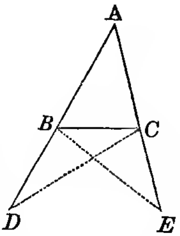


be determined. At E, lay off a right angle AED, and select a station at any point D of the line ED. Range a flag-staff between D and A, at some point as C, and lay off a line CF perpendicular to DE; measure the distances DF, CF and DE; then, from similar triangles,

$$DF : FC :: DE : AE; \therefore EA = \frac{FC \times DE}{DF}.$$

Second Method.

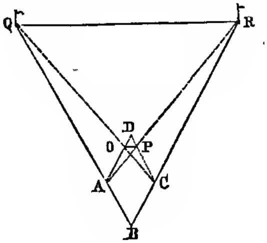
Let A be the inaccessible object, and B the given point. Select a station at some suitable point C, and then range the flag D in the line AB, and the flag E on the line AC. Measure the distances BC, CE, BE, BD, and DC.



In the triangle BCE all the sides are known, and the angle BCE may be found from the formulas given in plane trigonometry. From this we can find its supplement, which will be the angle ACB. In like manner we can find the angle ABC, and then in the triangle ABC we shall know two angles and the included sides, from which to find the side BA, the required distance.

2. To find the distance between two inaccessible objects.

Let Q and R be the positions of the objects. Assume some convenient point as B, and measure off in the lines BR and BQ, equal distances BC and BA; then find a



point D such that PD and CD shall be equal to AB and BC. Plant flags at O and P, the former at the intersection of AD and CQ, and the latter at the point of intersection of CD and AR; then will the auxiliary triangle ODP be similar to the triangle BQR. Measure the distances OD, OP, and PD: then will

$$RQ = \frac{OP \times BA^2}{OD \times DP};$$

for, from similar triangles,

$$PD : DA :: AB : BR \therefore BR = \frac{AB^2}{PD}, \quad \text{and}$$

$$OD : OP :: BR : RQ \therefore RQ = \frac{AB^2 \cdot OP}{OD \cdot DP}.$$

3. To find the distance from a point to any distant object, by means of a micrometer attached to a telescope.

Instruments of this kind have been constructed, by means of which very small angles can be measured. In employing them for measuring distances, all that is necessary to know is the angle subtended by an object of known dimensions, placed either horizontally or vertically at the remote extremity of the distance which we wish to measure. Approximate results may be obtained if there is

a house at that extremity built of bricks of the ordinary size, by regarding four courses in height as equal to one foot, or four in length as equal to one yard. Distances thus found will be tolerably accurate, if care is taken to make the line whose subtended angle is measured, exactly at right angles with the telescope. The best method, when practicable, is to plant a staff at one extremity of the distance, of known length, and exactly vertical, whilst the observer, with his micrometer, stands at the other. If h denote the height of the object, a the angle subtended, and D the distance, then will

$$D = \tfrac{1}{2} h \cot \tfrac{1}{2} a\,; \quad \text{or,} \quad D = h \cot a.$$

The first formula is used when the angle is a large one, and the eye opposite the middle of the object whose angle is measured; the latter one will apply when the eye is opposite one extremity of the line whose angle is measured, or when the angle is very small.

When a table of natural tangents is not at hand, a very close approximation for all angles less than a half degree, and a tolerable one for all angles up to a degree, will be furnished by the following rule:

If the distant object whose subtended angle is measured, is equal to one foot, and if n' denote the number of minutes, or n'' the number of seconds in the measured angle, the distance will be given by the formulas

$$D = n' \times 3437^{\text{ft}}, \text{ and } D = n'' \times 206264^{\text{ft}}.$$

If the distant object be 3, 6, 9, &c., feet, multiply the values thus obtained by 3, 6, 9, &c., respectively.

III. *By the Velocity of Sound.* It is found both by theory and experiment that at the same temperature, the velocity of sound through the atmosphere is constant, and at a standard temperature of 32° is equal to $1089^{f}\cdot 42$ per second. For any other temperature, the velocity is given by the formula,

$$V = 1089^{f}.42\sqrt{1 + (t - 32^\circ) \times 0.00208},$$

in which V denotes the velocity, and t the number of degrees indicated by a Fahrenheit thermometer.

To use this formula to compute distances, let a gun be fired at the remote station, and by the aid of a watch, which beats half or quarter seconds, note the length of time which elapses between seeing the flash and hearing the report; note also the temperature, thus obtaining the value of t. The interval expressed in seconds multiplied by

$$1089^{f}.42\sqrt{1 + (t - 32^\circ) \times 0.00208},$$

will give the distance required in feet. When the data have been noted carefully, the formula gives a very close approximation to the true distance.

When it is not desirable to attain great accuracy, the following rule will be sufficiently accurate:

Assume the velocity at 32° equal to 1090 feet, and add a half foot for each degree above 32°, or subtract a half foot for each degree below 32°; multiply this by the interval in seconds, and the product will be the distance required in feet.

DI-VERG'ING. Receding, separating. Diverging Series is a series in which each term is numerically greater than the preceding one, as

1 : 3 9 . 27 : 81 . . &c.

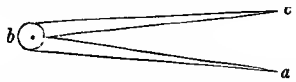
DI-VĪDE'. [L. *divido*, to part]. To resolve or separate into parts or factors. One quantity is said to be *divisible* by another, when it can be resolved into two entire factors, one of which is the first quantity or divisor. When *divisible* is used, it implies the second factor is an entire quantity, or sometimes that it is a commensurable quantity. This is the elementary idea, but by extension, the term has lost much of its original signification, and we say that one quantity can be divided by another, when a third quantity can be found, which, multiplied by the first, will produce the second. The first and third quantities are called *factors*.

In Topography, a *divide* is a ridge of land which separates the affluents of one stream from another. It is an irregular line, and the crest of the dividing ridge. The divide between any two streams, may be traced upon a map by drawing a line so that it shall head all of the affluents of both streams.

DIV'I-DEND. A quantity which is to be divided by another, called the *divisor*.

DI-VĪD'ERS. A mathematical instrument used in laying off distances, and describing circles in drawing.

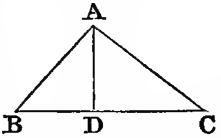
It is made of steel, or of a combination of steel and other metals, and consists essen-



tially of two legs ab and cb, which turn about a hinge at b. The legs are terminated in sharp points, and are sometimes so constructed as to admit of one or both legs being removed, so as to be replaced by a pencil or pen. The use of the instrument is apparent.

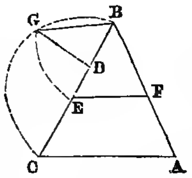
Dividing Land. In Surveying, is the operation of marking out upon a field lines, so that the portions thus marked out, may contain either certain given areas, or may bear fixed ratios to the whole tract to be divided. Fields are so differently shaped, that it is difficult to give any system of rules that will apply in all cases of the division of estates. The following principles, when judiciously combined, will serve to guide the surveyor in most cases.

I. To run a line from the vertex of a triangular field, that shall divide it into two parts, which shall have to each other the ratio of m to n.



Let ABC be a plot of the field, and A the vertex of the angle from which the dividing line is to be run. Measure the side BC, and divide it into two parts, which shall bear to each other the ratio of m to n. Let D be the point of division. In the field, measure off the distance BD equal to its value found, and at D plant a staff; then will the line run from A to D be the dividing line required.

II. To run a line parallel to one of the sides of a triangular field, so as to divide it into parts, which are to each other as m is to n.



Let ABC be a plot of the field, and let it be required to run the dividing line parallel to the side AC, so that the part BEF shall be to the part EFAC, as m is to n. On CB, as a diameter, describe a semi-circle. Divide the side BC into two segments at D, such that

$$BD : DC :: m : n;$$

at D erect the perpendicular DG, cutting the circle in G; draw the chord GB, and with B as a centre, and with GB as a radius, describe the arc GE, cutting the side in E; through E draw the line EF parallel to CA, and it will be the dividing line required. For,

$$BD \cdot DC :: m : n; \text{ whence,}$$

$$BD : BD + DC :: m : m + n \cdot (1),$$

but from a known property of the circle, we have

$$GB^2 \text{ or } BE^2 \cdot BC^2 : \cdot BD . BD+DE,$$

$$\text{or } \quad BE^2 . BC^2 : m : m + n .$$

But the triangles BEF and BCA, being similar, give the proportion,

$$BEF : BCA :: BE^2 . BC^2 :: m . m+n,$$

whence, by division,

$$BEF \cdot BCA - BEF :: m : n,$$

which agrees with the construction.

The distance BE may be computed from the above data, and the distance BF may be found from the proportion

$$BE : BC \cdot . BF : BA,$$

$$\therefore BF = \frac{BE \times BA}{BC}.$$

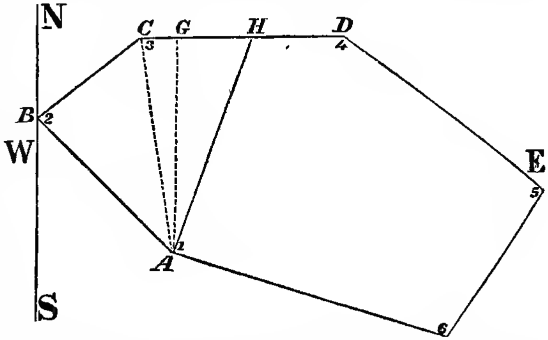
Measure off, in the field, on the side BC the distance BE, and plant a staff at E; in like manner, measure off the distance BF, in the line BA, and plant a staff at F; then will the line run from one staff to the other be the dividing line required.

III. To run a line from a point in the boundary of a polygonal field, so as to cut off from the field a given area.

Let ABCD . . . A be a plot of the field, obtained according to the rules laid down for field surveying. and let it be required to run a line from a point A, so that the portion ABCHA cut off, shall be equal to a given area S.

We may find, by examining the plot, or by a rough computation of the area by the aid of the dividers and a scale of equal parts, the angular point of the field nearest to which the dividing line will terminate. Suppose this point to be C. Draw the line AC; this is called the first closing line. From the field notes the bearing and length of the course CA may be determined,

and then the area ABC may be computed as though it were a separate field. Suppose that the area thus found is less than S, and denote it by S′. It now remains to find the point H, such that the triangle ACH will be equal to S − S′. In the plot draw AG perpendicular to CD; knowing the bearings of AC and CD, we can find the angle ACG, and the sine of this angle multiplied by the distance CA, will give the length of the perpendicular AG, which denote by h; then will the line CH be equal to the area of the triangle ACH divided by half of its altitude, or $CH = \frac{S - S'}{\frac{1}{2}h}$.



Measure off in the field, upon the side CD, the distance CH, and plant a staff at H; then will the line run from A to H be the dividing line required. If $S' > S$, the point H will fall upon the side CB, and we shall then have to use the angle BCA, and a perpendicular to the side BC. The process will be entirely analogous to that already considered.

If the point from which the dividing line is to be run, is not at an angular point of the field, but on a side, we may regard the side upon which it is situated as two separate courses, meeting at that point, both having the same bearing.

DI-VĬ″SION,′ Is the operation of finding from two quantities a third, which multiplied by the first shall produce the second. The *first* is called the *divisor*, the *second* the *dividend*, and the *third* the *quotient*. Both divisor and quotient are *factors* of the dividend. Division might then be defined to be the operation of finding the second factor of a given quantity, knowing the first.

We shall consider in their order, 1st. *Arithmetical division*, and second, *Algebraical division*.

I. Arithmetical Division.

1. *When the numbers are expressed in the scale of tens.*

Write down the dividend, and on its left write the divisor, separating them by a line. Beginning with the highest order of units of the dividend, pass on to the lower orders until the fewest number of figures is found that will contain the dividend; see how many times the divisor is contained in these figures and write the number on the left for the first figure of the quotient; the unit of this figure will be the same as the lowest unit used in the dividend. Multiply the divisor by this figure of the quotient, and place the product under the figures of the dividend used; subtract this from the figures used, and to the remainder bring down the next figure of the dividend; see how often the divisor is contained in this result, and write the number as the second figure of the quotient; proceed as before and continue the operation until all the figures of the dividend have been used; the final result will be the quotient sought.

If the final remainder is 0, the division is said to be exact, or the dividend contains the divisor an exact number of times; if the remainder is not 0, the division is not exact, and the quotient is true to within less than 1. The operation may be verified by multiplying the quotient obtained by the divisor, and adding to the product the remainder, if there is one; this last result should be the same as the dividend, if the operation has been performed correctly.

2. *When the numbers are expressed in varying scales.*

There may be several cases, according as the numbers are expressed in terms of ordinary fractional units, or in some of the varying scales of commerce.

If the numbers to be divided are fractions, they will be either vulgar or decimal.

1*st. Division of vulgar fractions.*

Reduce both dividend and divisor to the form of simple fractions, if one or both are mixed fractions; invert the terms of the divisor and multiply the dividend by the resulting fraction; the product will be the fractional quotient, which should be reduced to its lowest terms, or sometimes to a whole number by the rules for the transformation of fractions.

1. Divide $5\frac{3}{8}$ by $2\frac{1}{7}$; $5\frac{3}{8} = \frac{43}{8}$ and $2\frac{1}{7} = \frac{15}{7}$, hence the quotient is equal to

$$\frac{43 \times 7}{8 \times 15} = \frac{301}{120},$$

which cannot be reduced to lower terms.

If one of the numbers is a whole number, it may be regarded as a vulgar fraction whose denominator is 1.

2*d. Division of decimal fractions.*

Both dividend or divisor, or either one may be a mixed decimal, but the rule applies to all.

Write down the dividend and divisor as in whole numbers, annexing as many 0's to the dividend as may be necessary; perform the division as in whole numbers, continuing the process to any desirable extent, or till a remainder is found equal to 0; then point off from the right hand as many decimal places as the number of decimal places in the dividend exceeds that in the divisor; if there are not so many in the quotient, prefix 0's till the requisite number is obtained; the result will be the quotient.

When the numbers are expressed in any of the varying commercial scales, as *pounds*, *shillings*, and *pence*, or in any other of the irregularly varying scales, there may arise two cases; 1st. when the dividend is expressed in the varying scale, and the divisor in the scale of tens; and 2d. when both dividend and divisor are expressed in varying scales.

1*st. When the divisor is expressed in the scale of tens.*

In this case the quotient will be expressed in the same scale as the dividend.

Divide the number of units of the highest order in the dividend by the divisor, the quotient will be the number of units of the same order in the quotient sought. Multiply the remainder by the number of units of the next lower order which make one of this order, and to the product add the number of units of the next lower order in the given number; divide this sum by the divisor and the quotient will be the number of units of this order in the quotient sought; continue this operation till the lowest order of the scale is reached, and the result will be the quotient sought.

1. Let it be required to divide £25 8*s.* 6*d.*, by 3. The operation is as follows:

```
     £  s. d.  £  s. d.
   3)25  8  6( 8  9  6
     24
     --
      1
      20
      --
      28
      27
      --
       1
       12
       --
       18
       18
       --
        0
```

The operation may be much simplified in practice.

2. *When both dividend and divisor are expressed in varying scales.* In this case it is necessary that they should both be reduced to the same unit, and then the quotient will be expressed in the scale of tens.

Reduce both dividend and divisor to the same absolute unit, generally the lowest unit of the scale; then divide as in whole numbers.

1. Divide £25 10*s.* 9*d.* by 18*s.* 11*d.* If both be reduced to pence, we have

£	s.	d.		d.		s.	d.		d.
25	10	9	=	6129	and	18	11	=	227,

whence

$$\frac{6129}{227} = 27 \text{ the quotient required.}$$

Or both might have been reduced to decimals of a pound, thus

£	s.	d.		£	s.	d.		£
15	10	9	=	25.5375,	18	11	=	.94583 +

whence

$$\frac{25.5375}{.94583} = 27,$$

the same quotient as before. These rules will enable us to perform any case of arithmetical division. For a method of verification of division, see *Properties of the* 9's.

II. Algebraical Division.

1st. Division of monomials.

Divide the co-efficient of the dividend by the co-efficient of the divisor, for the co-efficient of the quotient; after this, write all the letters which enter the dividend and divisor, giving to each an exponent equal to the excess of its exponent in the dividend over that in the divisor; the result is the quotient sought.

If the signs of the dividend and divisor are *alike*, the sign of the quotient will be *plus*; if they are *unlike* it will be *minus*.

The *exact* division is impossible: 1st, when the co-efficient of the dividend is not exactly divisible by the co-efficient of the divisor: and 2d. when the exponent of any letter in the divisor is greater than it is in the dividend. This last case includes that in which the divisor contains a letter which does not enter the dividend.

As to the co-efficient, if the exact division is not possible, it may be indicated, and by the employment of negative exponents the remaining operation may always be indicated.

2d. Division of a polynomial by a monomial.

Divide each term of the polynomial by the divisor separately, and the algebraic sum of the separate results will be the quotient sought.

3d. Division of polynomials.

Arrange both dividend and divisor with reference to the same leading letter; divide the first term on the left of the dividend by the first term on the left of the divisor, and the quotient will be the first term of the quotient; multiply the divisor by the term of the quotient found, and subtract the product from the dividend for the first remainder: divide the first remainder by the first term of the divisor for the second term of the quotient; multiply the divisor by this term of the quotient, and subtract the product from the first remainder for the second remainder; continue the operation till a remainder is found which is 0, or till one is found whose first term is not exactly divisible by the first term of the divisor.

In the former case the division is exact, in the latter the exact division cannot be performed.

The following rules serve to indicate, by inspection, certain cases in which the division cannot be exactly performed.

1st. When the term of the dividend which contains the highest or lowest power of any letter, is not exactly divisible by the term of the divisor which contains the highest or lowest power of the same letter.

2d. When the divisor contains a letter which does not enter the dividend.

3d. When the dividend is a monomial, and the divisor a polynomial.

If the dividend contains a letter which does not enter the divisor, the division cannot be performed unless the co-efficients of the different powers of this letter are separately divisible by the divisor. In some cases, when the exact division is not possible, the operation may be continued, and a result obtained in the form of a series, having an infinite number of terms.

The operation of the division of polynomials may sometimes be shortened, by performing the multiplication of each term of the divisor by each term of the quotient, and subtracting the product mentally from the corresponding term of the dividend, writing down only the last result. It may, in some cases, be considerably abbreviated by a method called

Synthetic Division.

This method is applicable when the dividend and divisor are homogeneous, and contain but two letters.

In the common method of division, each term of the divisor is multiplied by the first term of the quotient, and the products subtracted from the dividend; but the subtractions are performed by first changing the sign of the product, and then adding. If, therefore, the signs of the terms of the divisor were first changed, we should obtain the same result by adding the products, instead of subtracting as before, and the same for the subsequent operations.

By this process, the second dividend would be the same as by the common method. But since the second term of the quotient is found by dividing the first term of the second dividend by the first term of the divisor; and since the sign of the latter has been changed, it follows that the sign of the second term of the quotient will also be changed. To avoid this change of sign, the sign of the first

term of the divisor is left unchanged, and the products of all the terms of the quotient by the first term of the divisor are omitted, because in the usual method, the first terms in each successive dividend are cancelled by these products.

Having made the co-efficient of the first term of the divisor 1, proceeding by the method of detached co-efficients, and omitting these several products, the co-efficient of the first term of any dividend will be the co-efficient of the succeeding term of the quotient Hence, the co-efficients in the quotient are, respectively, the co-efficients of the first terms of the successive dividends.

The operation thus simplified, may be further abridged by omitting the successive additions, except so much only as may be necessary to show the first term of each dividend; and also by writing the products of the several terms of the quotient by the modified divisor diagonally, instead of horizontally, the first product falling under the second term of the dividend. The literal part may be written immediately, since the exponent of the leading letter will go on decreasing from left to right, the sum of the exponents of both letters in any term being constant.

We may then write the following rule:

Divide both divisor and dividend by the co-efficient of the first term of the divisor.

Write in a horizontal line the co-efficients of the dividend with their proper signs, and place the co-efficients of the divisor with all their signs changed, except the first, on the right, observing when any term is wanting, to supply its co-efficient by 0.

Divide the co-efficients of the dividend by those of the divisor, after the manner of algebraic division, except that no term of the quotient is multiplied by the first term of the divisor; write the products diagonally to the right, under the terms of the dividend to which they correspond.

The first term of the quotient is the same as that of the dividend; the second term is the sum of the numbers in the second column; the third term, the sum of the numbers in the third column; and so on.

When the division can be exactly made, columns will be found to the right, whose sum will be 0: when the co-efficients are found, annex to each the literal part, as already explained.

1. Divide

$$a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$$

by $\quad a^2 - 2ax + x^2.$

OPERATION.

$$\begin{array}{llllll|l} 1 & -5 & +10 & -10 & +5 & -1 & \underline{1, +2 - 1} \\ & +2 & -6 & +6 & -2 & & \\ & & -1 & +3 & -1 & +1 & \\ \hline 1 & -3 & +3 & -1 & 0, & 0. & \end{array}$$

Hence, the quotient is

$$a^3 - 3a^2x + 3ax^2 - x^3.$$

The first term of the divisor being always 1, need not be written.

In the above operation, the first term of the quotient is 1, which we write in the last line under the first term of the dividend; multiplying $+2$ and -1 by this, and placing the results diagonally in the 2d and 3d columns, we get $+2$ and -1; the sum of the terms in the second column is -3, which we write in the last line as the second term of the quotient; multiplying $+2$ and -1, by this we get -6 and $+3$, which are written diagonally in the 3d, and 4th columns: the sum of the numbers in the 3d column is $+3$, which we write in the last line, and multiply by $+2$ and -1, as before; the sum of the fourth column is -1; by continuing in this manner, we find that the sum of all the remaining columns are 0.

To annex the literal parts, we notice that the literal part of the first term is a^3, and that the exponents of a must go on diminishing by 1 in each term to the right, whilst the exponents of x go on increasing by 1 in each term to the right.

2. Divide

$$x^6 - 5x^5 + 15x^4 - 24x^3 + 27x^2 - 13x + 5$$

by $\quad x^4 - 2x^3 + 4x^2 - 2x + 1.$

OPERATION.

$$\begin{array}{lllllll|l} 1 & -5 & +15 & -24 & +27 & -13 & +5 & \underline{1+2-4+1-1} \\ & +2 & -6 & +10 & & & & \\ & & -4 & +12 & -20 & & & \\ & & & +2 & -6 & +10 & & \\ & & & & -1 & +3 & -5 & \\ \hline 1 & -3 & +5 & +0 & +0 & +0 & +0. & \end{array}$$

Hence, the quotient is $x^2 - 3x + 5$.

4. *Division of Radicals.*

First reduce the radicals to equivalent ones of the same degree; then divide the co-efficient of the dividend by that of the divisor for the co-efficient of the quotient; write after this the common radical sign, under which place the quotient obtained by dividing the quantity under the radical sign in the dividend by that in the divisor.

5. *Division by means of Logarithms.*

One number may be divided by another by means of logarithms, as follows:

Find, from a table, the logarithm of the dividend and of the divisor; subtract the latter from the former, and find from the table the number corresponding to the remainder; it will be the quotient required.

DI-VĪ'SOR. One of the factors of the dividend; that factor by which the dividend is to be divided.

DIVISORS OF A NUMBER, are those numbers by which it is exactly divisible; thus, 1, 2, 3, 4, 6, and 12, are divisors of 12, because 12 may be divided by each of them without a remainder. A prime number can only be divided by 1 and the number itself.

1. Composite numbers admit of several divisors. If we denote a composite number by N, and its prime factors respectively by a, b, c, &c; and by m, n, p, &c., positive whole numbers, we shall have

$$N = a^m b^n c^p \cdots$$

and the whole number of its divisors will be equal to

$$(m + 1)(n + 1)(p + 1) \cdots \text{\&c.}$$

For example, $360 = 2^3 \cdot 3^2 \cdot 5^1$, Here, $m = 3$, $n = 2$, $p = 1$, and the whole number of divisors of 360 is $4 \times 3 \times 2$, or 24; that is, there are 24 different numbers which will divide 360 without a remainder.

2. If required, we can find a number which shall have any number of divisors

Denote the required number of divisors by w, resolve w into any number of factors possible, so that

$$w = x \times y \times z \cdots;$$

then denote $x - 1$ by m, $y - 1$ by n, $z - 1$ by $p \cdots$ &c., and we shall have

$$N = a^m b^n c^p \cdots \text{\&c.},$$

in which a, b, c, &c., may be any prime numbers whatever, which are not equal to each other. For example: let it be required to find a number which has 30 different divisors. We have $30 = 2 \cdot 3 \cdot 5$; whence, $m = 1$, $n = 2$, $p = 4$ and $N = a^1 b^2 c^4$.

Assume	$a=2, b=3, c=5$,	we find	$N=11250$
"	$a=5, b=3, c=2$,	"	$N=720$
"	$a=5, b=2, c=3$	"	$N=1620$
&c.	&c.	&c.	&c.

there will be an infinite number of solutions.

3. If we still suppose

$$N = a^m b^n c^p \cdots$$

the sum of all of the divisors of N will be expressed by the formula

$$S = \frac{a^{m+1} - 1}{a - 1} \times \frac{b^{n+1} - 1}{b - 1} \times \frac{c^{p+1} - 1}{c - 1} \times \cdots$$

For example: the sum of all the divisors of 360 will be

$$S = \frac{2^4 - 1}{2 - 1} \times \frac{3^3 - 1}{3 - 1} \times \frac{5^2 - 1}{5 - 1}$$
$$= 15 \times 13 \times 6 = 1170.$$

This sum includes 1 and the number itself.

4. If a number is of the form $t^2 + u^2$, in which t and u are prime with respect to each other, then will every divisor be of the same form; that is, of the form $t'^2 + u'^2$, in which t' and u' are prime with respect to each other. For example, $65 = 8^2 + 1^2$, and its divisors are $5 = 4^2 + 1^2$ and $13 = 3^2 + 2^2$. Also, $50 = 49^2 + 1^2$, and its divisors are $2 = 1^2 + 1^2$, $5 = 2^2 + 1^2$, $10 = 3^2 + 1^2$, and $25 = 4^2 + 3^2$.

5. If a number is of the form $t^2 + 2u^2$, t and u being prime with respect to each other, its divisors will also be of the same form.

For example: $99 = 1^2 + 2 \cdot 7^2$, and its divisors are $3 = 1^2 + 2 \cdot 1^2$, $9 = 1^2 + 2 \cdot 2^2$, $11 = 3^2 + 2 \cdot 1^2$ and $33 = 1^2 + 2 \cdot 4^2 \cdots$

6. If a number is of the form $t^2 - 2u^2$, its divisors are also of the same form.

Thus, $98 = 10^2 - 2 \cdot 1^2$, and its divisors are $2 = 2^2 - 2 \cdot 1^2$, $7 = 3^2 - 2 \cdot 1^2$, $14 = 4^2 - 2 \cdot 1^2$, and $49 = 9^2 - 2 \cdot 4^2$.

7. If a number is of the form $t^2 + 3u^2$, every odd divisor of it is also of the same form.

Thus, $133 = 5^2 + 3 \cdot 6^2$, and its odd divisors are $7 = 2^2 + 3 \cdot 1^2$, and $19 = 4^2 + 3 \cdot 1^2$.

8. If a number is of the form $t^2 - 5u^2$, every odd divisor of it is also of the same form.

For example: $95 = 10^2 - 5 \cdot 1^2$, and its

odd divisors are $5 = 5^2 - 5 . 2^2$ and $19 = 8^2 - 5 . 3^2$.

The above properties of divisors of numbers were discovered by Lagrange.

Various other properties of divisors might be enumerated, but we shall only enunciate some of the more useful ones, which are employed in the theory of numbers.

1. If a is a prime number, and if x is not divisible by a, then will a divide $(x^{n-1} - 1)$.

2. If a is a prime number it will divide the expression

$$[1 \cdot 2 \cdot 3 \cdots (n-1)] + 1.$$

3. If a is a prime number, and of the form $4n + 1$, it will divide the expression

$$\left[1^2 \cdot 2^2 \cdot 3^2 \cdots \left(\frac{a-1}{2}\right)^2\right] + 1.$$

4. If a is a prime number, and of the form $4n - 1$, it will divide the expression

$$\left[1^2 \cdot 2^2 \cdot 3^2 \cdots \left(\frac{a-1}{2}\right)^2\right] - 1.$$

5. If the sum of the digits of a number is divisible by either 3, or 9, the number itself will also be divisible by the same number, 3 or 9

6. If a number is divisible by 11, the sum of the digits in the even places is equal to the sum of the digits in the odd places.

7. If the number expressed by the n right hand digits of any number is divisible by 2^n, the number itself is divisible by 2^n.

DO-DEC′A-GON. [Gr. δωδεκα, twelve, and γωνια, angle]. In Elementary Geometry, a polygon of twelve sides or twelve angles.

To inscribe a regular dodecagon in a circle, apply the radius 6 times to the circumference as a chord and the inscribed figure will be a regular hexagon; bisect the arcs subtended by each side of the hexagon, and join each of the points of bisection with the vertices of the consecutive angles; the figure inscribed will be a regular dodecagon.

If one side of a regular dodecagon be taken as the unit of measure, the entire area will be equal to 11.1961524. If it is required to find the area of any other regular decagon, since similar figures are to each other as the squares of their homologous sides, we have simply to multiply the square of one side by 11.1961524, the product will be the area required.

DO-DEC-A-HĒ′DRAL. Pertaining to a dodecahedron. Having twelve faces.

DO-DEC-A-HĒ′DRON. [Gr. δοδεκα, twelve, and εδρα, base]. A polyhedron bounded by twelve faces. The regular dodecahedron is bounded by twelve equal and regular pentagons, and is one of the *five regular* polyhedrons of geometry. The diedral angle included between any two adjacent faces is 116° 33′ 54″. If we denote the length of one of its edges by 1, the area of its surface is 20.6457288, and its solidity 7.6631189. If we denote the length of an edge by l, the surface will be equal to $l^2 \times 20.6457288$, and its solidity $l^3 \times 7.6631189$.

If we denote by R the radius of the circumscribing sphere, we shall have the following formulas, in which l denotes the length of the edge, s the area of the surface, and v the volume:

$$1. \quad l = R \cdot \left(\frac{\sqrt{15} - \sqrt{3}}{3}\right).$$

$$2. \quad s = R^2 \cdot 10 \left(\sqrt{2 - \frac{2}{5}\sqrt{5}}\right).$$

$$3. \quad v = R^3 \cdot \frac{20}{3} \sqrt[3]{\frac{3 + \sqrt{5}}{30}}.$$

In like manner, if r denote the radius of the inscribed sphere, we have the formula for the length of an edge,

$$l = r\sqrt{50 - 22\sqrt{5}}.$$

We have also, for the radius of the circumscribing and inscribed spheres,

$$R = \left(\frac{\sqrt{15} + \sqrt{3}}{4}\right) \times l, \qquad \text{and}$$

$$r = \left(\frac{\sqrt{250 + 110\sqrt{5}}}{20}\right) \times l$$

If we have a regular dodecahedron and a regular hexahedron, or cube, inscribed in the same sphere, then if the edge of the cube be divided in extreme and mean ratio, the greater segment will be equal to the edge of the dodecahedron.

If a line be divided in extreme and mean ratio, and the lesser segment be taken as the edge of a regular dodecahedron, then will the other segment be equal to the edge of a cube

inscribed in the same sphere. These properties are demonstrated in Hutton's Mensuration. See *Regular Polyhedron.*

DOL'LAR. A silver coin of the United States, whose value in the scale of coins adopted is equal to 100 cents, 10 dimes or $\frac{1}{10}$ of an eagle. See *Money.*

DOU'BLE. [L. *duplus*]. Twice as much; a quantity repeated once.

DOZ'EN. Twelve in number.

DRAW'ING. The operation of making such a representation of an object or of objects upon a surface as to present to the eye taken at a particular point, the same appearance as that presented by the object itself. See *Perspective* and *Practical Geometry.* The delineation is called a *drawing.*

DRUM'MOND LIGHT. An intense light, produced by directing the flame of the oxy-hydrogen blowpipe upon a piece of quick lime. The light thus produced rivals in intensity that of the sun itself, and has been used for photographic and other purposes usually requiring the light of the sun.

The light was originally used by Drummond for illuminating signals, so as to render them visible at a great distance, and with such success that they might be seen at distances of 70 to 90 miles. At a distance of 7 miles, the light of Drummond cast a well defined shadow of an opaque object upon a screen 4 feet from the body. A pretty good substitute for the Drummond light may be formed by directing a stream of oxygen gas through the flame of a spirit-lamp upon a lump of quick lime. The Drummond light is still used in extensive triangulations.

DU-O-DEC'I-MAL. [L. *duodecim*, twelve]. A system of numbers in arithmetic, whose scale is 12; hence, the unit of each order is equal to twelve times a unit of the next lower order. This system is usually employed by artificers in estimating the superficial and solid contents of their work. See *Arithmetical Scale.*

DŪ'PLI-CĀTE. [L. *duplex*, double]. Double, two-fold.

Duplicate Ratio. The same as the square of the ratio; thus, the duplicate ratio of a to b is $\frac{b^2}{a^2}$.

DU-PLI-CĀ'TION OF THE CUBE. The operation of finding a cube whose volume is equal to double that of a given cube. The solution of this problem cannot be effected geometrically, as it requires the construction of two mean proportionals between two given lines. It may be solved by higher geometry, but its solution in this manner is rather curious than useful.

E. The fifth letter of the English alphabet; amongst the ancients it stood for 250. In surveying, it is employed as an abbreviation for East, one of the points of the compass.

EARTH. The name of the planet upon which we dwell. It is the third in order from the sun, and revolves about the sun in an elliptical orbit, of which the centre of the sun is one focus, in about 365¼ days. It also revolves about its own axis in 24 sidereal hours, or once in each day. The plane of the earth's orbit is called the ecliptic, and is inclined to the plane of the equator by an angle of about 23° 28′, which angle is constantly undergoing a small secular variation. This inclination is called the obliquity of the ecliptic, and it is to this cause that we are indebted for the succession of the seasons.

The figure of the earth is that of an oblate spheroid of revolution, the polar diameter being to the equatorial diameter in the ratio of 300 to 301. The equatorial diameter is nearly 7925 English miles, and the polar diameter about 7898 miles. The entire surface of the earth contains more than 197½ millions of square miles.

The knowledge of the true figure of the earth has been obtained from the combined results of mathematical theory, actual measurements of the lengths of arcs of the meridian, and careful experiments with the seconds pendulum, made at different points on the earth's surface. As a great portion of the earth's surface is covered by the ocean, the general form of that surface must be such as to conform to the principles of hydrostatic equilibrium. If the earth were a fluid, and had no motion of rotation, its form would, from the principles of mechanics, be nearly that of a perfect sphere, but rotary motion gives rise to a centrifugal force, one component of which is opposed to gravity, and the other acting tangentially, has a tendency to

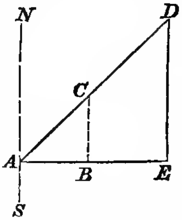
heap up the matter about the equator, and to flatten the sphere at the poles. This tendency must continue until a state of equilibrium is produced, which, according to theory, will happen when the form assumed is that of an ellipsoid of revolution. Such theoretical considerations as these, render the ellipsoidal form of the earth probable, and careful measurements and experiments of various kinds, fully confirm the conclusion. See *Figure of the Earth.*

EAST. In Surveying and Navigation, one of the cardinal points of the compass. The direction in which the sun appears to rise at the equinox. If an observer stand with his face towards the north, then will his right hand be towards the east.

An east and west line through a point, is a line which is perpendicular to the plane of the meridian at that point, and a vertical plane through this line is called the prime vertical.

Since the plane of the circle of latitude is perpendicular to the axis of the earth, it follows that an east and west line does not coincide with a circle of latitude except for a very small distance from a given point. Where great accuracy is required, as in tracing a parallel of latitude, correction has to be made for the deviation, and the correction will be an increasing function of the latitude, and also of the length of the course run. At the equator the prime vertical coincides with the plane of the circle of latitude, and the correction is 0. At the pole, the prime vertical is perpendicular to the plane of the circle of latitude, and the correction is a maximum. For all intermediate stations, the corrections may be determined by formulas given in treatises on geodesy. For limited portions of the earth's surface, the correction is almost inappreciable, and may, unless great accuracy is required, be entirely neglected.

EAS'TING. In Surveying, the perpendicular distance between two meridians drawn through the extremities of a course.



If AD represent any course run from A to D, NS and DE the meridians through its extremities, and AE perpendiculars to NS; then is AE the easting of the course. The easting is also called the *departure* of the course in this case. Had the course been run from D to A, the course would have made *westing* instead of easting, and the departure would be reckoned the same as before, with its sign changed. See *Departure*. In any field survey, the algebraic sum of the eastings and westings of all the courses, is equal to 0.

EC-CEN'TRIC. [L. *eccentricus*, deviating from the centre]. Two circles, ellipses, spheres, or spheroids, are said to be eccentric, when one lies within the other, but has not the same centre. The term stands opposed, in signification, to concentric, which signifies that one lies within the other, and that the two have a common centre.

Two magnitudes are not properly spoken of as concentric or eccentric, unless they are similar.

In machinery, a circle is said to be an eccentric when it revolves on an axis which does not pass through its centre. Such an arrangement is often made for the purpose of converting rotary into reciprocating motion.

EC-CEN-TRIC'ITY of a conic section, is the ratio of *the semi-transverse axis to the distance from the centre to the focus*. If the semi-transverse axis be taken as 1, the eccentricity becomes equal to *the distance from the centre to the focus;* under this supposition, it forms a very important element in astronomical computations.

In order to find an expression for the eccentricity, let us take the general equation of the conic sections

$$y^2 = r^2x^2 + 2px,$$

in which the axis of abscissas coincides with the principal axis of the curve, the origin of co-ordinates being at the principal vertex. If, in this equation, we make $y = 0$, the corresponding values of x, are 0 and $-\frac{2p}{r^2}$, the arithmetical mean of which gives for the semi-transverse axis $-\frac{p}{r^2}$. If we again make $y = p$, and deduce the corresponding values of x, we have

$$x = -\frac{p}{r^2} \pm \frac{p}{r^2}\sqrt{r^2+1};$$

if we subtract the first of these values from the second, and divide the result by 2, we

shall have, for the distance from the centre to the focus,

$$-\frac{p}{r^2}\sqrt{r^2+1}.$$

If now we divide this last distance by that previously found, and denote the eccentricity by e, we shall have the formula

$$e = \sqrt{r^2+1}.$$

For the *parabola* $r^2 = 0$, whence $e = 1$; that is, *the eccentricity of the ordinary parabola is always equal to* 1.

For the *ellipse*, $r^2 = -\frac{b^2}{a^2}$, in which a and b are positive, and represent the semi-axes, a being always greater than b for the *real* ellipse. Making the substitution, and reducing, the formula becomes

$$e = \frac{\sqrt{a^2-b^2}}{a}.$$

If $a = b$, the ellipse becomes a circle, and we find $e = 0$; that is, *the eccentricity of a circle is equal to* 0. If we suppose $b > a$, the ellipse becomes imaginary, and the expression for the eccentricity also becomes imaginary; that is, *the eccentricity of the imaginary ellipse is imaginary.* For every value of $b < a$, the expression for the eccentricity is less than 1, and greater than 0; that is, *the eccentricity of the ordinary ellipse is always found between* 0 *and* 1. If $b = 0$, the ellipse reduces to a limited straight line equal to the transverse axis, the foci being at the extremities; in this case, also, we have $e = 1$; that is, *the eccentricity of the limited straight line is* 1.

If the ratio $\frac{b}{a}$ remains constant, whilst a and b decrease, the value of e will remain constant, which shows that *the eccentricity of similar ellipses is always the same.* If a and b go on decreasing, but retaining a constant ratio to each other, they will both become 0 together; the ellipse will become a point, and *its eccentricity will be the same as that of an ellipse cut out of the right cone, with a circular base, by a plane parallel to the plane through the vertex which cuts out the point.* If b is infinite, a being finite, we have the extreme case of the imaginary ellipse, which corresponds to the two imaginary parallel lines of the parabola; in this case, the value of e becomes $\sqrt{-\infty^2}$, which is imaginary; hence, *the eccentricity of the imaginary parabola is*, like that of the imaginary ellipse, *imaginary.*

If we make a infinite, b being finite, we have another extreme case of the ellipse, which also coincides with an extreme case of the parabola; that is, we have two parallel straight lines drawn through the extremities of the conjugate axis, and parallel to the transverse axis. In this case, since b^2 is infinitely small in comparison with a^2, the numerator will be exactly equal to the denominator, and we shall again have $e = 1$. If a still continues infinite, and b goes on diminishing, the straight lines approach each other, till finally, when $b = 0$, they coincide, and we have another case of the ellipse coincident with another extreme case of the parabola, in which the eccentricity is likewise equal to 1.

We see, then, that for the eccentricity of the extreme cases of the parabola, we have for that of the imaginary parallels, an imaginary expression, and for the other particular cases, 1.

For the *hyperbola*, $r^2 = +\frac{b^2}{a^2}$, in which a and b are the semi-axes, and may have any ratio to each other. Substituting this value, the formula gives

$$e = \frac{\sqrt{a^2+b^2}}{a},$$

which can never be less than 1.

For $b = 0$, the hyperbola becomes a straight line, limited towards the centre, the foci being at the limiting points, and the line extending from these points outwards, indefinitely. In this case, $e = 1$; that is, *the eccentricity of a straight line limited towards the centre, is* 1.

For all values of b greater than 0 and less than a, the hyperbola is acute and the eccentricity is between 1 and $\sqrt{2}$. When $b = a$, the hyperbola is equilateral, and the eccentricity is equal to $\sqrt{2}$.

For all values of b greater than a the hyperbola is obtuse, and eccentricity is greater than $\sqrt{2}$, until we come to the value $b = \infty$, when the hyperbola passes to its extreme case and becomes two straight lines parallel to the conjugate axis, and drawn through the extremities of the transverse axis. In this case the eccentricity is infinite. If whilst b re-

mains infinite, a diminishes, the lines will approach each other, and when $a = 0$ they will coincide, and the eccentricity is still infinite. Hence, the eccentricity of two straight lines, or of one straight line perpendicular to the transverse axis, is infinite.

If b remains constant and of any infinite value whilst a diminishes till it becomes 0, the hyperbola approaches a straight line and finally coincides with the conjugate axis, and the eccentricity is again infinite.

If a and b vary together so that the ratio $\frac{b}{a}$ is constant, as a and b diminish the hyperbola approaches its asymptotes, and finally when a and b become 0, which they will together, we shall have the particular case of two straight lines intersecting, and the eccentricity is the same as that of the similar hyperbolas which have these lines for asymptotes.

By changing b into a and a into b, and denoting the eccentricity of the conjugate hyperbola by e' we get

$$e' = \frac{\sqrt{b^2 + a^2}}{b},$$

and by comparison,

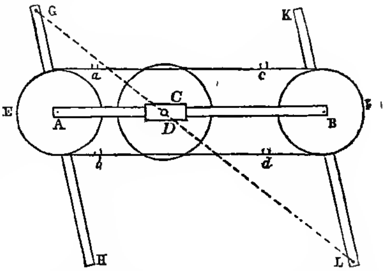
$$e : e' :: \frac{1}{a} : \frac{1}{b};$$

that is, the eccentricities of conjugate hyperbolas are inversely as their transverse axes. From the preceding discussion we infer that the eccentricity of the circle is 0: passing through the ordinary ellipse as it becomes elongated, the eccentricity increases till at the limit it becomes 1; passing through the parabola where it is 1, it continues to increase through the acute hyperbola till at the equilateral hyperbola it becomes equal to $\sqrt{2}$; still continuing to increase, it finally, in the last case, become infinite. It may be inferred, in general, that the eccentricity of a conic section is the measure of its departure from the circle. It is also evident, that the value of the eccentricity determines the kind of conic section.

E-CLIP'TIC. [Gr. εκλιπτικος; L. *Eclipticus*]. In spherical projections, a great circle whose plane makes an angle of about 23° 28′ with the plane of the equator. The plane of the ecliptic is the plane of the earth's orbit about the sun, and its inclination to the plane of the equator is constantly undergoing a slight secular change. See *Spherical Projections.*

EDGE *of an Angle*, in Geometry, the line in which two faces of a polyedral angle meet each other. An edge of a polyhedron, is the line in which two adjacent faces meet each other. In speaking of the edge of a polyhedron, the line is supposed to be limited to that portion which lies between the vertices of the two polyhedral angles which it joins.

EĪDO'-GRAPH. A mathematical instrument, invented by Wallace, and like the pantograph, serves to copy plans and drawings on the same or on different scales.

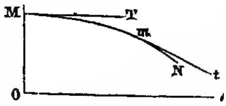


A rod or beam of brass AB, 30 inches long and $\frac{5}{8}$ of an inch square, and made hollow for the sake of being light, slides freely through a hollow rectangular socket C, whose length is $4\frac{1}{8}$ inches. From the lower surface of this socket projects a steel pin, of a conical shape, serving as an axis; the pin entering into a tube of a corresponding form which stands vertically on a cylindrical mass of metal, D. The mass serves as a base for the whole instrument; and whilst the beam AB may slide horizontally in the socket C, it is capable of turning with the socket upon the vertical axis in the tube. Each end of the beam AB carries a short tube, in a vertical position, and through this passes the conical axis of a wheel or pully E, F, which is placed below the beam; these wheels are precisely equal in diameter, and are capable of turning freely on their axes in a horizontal plane. The edges of these wheels are grooved so as to receive a piece of very thin watch spring a E b, c F d; and the ends a and c, b and d, are connected by a steel wire; the pieces of watch spring are made fast near E and F to

the circumferences of the wheels, in order to prevent them from slipping on these circumferences, a small movement for the purpose of adjustment only being allowed. Small screws at c and d serve to tighten and relax the band as may be necessary. Under each of the wheels E, F, are two rectangular sockets similar to C, and in these slide, horizontally, the rectangular arms GH, KL, each of which is $27\frac{1}{2}$ inches long. These arms, which turn with the wheels E and F, are adjusted by means of the screws at c and d, so as to be always parallel to each other. At L is fixed a tracing point, like that of the pantograph; at G, a pencil in a socket or tube; the tracer and pencil are to be always in a straight line, passing through the common axis of the mass D, and of the socket C. The pencil is made to press gently upon the paper by weights, but it is capable of being raised from it by means of a lever, one end of which is connected with the socket which carries it, and to the other is attacked a string which is to be pulled by the operator when necessary: this movement of the pencil carrier is facilitated by the aid of small friction rollers.

The beam AB is graduated on its upper surface into 100 or 1000 equal parts, and divisions equal to them are made on the upper face of each of the arms GH and KL. By these divisions, the distances of A and B from the axis D may be made to have any given ratio to one another, and AG, BL, may be made respectively equal to the last distances. Thus the isoceles triangles GAD, DBL, will always be similar, and the figure described by the movement of the pencil at G will be similar to that over which the tracer at L is made to pass.

E-LAS'TIC CURVE. [Gr. ελαστρεω, *to impel*]. The curve taken by an elastic filament, fixed horizontally at one end, and loaded with a weight applied to it.



Let the filament MN be fixed at the point M, so that the direction of the tangent at M shall be horizontal in whatever manner the filament may be bent, and let it be acted upon by a weight at its extremity. The plane of curvature will be vertical, and the plane of the co-ordinate axes OMN may be assumed as coinciding with it.

Denote by P the weight; its line of direction will be parallel to OM, supposed vertical; denote the distance of the point of application of this force from the axis OM, by p. At any point m of the curve, an equilibrium will exist between the force P which tends to turn the curve about m, and the elasticity which acts in a perpendicular to the tangent MT, and which resists this tendency. If we denote by E the moment of elasticity, and call the abscissa of the point m, x, we shall have

$$P(p-x) = E.$$

It is generally assumed that the E varies so as to be proportional to the tension, or inversely as the radius of curvature at any point: denoting the radius of curvature by r, and the elasticity at the point when the radius of curvature is 1 by e, the equation of the elastic curve will be

$$P(p-x) = \frac{e}{r}, \quad \text{or since,} \quad r = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$P(p-x) = e\frac{\frac{d^2y}{dx^2}}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}$$

If the curvature is very small $\left(\frac{dy}{dx}\right)^2$ will be very small in comparison with 1, and may be neglected; whence,

$$P(p-x) = e\frac{d^2y}{dx^2},$$

and by integrating twice,

$$y - k = \frac{P}{2 \cdot 3 \cdot e}(3lx^2 - x^3),$$

k being constant and equal to OM. The elastic curve is the same as that assumed by a spider's web, when fixed at its extremities and blown by a uniform breeze; or, it is the curve assumed by a perfectly flexible and hollow line, fastened at its extremities and filled by a fluid filling its entire cavity.

EL'E-MENT. [L. *elementum*, element]. If we suppose a surface to be generated by a right line moving according to some fixed law, every position of the moving line is

called an element. Thus, a conic surface is one which may be generated by a straight line moving in such a manner as to pass through a fixed point, and constantly touch a given curve. Any position of the moving line is an element of the surface.

To find an element of a conical surface; draw a straight line from the vertex to any point of the base or directrix, and it will be an element. To find an element of a cylindrical surface, draw a straight line through any point of the base, or directrix, parallel to the axis; it will be an element.

To find an element of a warped surface of the first kind; pass a plane parallel to the plane director, and find the points in which it cuts the directrices; the straight line through these points is an element.

To find an element of a warped surface of the second kind; assume any point of the first directrix as the vertex of a conic surface, and take the second directrix as its base; join the point in which the conic surface cuts the third directrix by a straight line, and it will be an element of the surface.

If the surface is generated by a curved line, every position of the generatrix is a curvilinear element.

In the application of Calculus to Geometry, the term element is often used as synonymous with differential. Thus, the differential of a plane area is often called an element of the area, or an elementary area. It is the infinitely small space included between the axis of abscissas, the curve and two ordinates, whose distance from each other is equal to the differential of x.

The element of a solid of revolution, or the elementary solid, is that portion of the solid included between two planes, both perpendicular to the axis of revolution, and which are distant from each other equal to the differential of x. The surface included between the same two planes is an elementary surface, and so on.

In this latter sense of the term, an element is the same as an infinitely small particle of the same nature as the entire magnitude considered.

EL-E-VĀ'TION. [L. *elevatio*, lifting up]. In Descriptive Geometry, the same as *Vertical Projection.*

Elevation, Angle of. In Surveying, a vertical angle, one of whose sides is horizontal, the inclined side lying above the horizontal one.

In Shades and Shadows, and in Architecture, the elevation of a body is the same as its orthographic projection upon a vertical plane.

E-LIM-I-NĀ'TION. [L. *e*, from, and *limen*, threshold]. In Analysis, the operation of combining several equations containing several unknown quantities, so as to deduce therefrom a less number of equations, containing a less number of unknown quantities. There are several different processes of elimination: we shall consider first, those which are applicable chiefly to equations of the first degree, supposing all the unknown terms to be in the first member.

1. *The Method by Addition or Subtraction.*

Find the least common multiple of the co-efficients of the quantity to be eliminated in the two equations; multiply every term of each equation by the quotient found by dividing this multiple by the co-efficient of the quantity to be eliminated in that equation: If the signs of the terms containing this quantity in the two equations are alike, subtract one equation from the other, member from member; if they are unlike, add them member to member; the resulting equation will be independent of that quantity.

1. Eliminate y between the equations

$$3x + 8y = 25$$
$$5x + 6y = 13.$$

The least common multiple of 6 and 8 is 24; multiply both members of the first equation by 3, and of the second by 4, and subtracting, we shall have

$$\begin{array}{r} 9x + 24y = 75 \\ 20x + 24y = 52 \\ \hline 11x = -23, \end{array}$$

which does not contain y.

2. *The Method by Substitution.*

Find, from one of the equations, the value of the quantity which we wish to eliminate, in terms of the other, and substitute this for that quantity in the second equation; the resulting equation will be independent of that quantity.

1. Eliminate y between the equations

$$4x - 2y = 6$$
$$5x + 5y = 14.$$

From the first we find $y = 2x - 3$, and this, substituted for y in the second equation, gives

$$5x + 10x - 15 = 14, \quad \text{or} \quad 15x = 29,$$

which does not contain y.

3. *The Method by Comparison.*

Find, from each equation, the value of the quantity which we wish to eliminate, in terms of the other, and place these values equal to each other; the resulting equation will be independent of that quantity.

1. Eliminate y between the equations

$$3x + 2y = 5 \cdots (1),$$
$$4x - 7y = 11 \cdot\cdot (2).$$

From the first we find

$$y = \frac{5 - 3x}{2}, \text{ and from the second}$$

$$y = \frac{4x - 11}{7}; \text{ whence } \frac{5 - 3x}{2} = \frac{4x - 11}{7};$$

or, $29x = 13$, which is independent of y.

4. *The Method of Arbitrary Multipliers.*

Multiply both members of the first equation by an arbitrary quantity, and add the resulting equation to the second, member to member; place the co-efficient of the quantity which we wish to eliminate equal to 0, and deduce from this the value of the arbitrary multiplier; substitute this value for the arbitrary quantity in the preceding equation, and the resulting equation will be independent of the quantity to be eliminated.

1. Eliminate y between the equations

$$3x + 2y = 5 \cdots (1),$$
$$4x - 5y = 10 \cdot\cdot (2).$$

Multiplying both members of the first by p,

$$3px + 2py = 5p \cdot\cdot (3),$$

and adding to the second

$$(4 + 3p)x + (2p - 5)y = 10 + 5p \cdot\cdot (4),$$

placing $2p - 5 = 0$, we find

$$p = \frac{5}{2}, \text{ this substituted in (4) gives}$$

$$\left(4 + \frac{15}{2}\right)x = 10 + \frac{25}{2},$$

which is independent of y.

Either of these methods may be employed, and the equations will indicate which is most convenient: by repeated applications of the rules, a number of quantities may be eliminated from a group of simultaneous equations, equal to the number of equations in the group, diminished by 1.

When the equations are of a higher degree than the first, the preceding methods will in general be inapplicable. Other methods must then be resorted to.

1. When there are two equations containing two unknown quantities, both being of the second degree and homogeneous with respect to these quantities, one of them may be eliminated as follows:

Substitute, in the two equations, for one of the unknown quantities, a new unknown quantity multiplied by the other; from the resulting equations find the value of the auxiliary unknown quantity, and substitute this value, in either the third or fourth equation, for this quantity; the resulting equation will be independent of the quantity to be eliminated.

1. Eliminate y between the equations

$$x^2 + xy - y^2 = 5 \cdots (1),$$
$$3x^2 - 2xy - 2y^2 = 6 \cdots (2).$$

Substitute for y, in (1) and (2), px; there will result

$$x^2 + px^2 - p^2x^2 = 5 \cdots (3),$$
$$3x^2 - 2px^2 - 2p^2x^2 = 6 \cdots (4).$$

Finding the values of x^2 in terms of p from (3) and (4), and placing them equal to each other, we get

$$\frac{5}{1 + p - p^2} = \frac{6}{3 - 2p - 2p^2};$$

or, by reduction,

$$p^2 + 4p = \frac{9}{4}, \text{ from which}$$

$$p = \frac{1}{2} \quad \text{and} \quad p = -\frac{9}{2}.$$

The first value of p in equation (3), gives

$$x^2\left(1 + \frac{1}{2} - \frac{1}{4}\right) = 5;$$

and the second value of p in the same equation, gives

$$x^2\left(1 - \frac{9}{2} + \frac{81}{4}\right) = 5,$$

both of which are independent of y.

2. When there are two equations of any

degree whatever between two unknown quantities. The most usual method is

The Method of the greatest Common Divisor.

In the first place, suppose all the terms in both equations to have been transposed to the first members; they may be written under the form

$$f(x, y) = 0, \quad \text{and} \quad f'(x, y) = 0.$$

It is to be observed, that if the equations admit of a finite number of solutions, their first members cannot have a common divisor which is a function of both x and y, or which is a function of either of them. If they have a common divisor which does not depend upon either x or y, we will suppose that both members of each equation have been divided by it.

Having made this preliminary preparation, and arranged both with respect to the quantity which we wish to eliminate, *apply to the first members of the given equations the rule for finding their greatest common divisor, and continue the operation till a remainder is found which is independent of the leading letter; place this remainder equal to* 0, *and it will be the required equation, which will be independent of the leading letter.*

This equation is called the *final equation*, and the values of the unknown quantity deduced from it are called *compatible values.* The objection to this method of elimination is, that the final equation may give values which do not correspond to roots of the given equations. This arises from the circumstance that during the operation for finding the greatest common divisor, it may be necessary to multiply one of the polynomials by a function of that unknown quantity which enters the final equation, and this operation may introduce extraneous values. As these values may be tested, however, it does not make much difference in a practical point of view. In like manner, if we strike out any factor which is a function of that unknown quantity, it may happen that the final equation will not give all the *compatible values.*

Example of elimination by the method of the greatest common divisor.

Having given the equations

$$x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y = 0,$$

and $\quad x^2 - 2yx + y^2 - y = 0,$

to find the final equation in y:

First operation.

$$\begin{array}{l|l}
x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y & \\
\underline{x^3 - 2yx^2 + (y^2 - y)x} & \underline{x^2 - 2xy + y^2 - y} \\
-yx^2 + (2y^2 + 1)x - y^3 + y^2 - 2y & x - y \\
\underline{-yx^2 + 2y^2x \qquad -y^3 + y^2} & \\
\qquad\qquad x - 2y &
\end{array}$$

Second operation.

$$\begin{array}{l|l}
x^2 - 2xy + y^2 - y & \underline{x - 2y} \\
\underline{x^2 - 2xy} & x \\
\qquad\quad y^2 - y &
\end{array}$$

Hence, $y^2 - y = 0$, is the equation sought.

EL-LĬPSE′. [Gr. ελλειψις, an omission or defect]. One of the conic sections, and including its particular case, the circle, by far the most important curve considered in analysis. The ellipse may be cut from a right cone with a circular base, by a plane which makes with the plane of the base an angle less than that made with the same plane by one of the elements. All the elements are cut in one nappe, and therefore the curve returns upon itself, or is a *closed curve.* It is of an oval form, and has but one branch. By giving the cutting plane different positions, so as to satisfy the conditions for cutting out the ellipse, and by varying the angle of the cone, every variety of the curve may be found

If the cutting plane is parallel to the base of the cone, the section is a *circle;* hence the *circle* is a particular case of the ellipse. If the cutting plane is passed through the vertex, satisfying the conditions for cutting the ellipse, the section reduces to a *point*, which is therefore regarded as a particular case of the ellipse. If a plane be passed cutting out an ellipse, all parallel planes will cut similar ellipses.

If we regard an oblique cone, with a circular base, any plane which cuts all the elements will cut out an ellipse. If we suppose the vertex to approach the plane of the base, and finally to reach that plane, falling without the base, the cone will reduce to a portion of a plane determined by drawing two lines tangent to the base.

In this case, if the cutting plane cuts all of the elements, it will cut out a limited straight line, which is therefore another particular case of the ellipse.

Although the ellipse was first suggested to geometers from considering the sections of

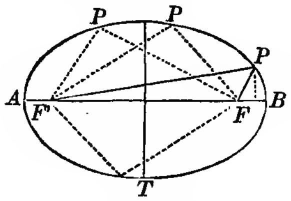
right cone, it may be defined in various ways, and all its properties deduced without any reference whatever to the cone.

It may be defined from some one of its characteristic properties, or it may be defined by its equation.

We shall enumerate some of the definitions of the curve, and then proceed to mention its most remarkable properties.

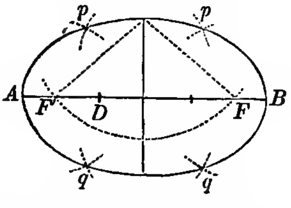
1. One of the most common definitions of the ellipse is the following: An ellipse is a plane curve, such that the sum of the distances from any point to two fixed points is equal to a given distance. The fixed points are called *foci*, and the given distance is equal to that portion of the straight line through the foci which is included within the curve.

This definition gives rise to the following method of constructing the curve by a continuous motion:



Let F and F′ represent the foci, and AB any given distance, greater than the distance between F and F′. Take a string equal in length to AB and fasten one extremity at F and the other extremity at F′; press a pencil against the string, so as to stretch it and move it about F and F′; the point of the pencil will describe the curve, for in any position we shall have $PF + PF' = AB$, which is a characteristic property of the curve.

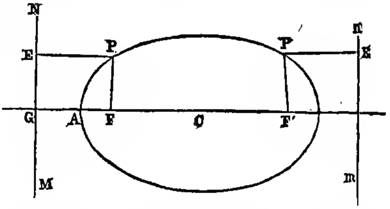
The same property also enables us to con-



struct the curve by points. Let F and F′ be the foci, AB the given distance FB being equal to F′A. Assume any point on FF′ as D; with either focus as a centre, and with a radius equal to the segment AD describe an arc of a circle; with the remaining segment DB as a radius, and with the other focus as a centre, describe an arc, cutting the first in the points p and q; these will be points of the ellipse. In like manner any number of points may be constructed, and having a sufficient number, a curve drawn through them will be the curve required. The reason for this construction is evident.

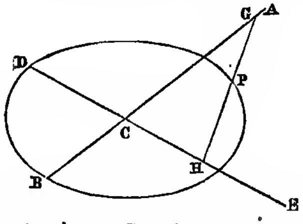
2. A second definition of the ellipse is as follows:

If F be a given point, MN a given straight line, and if we suppose a point P to move in the same plane, so that the ratio of its dis-



tances PF and PE from the fixed point and the fixed line shall be constant, PF being always less than PE, the point will describe an ellipse. The fixed point is called the focus, the straight line MN the directrix, and the moving point is the generatrix. It is to be observed that this definition only corresponds to one half of the curve. In order to generate the other half we must take the other focus and another directrix mn, at the same distance from F′ that MN is from F. The distance from the centre C to the directrix is a third proportional to CF and CA, that is CF : CA : : CA : CG.

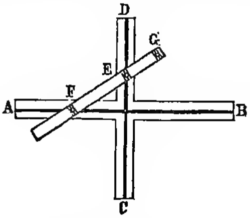
3. If two straight lines AB and DE inter-



sect each other at C, and a limited straight

line GH be moved so that its extremities G and H shall lie always in these two lines respectively, then will any point P of the moving line describe an ellipse whose centre is at C.

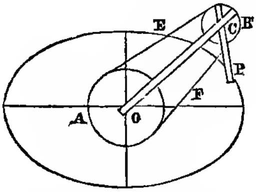
If the lines AB and DE are at right angles, the axes of the ellipse will coincide with them, and the part PH will be equal to the semi-conjugate, and PG will be equal to the semi-transverse axis, when PH < PG, and the reverse when PH > PG. It is to be observed that the point P may lie either between the two points G and H, or it may lie anywhere on the prolongation of GH. It is in accordance with this property of the ellipse that instruments for describing ellipses and lathes for turning ovals, are constructed. One of the instruments for describing ellipses is called the elliptical compasses or the trammel. It consists of two rulers framed at right angles to each other, in which dovetail grooves CD and AB are cut. A third rule FG carries



three movable slides, F, E and G, which may be set at any points of the ruler; the slides F and E carry pins with dovetail heads, which fit accurately into the grooves AB and CD, and the third slide carries a pencil. Each slide may be made fast to the ruler by a clamp screw. To use this instrument for describing an ellipse, we lay the cross upon the paper, so the direction of the groove AB shall coincide with the transverse axis, the groove CD with the conjugate axis. The slides are next set so that the distance GF shall be equal to the semi-transverse, and the distance GE equal to the semi-conjugate axes of the ellipse, and all three are clamped. Now if the rule be moved in such a manner that the pins shall slide in the grooves, the pencil will trace an ellipse.

4. If a circle roll upon the concave arc of a second circle in the same plane, if the radius of the first is half that of the second, any point in the plane of the moving circle will generate an ellipse. The fixed circle is called the *directing* circle, the moving circle the *generating* circle, and the point the generatrix. This property of the circle gives rise to a very ingenious instrument for describing ellipses, invented by Prof. Wallace of Edinburgh.

"A and B are two wheels, the axes of



which turn in two holes OC, near the ends of the connecting bar OC. The diameter of one of the wheels B is just half that of the other wheel A, which may be of any size, and a band EF goes round them outside; one arm CP is attached to the wheel B, and admits of being lengthened or shortened by sliding along its surface in a socket which may be anywhere on the wheel. Suppose now that the wheel A is fixed or kept from turning, and that the bar OC is turned around the centre O, carrying at its other extremity the wheel B; the action of the band EF will then turn this wheel around its centre C, and while the bar makes one revolution round the centre of the fixed wheel, the other wheel will make two revolutions around its centre, and the point P will trace an ellipse."

5. The ellipse may be defined by any one of its equations; of these we shall only mention the two most commonly employed.

First. When the curve is referred to its centre and axes, its equation is

$$a^2y^2 + b^2x^2 = a^2b^2,$$

in which a and b are the semi-axes, and x and y the co-ordinates of any point of the curve.

Second. The ellipse is the path described by the planets in their revolutions about the sun, and its properties enter into almost every investigation of physical astronomy. In these investigations, it has been found most convenient to define the curve by its polar equation, which is

$$r = \frac{a(1 - e^2)}{1 + e\cos\phi};$$

the pole is taken at one focus: r denotes the radius vector or distance from any point of the curve to the focus, ϕ is the angle which the radius vector makes with the transverse axis, a is the semi-transverse axis, and e is the eccentricity of the curve. The angle ϕ is sometimes called the *anomaly*.

Properties of the Ellipse and useful constructions. The following definitions will be found useful: some that have been given are here repeated, so as to bring them all together.

1. The ellipse is a curve, such that the sum of the distances from any point to two fixed points is constantly equal to a given distance. The fixed points are called *foci*. and the given distance is equal to the length of the transverse axis.

2. The point midway between the foci, on the straight line joining them, is the *centre*. The centre bisects all the diameters.

3. Any straight line which bisects a system of parallel chords, is a *diameter*. Every diameter passes through the centre. If the diameter is perpendicular to the chords which it bisects, it is an *axis* of the curve; there are two axes, the one which passes through the foci is the *transverse axis*, the one which is perpendicular to it, is the *conjugate axis*. The transverse axis is the longest diameter of the curve, and the conjugate axis is the shortest.

4. Two diameters are *conjugate*, when each bisects a system of chords parallel to the other. There are an infinite number of pairs of conjugate diameters in the ellipse; every diameter has one conjugate, and has but one.

5. The points in which a diameter meets the curve, are called *vertices* of the diameter. The left hand vertex of the transverse axis is called the *principal vertex of the curve*.

6. The *parameter* of any diameter is a third proportional to that diameter and its conjugate. The parameter of the transverse axis is called the parameter of the curve, and is equal to the double ordinate through the focus.

6. If a chord be drawn through a focus perpendicular to the transverse axis, and a tangent be drawn to the curve at the point in which it cuts the curve, this tangent is called the *focal tangent*.

7. If a perpendicular be erected to the transverse axis produced, at the point in which the focal tangent intersects it, this line is the *directrix*.

8. If any point be assumed in the plane of the curve, and chords be drawn through it, each cutting the curve in two points, then will the tangents to the curve at the extremities of each chord intersect each other upon a straight line called the *polar line of the point*; the point is called the *pole* of the line. The directrix is the polar line of the focus.

9. *An ordinate to a diameter* is a straight line drawn from any point of the diameter to the curve, and parallel to the conjugate of the diameter. The ordinates to the axes are perpendicular to them.

10. *A tangent to the curve* at any point, is the limit of all secants drawn through the point. If a secant be drawn, cutting the curve in two points, and then be revolved about one of the points till the points of secancy unite in one, the secant passes to its limit and becomes a tangent, and the two coincident points become the *point of contact*. A subtangent on any diameter, is that portion of the diameter included between the point in which the tangent cuts the diameter, and the foot of the ordinate to the diameter through the point of contact.

11. *A normal to the curve*, at any point, is a straight line perpendicular to the tangent at the point of contact.

A subnormal on any diameter, is that portion of the diameter included between the point in which the normal cuts the diameter, and the foot of the ordinate to the diameter through the point of contact.

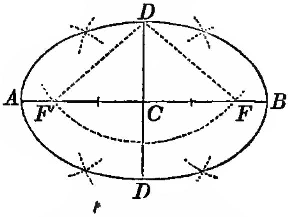
12. *Supplementary Chords* are chords drawn through the vertices of any diameter, and meeting each other on the curve.

13. The two points into which any ordinate to a diameter divides the diameter, are called *segments*, and sometimes abscissas of the diameter.

14. The *eccentricity* of the ellipse is the distance from a focus to the centre, expressed in terms of the semi-transverse axis as the unit of measure, or it is the distance from one focus to the centre divided by the semi transverse axis.

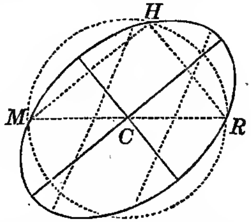
15. Two ellipses are conjugate with each other when the transverse axis of the one is the conjugate axis of the other, and the reverse.

The manner of constructing the curve, when its foci and transverse axes are given, has already been explained. The following constructions will serve to determine these elements of the curve in certain cases:



1. *When the two axes are given.* Let AB represent the transverse axis and CD the semi-conjugate axis. With D as a centre, and CB as a radius, describe an arc cutting the transverse axis in the two points F and F′; these will be the foci.

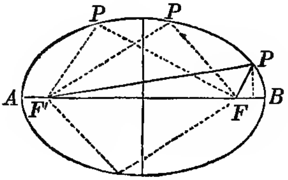
2. When the foci and conjugate axis are given. Through the foci draw an indefinite straight line AB (last figure); lay off from the point C in which it cuts the conjugate axis the distances CA and CB, each equal to the distance DF; then will AB be the transverse axis.



3. *When the curve is traced upon a plane to find the axis.* Draw any two parallel chords, and bisect them by a straight line RM; this will be a diameter; bisect RM in C, and C will be the centre; on RM as a diameter, describe a semi-circle cutting the ellipse in H; draw the supplementary chords RH and HM; they will be at right angles to each other, and two straight lines through O, parallel to them, will be the axes.

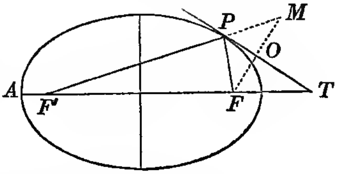
The foci may be found as before. The diameter parallel to the chords first drawn, is conjugate with KM.

4. When the foci and one point of the curve are given, the curve mav be constructed. Through the foci draw a straight line indefinite in extent; the point of this line mid-



way between the foci is the centre; from this point lay off on each side a distance equal to half the sum of the distances from the foci to the given point; then will the line so determined be the transverse axis; the construction may be completed as explained in the last case.

5. When the foci and any tangent to the curve are given, the curve may be constructed.

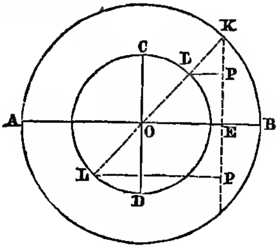


Let F, F′ be the foci, and TP a tangent. Draw FM perpendicular to the tangent, and make the prolongation OM equal to FO; through M and F′ draw a straight line cutting the tangent in P; then will P be a point of the curve, and the construction may be completed as explained in the preceding cases.

The following properties give rise to important constructions of the curve, and of tangents to it.

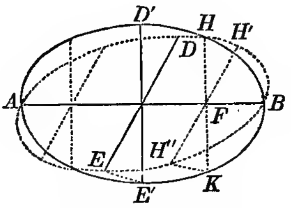
1. If a circle be described upon the transverse axis of an ellipse, as a diameter, and a second circle be described upon the conjugate axis as a diameter, the first is said to be circumscribed about, and the second inscribed within, the ellipse. Any ordinate to the transverse axis of the ellipse, is to the corresponding ordinate of the circumscribed circle as the semi-conjugate is to the semi-transverse axis; also, any ordinate to the conjugate axis of the ellipse is to the corresponding abscissa of the inscribed circle as the semi-transvei se

is to the semi-conjugate axis. Upon this property the construction of the *trammel* depends; it may also be used to construct the curve by points.



Let AB and CD be the axes. On these lines as diameters describe two circles; assume any abscissa as OE, and at E erect a perpendicular to AB, prolonging it till it meets the outer circle in K; join K and O; through L, where this line intersects the inner circle, draw a line parallel to AB, cutting EK in P; the point P will be a point of the ellipse; find a sufficient number of points in this manner, and draw a curve through them, it will be the required ellipse.

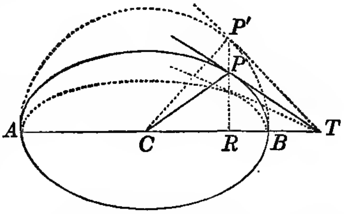
2. The squares of the ordinates to any diameter are to each other as the rectangles of the segments into which they divide the diameter. This property enables us to construct the curve, when any pair of conjugate diameters is given, and the angle which they make with each other is known.



Let AB and ED be any pair of conjugate diameters. Revolve ED about C till it becomes perpendicular to AB; on AB and E′D′ as axes, construct an ellipse AD′BE′. Take any double ordinate to the axis AB, as HK, and revolve it about F till it becomes parallel to DE; then will its extremities H′ and H″ be points of the required ellipse: having found a sufficient number of points, draw a curve through them, and it will be the required ellipse.

3. The subtangent upon the transverse axis of an ellipse, is entirely independent of the length of the conjugate axis. If, therefore, any number of ellipses be constructed, having a common transverse axis, and if points be taken on the same ordinate to the transverse axis, and tangents be drawn to the ellipses, these tangents will all pass through the same point on the transverse axis produced.

This property gives rise to a useful method of drawing a tangent to an ellipse, at a given point.



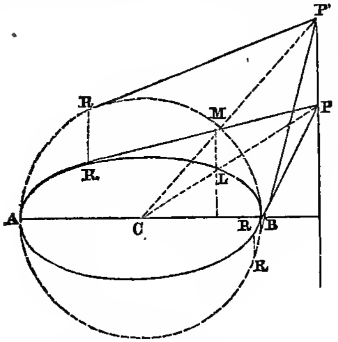
Let APB be an ellipse, and P any point upon it; on AB, as a diameter, describe a semi-circle; through P draw an ordinate to the transverse axis, and produce it till it cuts the semi-circle in P′; at P′ draw a tangent, cutting the line AB at T; unite P and T by a straight line: it will be the tangent required. This construction follows, because a circle is a particular case of the ellipse.

The same result might have been reached by the following course of reasoning: The orthographic projection of a circle is an ellipse; if, therefore, the semi-circle AP′B and its tangent P′T be revolved about the line AB as an axis, the circle will constantly be projected into an ellipse, and the tangent into a line tangent to the ellipse, at a point on the line P′P, and constantly passing through T, which agrees with the above construction.

This last view of the case suggests the following method of drawing a tangent to an ellipse, through a point without the curve.

Let AB be the ellipse, and P a point without the curve. On AB, as a diameter, describe a circle ARB: through P draw PC to the centre of the ellipse, cutting the curve in L, and through L draw a straight line LM, perpendicular to the transverse axis, cutting the circumscribing circle in M; draw

CM, and produce it till it intersects a perpendicular to the transverse axis, through P, in

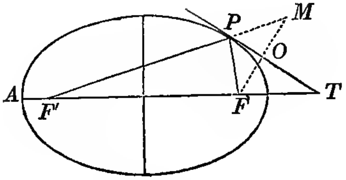


the point P′. From P′ draw P′R tangent to the circle at R, and through R draw a perpendicular to the transverse axis, cutting the ellipse in R′; then will PR′ be tangent to the ellipse: two such tangents can always be drawn, when the point P lies without the curve; one. when it lies upon the curve; none, when it falls within the curve.

4. If, at any point of an ellipse, a tangent be drawn to the curve. and two straight lines to the foci, then will these lines make equal angles with the tangent.

This property gives rise to the following constructions:

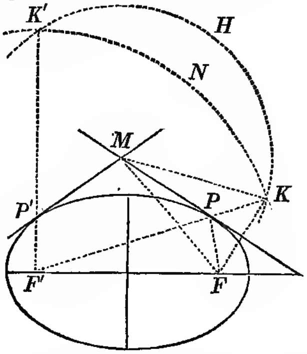
First. To draw a tangent to the curve, at a given point.



Let AHB be the given ellipse, P the given point, and F, F′, the foci. Draw the lines PF and PF′; produce PF′ till F′M is equal to the transverse axis; draw MF, and through P draw PT perpendicular to MF: it will be the tangent required.

Had we, in like manner, prolonged FP till it was equal to the transverse axis, and drawn through M and its extremities a straight line, cutting AB in T; then would the straight line PT have been the tangent required.

Second. To draw a tangent to the curve, through a point without the curve.



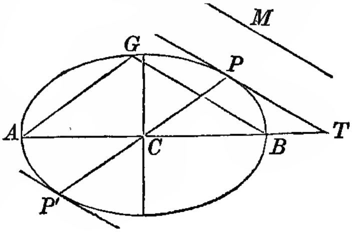
Let M be the point, and F, F′ the foci; with either focus, F′, as a centre, and a radius equal to the transverse axis, describe an arc KNK′; then, with M as a centre, and a radius equal to MF, the distance to the other focus, describe the arc FKHK′ intersecting the former in K and K′; through K and K′ draw KF′ and K′F′, and unite the points, where these lines intersect the ellipse, with M, by the straight lines MP and MP′: these will be tangent to the curve, and will be the lines required.

5. If a chord of the ellipse be drawn through the extremity of any diameter parallel to a given diameter, its supplementary chord will be parallel to the tangents through the vertex of that diameter; and conversely, if a chord is parallel to a tangent of the curve, its supplement will be parallel to the diameter through the point of contact.

The following constructions flow from this property:

First. To draw a straight line tangent to an ellipse, at a given point.

Let ABG be the ellipse, P the point and AB any diameter; draw through P the line PC



to the centre; through A draw the chord AG parallel to PC, and also its supplement GB; through P draw a straight line parallel to GB, and it will be the tangent required.

Second. To draw a line parallel to a given line, and tangent to the ellipse. Let M be the given line. Draw a chord BG parallel to M, and draw its supplement AG; draw the diameter PP′ parallel to AG, and at its extremities draw lines parallel to M; they will be the tangents required.

6. The tangent to an ellipse is parallel to the chords which the diameter through the point of contact bisects.

We may, therefore, draw a tangent to an ellipse, and parallel to a given line, as follows:

Draw two chords parallel to the given line, and bisect them by a straight line; through the points in which this line cuts the curve, draw lines parallel to the given line, and they will be tangent to the ellipse, and therefore the lines required. In any of the preceding constructions, a normal may be constructed by drawing a straight line perpendicular to the tangent at the point of contact.

The following properties are useful in analytical investigations:

1. The angle included between two conjugate diameters can never be less than a right angle. The least angle made by conjugate diameters, is that included between the axes, which is equal to a right angle.

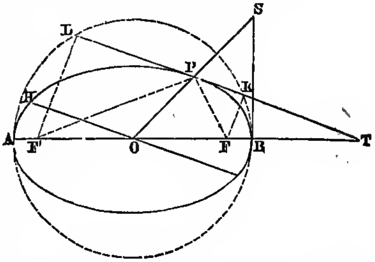
The greatest angle made by any pair of conjugate diameters, is that included by the two which coincide with the diagonals of the rectangle described upon the axes. The tangent of half this angle is equal to $\frac{b}{a}$, in which a and b are the semi-axes; the conjugate diameters, which make with each other the maximum angle, are equal to each other, and they are the only conjugate diameters which are equal, except in the circle, where every diameter is equal and perpendicular to its conjugate.

2. The parallelogram described upon any pair of conjugate diameters, is equal to the rectangle of the axes.

3. The sum of the squares of any pair of conjugate diameters, is equal to the sum of the squares of the axes.

4. If perpendiculars be drawn from the foci to any tangent to the curve, they will intersect it upon the circumference of the circle described upon the transverse axis as a diameter.

The rectangles of the two perpendiculars upon the same tangent, are equal to the square of the semi-conjugate axis.



Let AB represent an ellipse, F, F′ its foci, PT a tangent at any point, FK, F′K perpendiculars to the tangent, and ALKB a semi-circle described on AB as a diameter; then will the points K and L fall upon the circumference of the circle, and

$$LF' \times KF = CO^2.$$

The perpendiculars are also to each other as the focal distances of the point of contact, that is

$$LF' \cdot KF : \cdot PF' : PF.$$

The rectangle of the focal distances of any point, is equal to the square of half of the diameter which is conjugate with the diameter through the point of contact, or

$$FP \times F'P = OH^2.$$

5. If two tangents be drawn, one at the principal vertex, and the other at the vertex of any other diameter, each meeting the other diameter produced, the tangential triangles so formed will be equivalent. Let ALB be the ellipse [see last figure], B the vertex, BS and PT the tangents; then are the triangles OBS and OPT equal in area, or equivalent.

6. The area of an ellipse is equal to πab, in which $\pi = 3.1416$, a and b being the semi-axes. It is also equal to $\pi a'b' \sin a$, in which $\pi = 3.1416$, a', b', any pair of semi-conjugate diameters, and a the angle which these diameters make with each other.

7. The length of the entire circumference of an ellipse is given by the formula

$$l = 2\pi\left(1 - \frac{1}{2^2}e^2 - \frac{1^2\cdot 3}{2^2 4^2}e^4 - \frac{1^2\cdot 3^2\cdot 5}{2^2 4^2 6^2}e^6 - \frac{1^2\cdot 3^2\cdot 5^2\cdot 7}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}e^8 - \&c.\right);$$

in which l denotes the length, $\pi = 3.1416$, e the eccentricity, the semi-transverse axis being equal to 1.

If $e = 0$, the ellipse becomes the circle, and $l = 2\pi$.

8. The rectangle of two conjugate diameters is a maximum, when they are equal; it is a minimum, when the difference between them is the greatest; that is, the rectangle of the axes is the least possible rectangle of any pair of conjugate diameters. The sum of the equal conjugate diameters is greater, and the sum of the axes less than the sum of any other pair of conjugate diameters.

The following analytical expressions are much used.

Let us denote by x and y, the co-ordinates of any point of the curve; by x'' and y'', the co-ordinates of the point of contact; by a, the semi-transverse axis; by b, the semi-conjugate axis; by a' and b', any pair of semi-conjugate diameters; by e, the eccentricity.

1. The equation of the curve referred to its centre and axes, is

$$a^2y^2 + b^2x^2 = a^2b^2.$$

The equation of the curve referred to any pair of conjugate diameters, is

$$a'^2y^2 + b'^2x^2 = a'^2b'^2.$$

2. The equation of the curve referred to the transverse axis, and a tangent at the principal vertex, is

$$y^2 = \frac{b^2}{a^2}(2ax - x^2).$$

The equation of the curve referred to any diameter, and the tangent at its vertex, is

$$y^2 = \frac{b'^2}{a'^2}(2a'x - x^2).$$

3. The equation of a tangent to the curve, referred to the centre and axes, is

$$a^2yy'' + b^2xx'' = a^2b^2.$$

The equation of a tangent referred to any pair of conjugate diameters, is

$$a'^2yy'' + b'^2xx'' = a'^2b'^2.$$

The expression for the subtangent upon the axis of X is, in the first case,

$$sub\text{-}tan = \frac{a^2 - x''^2}{x''};$$

and, in the second case,

$$sub\text{-}tan = \frac{a'^2 - x''^2}{x''}.$$

4. The equation of a normal to the curve at any point x'', y'', when referred to the axes, is

$$y - y'' = \frac{a^2y''}{b^2x''}(x - x''),$$

and to any pair of conjugate diameters, it is

$$y - y'' = \frac{a'^2y''}{b'^2x''}(x - x'').$$

The expression for the sub-normal upon the axis of X, in the first case, is

$$sub\text{-}nor = \frac{b^2x''}{a^2};$$

and, in the second case,

$$sub\text{-}nor = \frac{b'^2x''}{a'^2}.$$

5. The equation of condition for conjugate diameters, is

$$\tan a \tan a' = -\frac{b^2}{a^2};$$

in which a and a' denote the angles which the conjugate diameters make with the transverse axis. The same equation is also the equation of condition for supplementary chords drawn from the extremities of the transverse axis. If they are drawn from the extremities of any diameter whose length is $2a'$, the equation of condition is

$$cc' = -\frac{b'^2}{a'^2},$$

in which c and c' are respectively the ratios of the sines of the angles which the chords make with the conjugate diameters.

6. Any equation of the form

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

will represent an ellipse, whenever

$$b^2 - 4ac < 0.$$

The co-ordinates of its centre, are

$$x' = \frac{2ae - bd}{b^2 - 4ac}, \quad \text{and} \quad y' = \frac{2cd - be}{b^2 - 4ac}.$$

7. The polar equation of the ellipse, when the pole is taken at the right hand focus, is

$$r = \frac{a(1 - e^2)}{1 + e\cos\phi};$$

in which r denotes the radius vector, and ϕ the angle which it makes with the transverse axis.

EL-LIP'SO-GRAPH. [Gr. ελλειψις, and

γραφω]. An instrument for describing an ellipse by a continuous movement.

EL-LIP'SOID. [Gr. ελλειψις, and ειδος, form]. A solid, all of whose plane sections are ellipses. Any plane which bisects a system of parallel chords is called a *diametral plane*. If the chords which it bisects are perpendicular to it, it is a principal plane. The ellipsoid has three principal planes at right angles to each other, whose lines of intersection are called *axes* of the ellipsoid. If we denote the lengths of the semi-axes in their order of magnitude by a, b and c, respectively; the co-ordinates of any point of the surface referred to the axes by x, y and z, the equation of the surface is

$$a^2 b^2 z^2 + a^2 c^2 y^2 + b^2 c^2 x^2 = a^2 b^2 c^2.$$

If $b = c$, the equation reduces to

$$a^2 z^2 + a^2 y^2 + b^2 x^2 = a^2 b^2,$$

which is the equation of an ellipsoid of revolution, which may be generated by revolving an ellipse about its transverse axis. Such an ellipsoid is called a *prolate spheroid.*

If $b = a$, the equation is that of an ellipsoid of revolution which may be generated by revolving an ellipse about its conjugate axis. Such an ellipsoid is called an *oblate spheroid.*

If $a = b = c$ the equation becomes

$$x^2 + y^2 + z^2 = a^2,$$

which is the equation of a sphere.

The ellipsoid is a solid of much importance, on account of its being the form assumed by the bodies of the planetary system.

ELLIPTICAL. Appertaining to the ellipse.

ELLIPTICAL ARC, a portion of the circumference of an ellipse.

ELLIPTICAL COMPASSES. See *Compasses.*

ELLIPTICAL SEGMENT. A portion of the area of an ellipse, lying between an elliptical arc and its chord.

ELLIPTICAL SPINDLE. A solid generated by revolving an elliptical segment around its chord as an axis.

EL-LIP-TIC'I-TY, of an oblate spheroid, like the earth, is the difference between its equatorial and polar semi-diameters, divided by the equatorial semi-diameter; or, regarding the equatorial semi-diameter as 1, it is the difference of these two semi-diameters.

If we denote the ellipticity by E, we shall have

$$E = \frac{a - b}{a} = 1 - \frac{b}{a}, \qquad \text{or} \qquad E = \frac{e^2}{2}$$

very nearly.

The value of the ellipticity of the earth's meridian, as assumed by the United States Topographical Engineers, is $\frac{1}{300}$, the eccentricity being 0.0816967.

E-LON"GA'TION. [L. from *elongo*]. Of a star, an astronomical term used in Common and Geodesic Surveying. It is the angle included between the meridian plane and a vertical plane through the star's place. See *Azimuth* and *Variation of the Needle.*

EN'NE-A-GON. [Gr. εννεα, nine, γωνια, angle]. A polygon of nine sides or angles, commonly called a nonagon.

E-NŪ'MER-ĀTE. [L. *enumero; e* and *numero, numerus*, number]. To count, to tell by numbers, to number.

E-NUN'CI-ĀTE. [*enuncio; e*, and *nuncio*, to tell]. To utter, to relate.

E-NUN-CI-Ā'TION. A concise statement of what is to be proved in a proposition, which is often made before commencing the demonstration.

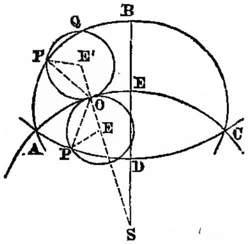
EP'I-CȲ-CLE. [Gr. επι, and κυκλος, a circle]. A small circle, whose centre is on the circumference of a greater circle; or a small orb, which being fixed in the deferent of a planet, is carried along with it, and yet by its own peculiar motion carries the body of the planet fastened to it around its proper centre. A term employed by Ptolemy.

EP-I-CȲ'CLOID. [Gr. επικυκλοειδης; επι, κυκλος, and ειδος, form]. If a circle be conceived to roll upon the circumference of another circle in the same plane, either internally or externally, any point of the first circumference will generate a curve called an epicycloid. At the same time any point not in the circumference, but lying in the same plane, will generate a curve called an epitrochoid.

The rolling circle is called the *generating circle;* the point which generates the curve is called the *generatrix;* and the circle upon which the generating circle rolls, is called the *directing circle* or the *directrix.* That portion of the directrix, or as it is sometimes

called, the *fundamental circle*, which lies between two successive points of concurrence of the fundamental circle and the curve, is called a *base*, and is equal in length to the circumference of the generating circle. When the generating circle rolls on the convex side of the directrix, the name of the generated curve is properly *epicycloid*, but when it rolls on the concave side, it is called a *hypocycloid*. When the generating circle has rolled once over, so that every point shall have been in contact with the directrix, the portion generated is called a *branch*.

Let AEC be an arc of the directing circle, S its centre, OQ the generating circle, and P



the generatrix, then is the arc AEC a base, ABC a branch, and the line SEB, which bisects the base, is an axis of the branch. ADC is a branch of a hypocycloid.

If the ratio of the lengths of the generating circle and the directing circle can be expressed in exact parts of 1, there will be a finite number of branches, and although the generating circle may continue to roll, the generatrix will, at regular intervals, only repeat the portion already generated. For example, if the generating circumference is contained n times in that of the directing circle, there will be n, and only n, branches of the epicycloid, and n branches of the hypocycloid. If the circumference of the generating circle is the $\frac{m}{n}$th part of that of the directing circle, $\frac{m}{n}$ being an irreducible fraction, then after the generating circle has rolled over mn times, the generatrix will have arrived at the point from which it started, and then it will continue to repeat the curve already described indefinitely.

When the circumference of the generating circle is an aliquot part of that of the directing circle, the epicycloid is always an algebraic curve.

If the ratio of the circumferences cannot be expressed in exact parts of 1, the number of branches is infinite, and the curve is then transcendental.

To construct the point of the epicycloid or hypocycloid, corresponding to any position of the generating circle: Let S, last figure, be the centre of the directing circle, AEC a portion of its circumference, and suppose the point A to be the point from which the generatrix started; let O be the point of contact of the generating and directing circles, corresponding to which the point is required. Draw SO, and make OE and OE′ each equal to the radius of the generating circle, and suppose that the circles which generated the epicycloid and hypocycloid are equal. In this position, from the nature of the curve, the arc AO is equal to the arc OP, and the angles at the centre will be to each other inversely as the radii. Hence, construct the angles OEP′ and OE′P′ so that

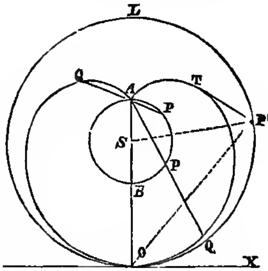
OE′P′ or OEP : OSA : : SO : OE or OE′;

then will the points P and P′ be the points required. This construction can always be made geometrically when the radii SB and OE′ are commensurable. If the chord OP′ or OP′ be drawn, it will be normal to the curve, and a line perpendicular to it through P or P′ will be a tangent to the curve.

If the radius of the generating circle is an aliquot part, as $\frac{1}{n}$th of the radius of the directing circle, then will the length of one branch of the epicycloid be equal to $8\left(\frac{n+1}{n}\right)$ times the length of the radius of the generating circle. Also the length of one branch of the corresponding hypocycloid will be $8\left(\frac{n-1}{n}\right)$ times the radius.

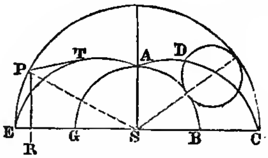
The area of one branch of the epicycloid, that is, the area between one branch and its base, will be equal to $\frac{3n+2}{n}$ times the area of the generating circle, and the area of the corresponding branch of the hypocycloid will be equal to $\frac{3n-2}{n}$ times the area of the generating circle.

If $n = 1$, the epicycloid becomes the cardioid AQO; its length is 16 times the radius SA, and the area between it and the directing circumference, is $2\frac{1}{2}$ times that of the generating circle. The hypocycloid in this case



reduces to a point. Let a circle OP′L, whose radius is equal to SO, or three times that of the generating circle, and whose centre is S, be described, and the radius AS be produced till it meets the circumference in O; draw any chord OP′, and the radius SP′, and make the angle SP′T = SP′O; then will PT be tangent to the cardioid. Hence the cardioid is the caustic curve of rays proceeding from R and reflected from the circle RPQ.

If $n = 2$, the length of the epicycloid is 12 times the radius of the generating circle, and its area is equal to 4 times that of the generating circle. In this case let S represent the centre of the directing circle, S B its radius, GAB half of the circumference, and ADC, ATE, respectively, halves of the two branches of the epicycloid. With S as a centre, and a radius SC = 2SB, describe a semi-circumfer-



ence CPE, and suppose the radius SA perpendicular to EC; draw any line RP parallel to SA, and draw the radius SP; make the angle SPT = SPR; then will PT be tangent to the epicycloid. Hence the part of the epicycloid drawn, is the caustic curve for rays parallel to SA, and reflected from the circumference CPE.

If $n = \infty$, the directing circumference becomes a straight line, and both the epicycloid and the hypocycloid become the ordinary cycloid. The length of one branch becomes 8 times the radius, or 4 times the diameter of the generating circle, and the area of one branch becomes 3 times that of the generating circle.

The chord drawn through the generatrix and the point of contact of the generating and directing circles, is called the tracing chord; denote its length by c, and the radius of curvature at the outer extremity of this chord, r; then for the epicycloid we shall have for determining the radius of curvature,

$$r = \frac{2n+2}{n+2}c, \text{ and for the hypocycloid}$$

$$r = \frac{2n-2}{n-2}c.$$

If $n = 1$, the first formula gives,

$$r = \frac{4}{3}c, \quad \text{and the second,} \quad r = 0.$$

If $n = 2$, the first formula gives,

$$r = \frac{3}{2}c, \quad \text{and the second,} \quad r = \infty,$$

which shows that the hypocycloid in this case is a straight line. Hence, if one circle is rolled upon the concave arc of another, having its radius double that of the first, every point of the circumference will generate a straight line which can easily be shown to be a diameter of the directing circle. This principle has been employed for the purpose of converting circular motion into rectilinear alternating motion in machines.

If $n = \infty$, we have the case of the common cycloid, and both formulas give $r = 2c$.

The involute of an epicycloid is a similar epicycloid.

A spherical epicycloid is a curve generated by a point of the circumference of a circle which rolls along the circumference of a directing circle, so that the plane of the generating circle shall make a constant angle with that of the directing circle.

The same distinction is drawn between trochoidal arcs as between epicycloidal arcs. When the generating point lies in the plane of a circle which rolls upon the convex side of the generating circumference, the curve is called an epitrochoid, when it rolls upon the

concave side of the arc, it is called a hypotrochoid.

If the generating circle roll upon the arc of an ellipse, parabola, &c., either upon the convex arc, or the concave arc, points in the plane of the generating curve generate elliptical, parabolic, &c., epicycloids, hypocycloids, epitrochoids, and hypotrochoids.

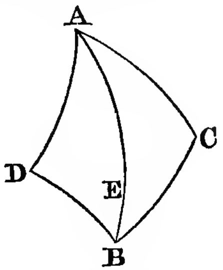
If an ellipse roll upon the circumference of an ellipse, analogous curves could be generated. These conditions may be infinitely varied.

EP-I-CȲ-CLOID'AL. Pertaining to the epicycloid.

EP-I-PE-DOM'E-TRY. [Gr. from επι, πους, μετρον]. The operation of measuring figures standing on the same base.

EP-I-TRO'CHOID. See Epicycloid.

E-QUAL. [L. *æqualis*]. In Geometry, two magnitudes are equal when they are so related that if they be properly placed, the one upon the other, they coincide throughout their whole extent. This is the fundamental meaning of the word *equal* in Mathematics. Besides this kind of equality, there is another kind in which two magnitudes are such that they may be arranged with respect to a given plane, so that for every point of one magnitude on one side of this plane, there will be a corresponding point of the other magnitude at an equal distance on the other side. Such magnitudes, when they cannot be placed so as to coincide, are called equal by symmetry.



Thus, in the two spherical triangles ABC and ABD, if AD = AC, BD = BC, and AB common, the two triangles are equal by symmetry but cannot be applied, the one to the other, so as to coincide.

In Algebra, two quantities are equal when their measures are equal, that is, when both contain the same unit taken the same number of times. The idea of equality in Algebra, is the same as that of equivalency in Geometry.

The following are some of the tests of equality between two quantities:

1. Things which are equal to the *same* or to *equal* things, are equal to each other.
2. If equals be added to equals the sums will be equal.
3. If equals be subtracted from equals the remainders will be equal.
4. Like parts of equal things are equal.
5. Like powers and like roots of equals are equal.

E-QUAL'I-TY. [L. *æqualitas*, equality]. The attribute of exact agreement of two things with respect to their quantity. In Mathematics, the symbol employed to denote this relation is $=$; thus, $a = x$, implies that a contains the same number of units of measure of a certain kind, that x does.

Equal Roots. An equation involving but one unknown quantity, is said to have *equal roots* when the second member being 0, its first member has two or more equal factors of the first degree, with respect to the unknown quantities.

When this is the case, the derived polynomial of the first member, which is the sum of the products of the m binomial factors of the first member taken in sets of $(m - 1)$, contains a factor which is also a factor of the first member of the given equation: hence,

There must be a common divisor between the first member of the proposed equation and its first derived polynomial.

Let the given equation be

$$X = 0,$$

and suppose that its first member contains n factors equal to $x - a$, n' factors equal to $x - b$, n'' factors equal to $x - c$, &c., and the simple factors $x - k$, $x - l$, &c. Then will the equation be of the form

$$(x-a)^n(x-b)^{n'}(x-c)^{n''}\cdot\cdot(x-k)(x-l)\cdot\cdot=0,$$

and the derived polynomial of the first member will be

$$\frac{nX}{x-a}+\frac{n'X}{x-b}+\frac{n''X}{x-c}+\cdot\cdot+\frac{X}{x-k}+\frac{X}{x-l}+\cdot\cdot$$

By comparing this with the first member of

the given equation, it is apparent that their greatest common divisor is,

$$(x-a)^{n-1}(x-b)^{n'-1}(x-c)^{n''-1}\dots$$

that is, it is equal to the product of the factors which enter two or more times into the first member of the given equation, each raised to a power whose exponent is one less than in the given equation.

In order, therefore, to discover whether a given equation, $X = 0$, has any equal roots, form the first derived polynomial of X and call it Y; then see if X and Y have any common divisor in terms of the unknown quantity; if they have one, the given equation contains equal roots; if they have no common divisor, the given equation has no equal roots.

Having found the greatest common divisor of X and Y, place it equal to 0: then will every single root of this new equation be twice a root of the given equation, every double root will be three times a root of the given equation, and so on. When the resulting equation cannot be solved, the given equation may be freed of its equal roots by dividing both members by the greatest common divisor already found. When the equation can be solved, the degree of the equation may be still further reduced.

Denote the greatest common divisor by D: then if D is of the form of $(x-h)^2$, there will be three roots equal to h, and the equation will be freed of them by dividing both members by $(x-h)^3$. If D is of the form $(x-h)\ (x-h')$, there will be two roots equal to h, and two equal to h', and the equation may be freed of them by dividing both members by $(x-h)^2\ (x-h')^2$, and so on.

Should the equation $D = 0$ contain equal roots, the same principles may be applied to it as to the given equation, and thus equations of a very high degree may often be solved.

Let us take the equation,

$$x^7 + 5x^6 + 6x^5 - 6x^6 - 15x^3 - 3x^2 + 8x + 4 = 0 \dots (1).$$

Its first derived polynomial is

$$7x^6 + 30x^5 + 30x^4 - 36x^5 - 45x^2 - 6x + 8.$$

The greatest common divisor between this and the first member of the given equation, placed equal to 0, gives

$$x^4 + 3x^3 + x^2 - 3x - 2 = 0 \dots (2).$$

Which cannot be solved directly, but by applying the principle of equal roots to it, that is, by seeking a common divisor between the first member and its first derived polynomial, we find that they have one equal to $x + 1$.

The first member of equation (2) has therefore a factor equal to $(x+1)^2$; and by factoring, it may be reduced to

$$(x+1)^2\ (x-1)\ (x+2) = 0.$$

Hence, the first member of equation (1) may be reduced to the form

$$(x+1)^3\ (x-1)^2\ (x+2)^2 = 0.$$

It has three roots equal to -1, two equal to $+1$ and two equal to -2.

E-QUĀ'TION. [L. *æquatio*, from *æquo*, to make equal]. In Analysis, an equation is the algebraic expression of equality between two quantities; thus,

$$x = a + b,$$

is an equation, and denotes that the quantity represented by x is equal to the sum of the quantities denoted by a and b. Every equation is composed of two parts, connected by the sign of equality. The part on the left of the sign of equality, is called the *first member*, that on the right, the *second member*. The second member is often 0.

Equations are divided into two grand divisions—*Algebraic* and *Transcendental.*

ALGEBRAIC EQUATIONS are those in which the relation between the quantities which enter them, are expressed by the ordinary operations of algebra; that is, *addition, subtraction, multiplication, division, raising to powers denoted by constant exponents, and extraction of roots indicated by constant indices.*

TRANSCENDENTAL EQUATIONS, are those in which the relations between the quantities cannot be expressed by the ordinary operations of algebra, but are expressed by *transcendental* relations, that is, by the aid of *logarithmic, trigonometrical,* or *exponential symbols.*

ALGEBRAIC EQUATIONS may involve *one,* or *more than one unknown quantity,* and are classified into *orders* depending upon the *degree of the equation.*

If the equation contains but one unknown quantity, its degree is denoted by the *highest exponent of the unknown quantity in any term.* If it contains more than one unknown quantity, the degree is indicated by *the greatest*

sum of the exponents of the unknown quantities in any term.

EXAMPLES.

$$\left.\begin{array}{l} ax + b = cx + d \\ ax + 3by + 4z = c \end{array}\right\} \text{First Degree.}$$

$$\left.\begin{array}{l} ax^2 + 2bx + c = o \\ ax^2 + bxy + z^2 = d \end{array}\right\} \text{Second Degree,}$$

$$\left.\begin{array}{l} cx^3 + dx^2 + cx + f = 0 \\ 4axy + 5z^3 - 3xyz = d \end{array}\right\} \text{Third Degree,}$$

and so on.

TRANSCENDENTAL EQUATIONS are divided into *exponential*, *logarithmic*, *trigonometrical* and *mixed*.

An EXPONENTIAL EQUATION is one in which the unknown quantity enters an exponent; as,

$$a^x = b, \ a^{b+x} + c^y = d.$$

A LOGARITHMIC EQUATION is one in which the unknown quantities enter a logarithm; as,

$$\log(e + ax) = d, \ \log(c + x) - \log(y - cz^2) = a.$$

TRIGONOMETRICAL EQUATIONS are those in which the unknown quantities enter into some one or more of the trigonometrical elements; as,

$$\tan(x + y) = \sin(x - y) + \sin z,$$
$$a + c\cos x = d + e\sin y.$$

MIXED EQUATIONS are those in which the unknown quantities enter two or more of the *transcendental* expressions, or when it enters algebraically into some terms, and transcendentally into others; as,

$$\log x + \sin y = d, \quad ax + b^x = c.$$

Algebraic or transcendental equations may be either *numerical* or *literal*.

NUMERICAL EQUATIONS are those in which all of the known quantities are expressed by numbers; as,

$$2x + 3y = 4, \ 2 + 3\log x = 6.$$

LITERAL EQUATIONS are those in which all or a part of the unknown quantities are expressed by letters; as,

$$ax + by + c = 0, \ a + \log x + cy = 0.$$

IDENTICAL EQUATIONS are those in which the second member is a repetition of the first, or in which the second member is the result of certain operations indicated in the first, this result being either expressed or indicated; as,

$$ax + b = ax + b, \ (a + x)^2 = a^2 + 2ax + x^2,$$

$$\log(1 + y) = M\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} \&c.\right)$$

It is a characteristic property of identical equations, that they are true for all values of the unknown quantity or quantities which enter them.

The *solution* of an equation, or group of equations, is the operation of finding such values for the unknown quantity or quantities which enter them, as when substituted for the unknown quantity or quantities will satisfy the equation or equations. These values are called roots of the equation.

We shall briefly explain the method of solving some of the principal algebraic equations which are met with in analysis; and first, it may be observed, that every equation containing but one unknown quantity, has at least one root, for if the two members are equal, they must be so for at least one value of the unknown quantity, either real or imaginary; this value is a *root*.

1. *Equations of the first degree, containing but one unknown quantity.*

These may be solved by the following method:

Perform all of the algebraic operations indicated; transpose all the known terms to the second member, and the unknown terms to the first member; resolve the first member into two factors, one of which shall be the unknown quantity, the other will be the algebraic sum of its co-efficients; divide both members by the co-efficient of the unknown quantity; the second member will be the value of the unknown quantity.

1. Find the value of x in the equation,

$$ax + b - cx = d - fx.$$

Transposing and factoring,

$$(a - c + f)x = d - b.$$

Dividing both members,

$$x = \frac{d - b}{a - c + f}.$$

2. *Groups of equations of the first degree, containing more than one unknown quantity.*

If the number of equations is less than the number of unknown quantities, the group is indeterminate. If the number of independent equations is greater than the number of the unknown quantities, the group is impossible. Hence, in order that the group may be determinate, the number of independent equations must be just equal to the number of unknown quantities. In this

case, the values of the unknown quantities may be found by the following rule:

Combine one of the m equations of the group with each of the $m-1$ others, eliminating the same unknown quantity; there will result a new group of $m-1$ equations, containing $m-1$ unknown quantities.

Combine one of the $m-1$ equations with each of the $m-2$ others, eliminating a second unknown quantity; there will result a new group of $m-2$ equations, containing $m-2$ unknown quantities.

Continue this operation of combination and elimination until a single equation containing a single unknown quantity is obtained; find from it the value of this unknown quantity, and substitute it in either of the two equations containing two unknown quantities, and find therefrom the value of a second unknown quantity; substitute the values of these two in either of the three equations containing three unknown quantities, and find the value of a third unknown quantity; continue this operation of successive substitution till the values of all the unknown quantities are determined.

If any or all the constants which enter the equations are arbitrary, it may happen that values assigned to them will reduce one or more of the roots to $\frac{0}{0}$. If such values are assigned in an equation containing but one unknown quantity, they will render it identical, and the value $\frac{0}{0}$ will truly represent the value of the unknown quantity. If such values be assigned in a group as to reduce the roots to the form $\frac{0}{0}$, they will cause some of the equations to depend upon the others, and thus render the group indeterminate, in which case the roots ought to be indeterminate.

If there are more equations than there are unknown quantities, and the constants are arbitrary, we may combine them so as to eliminate all the unknown quantities, and the resulting equations will be so many equations of condition, which will express the relations that must exist between the constants, in order that the group may be determinate.

In this case, some of the equations will become dependent upon the others.

The following example will illustrate the operation of solving a group of equations.

Required the values of x, y, and z, in the equations

$$5x - 6y + 4z = 15 \quad (1).$$

$$7x + 4y - 3z = 19 \quad (2).$$

$$2x + y + 6z = 46 \quad (3).$$

Eliminating z between equations (1) and (2), and between equations (2) and (3), there results

$$43x - 2y = 121 \qquad (4).$$

$$16x + 9y = 84 \qquad (5).$$

Eliminating y between equations (4) and (5), we have

$$419x = 1257 \qquad (6);$$

whence $x = 3$.

This value of x substituted in equation (5), gives $y = 4$, and these values being substituted in equation (1), give $z = 6$. Hence,

$$x = 3, \quad y = 4 \quad \text{and} \quad z = 6.$$

The principles here explained for solving a group of equations containing several unknown quantities, will apply, whatever may be the degree of the equations. The elimination is to be performed in accordance with the rules laid down under the head of *Elimination*. We shall, therefore, only discuss the rules for solving single equations containing one unknown quantity, in what is to follow.

3. *Equations of the second degree.*

Every equation of the second degree, containing but one unknown quantity, can always be reduced to the general form

$$x^2 + 2px = q,$$

by the following rule.

Perform all the algebraic operations indicated; transpose all the known terms to the second member, and the unknown terms to the first member.

Resolve the terms, containing x^2, into two factors, one of which shall be x^2, the other will be the algebraic sum of its co-efficients; resolve the terms containing x, into two factors, one of which will be x; then divide both members of the equation by the co-efficient of x^2, and the resulting equation will be of the required form,

$$x^2 + 2px = q.$$

Having reduced the equation to the above form, its two roots may be written by the following rule.

The first root is equal to half the co-efficient of the second term taken with its sign

changed, plus the square root of the second member increased by the square root of half the co-efficient of the second term.

The second root is equal to half the co-efficient of the second term with its sign changed, minus the square root of the second member increased by the square of half the co-efficient of the second term.

These rules, expressed algebraically, give

$$x = -p + \sqrt{q + p^2}, \qquad \text{1st root.}$$
$$x = -p - \sqrt{q + p^2}, \qquad \text{2d root.}$$

1. Find the values of x in the equation

$$2x + 2 = 24 - 5x - 2x^2.$$

Reducing to the required form,

$$x^2 + \frac{7}{2}x = 11;$$

whence, by the rule,

$$x = -\frac{7}{4} + \sqrt{11 + \frac{49}{16}} = 2, \quad \text{and}$$
$$x = -\frac{7}{4} - \sqrt{11 + \frac{49}{16}} = -\frac{11}{2}.$$

The following relations exist between constants entering an equation of the form

$$x^2 + 2px = q,$$

and its roots:

1st. The algebraic sum of the two roots is equal to the co-efficient of the second term with its sign changed.

2d. The product of the two roots is equal to the second member with its sign changed.

3d. The greatest possible value of the product of the two roots, is equal to the square of half the co-efficient of the second term.

These principles make known all the circumstances attending a change in the values of the constants, p and q.

With respect to the signs of p and q: they may both be positive; they may have contrary signs; or they may both be negative. These suppositions, with respect to signs, give rise to four forms:

$$\left.\begin{array}{ll} 1. & x^2 + 2px = q, \\ 2. & x^2 - 2px = q, \\ 3. & x^2 + 2px = -q, \\ 4. & x^2 - 2px = -q, \end{array}\right\} \text{FOUR FORMS.}$$

Besides these suppositions, we may suppose

$$p^2 > q, \quad p^2 = q. \quad p^2 < q;$$

we may suppose p and q, separately, equal to 0, and that both are equal to 0 together.

From the 1st and 2d principles, above enunciated, we see that both roots in the first and second forms are always real, and that they have contrary signs. In the first form, the negative root is numerically the greatest, and in the second form, the positive root is numerically the greatest.

In the third and fourth forms, the roots have the same sign in each, being both negative in the third, and both positive in the fourth.

In the third and fourth forms, if $p^2 > q$, the roots are real: if $p^2 = q$ the roots are equal in each form; if $p^2 < q$, the roots are imaginary.

If $p = 0$, the equation is *incomplete*, and in each form the roots are equal with contrary signs, real in the first and second, and imaginary in the third and fourth.

If $q = 0$, one root reduces to 0 in each form, and the equation may be reduced to one of the first degree by dividing both members by x.

If both p and q are equal to 0, all the roots in the four forms are 0.

Any supposition which reduces one root to ∞, reduces the equation to one of the first degree.

4. *Trinomial equations.*

Trinomial equations are those which involve but three kinds of terms, viz; terms containing two different powers of the unknown quantity and known terms By a method entirely analogous to that employed in reducing equations of the second degree, every trinomial equation may be reduced to the form

$$x^m + 2px^n = q,$$

Such equations can always be solved when $m = 2n$, in which case the above form becomes

$$x^{2n} + 2px^n = q, \qquad (1)$$

and its roots are

$$x = \sqrt[n]{-p \pm \sqrt{q + p^2}};$$

Hence, to solve a trinomial equation in the case specified

Reduce it to the form of equation (1); *then will its roots be found by extracting the* n*th root of half the co-efficient of the second term with its sign changed, plus and minus the square root of the second member, increased by the square of half the co-efficient of the second term.*

1. Find the values of x, in the equation

$$x^4 - 25x^2 = -144.$$

By the rule,

$$x = \pm\sqrt{\frac{25}{2} + \sqrt{-144 + \frac{625}{4}}} = \pm\sqrt{\frac{25}{2} \pm \frac{7}{2}}.$$

Hence,

$x = +4$, $x = -4$, $x = +3$, and $x = -3$.

5. *Cubic equations, or equations of the third degree.*

The method of solving cubic equations is given under the head of *cubic equations*, which see.

6. *Equations of the fourth degree.*

Every equation of the fourth degree may be reduced to the form,

$$x^4 + kx^3 + lx^2 + mx + n = 0, \qquad (1)$$

and making the second term disappear by a transformation about to be explained, it may be still further reduced to the form,

$$x^4 + px^2 + qx + r = 0. \qquad (2)$$

Assume

$$x = a + b + c:$$

then by squaring both members and transposing,

$$x^2 - (a^2 + b^2 + c^2) = 2(ab + ac + bc),$$

and squaring both members of the last equation,

$$x^4 - 2(a^2 + b^2 + c^2)x^2 + (a^2 + b^2 + c^2)^2$$
$$= 2(a^2b^2 + a^2c^2 + b^2c^2) + 8abc\,(a+b+c);$$

transposing and replacing the factor $(a+b+c)$ by x, we have finally

$$\left.\begin{array}{l} x^4 - 2(a^2 + b^2 + c^2)x^2 - 8abc\,x \\ +(a^2+b^2+c^2)^2 - 4(a^2b^2+a^2c^2+b^2c^2) \end{array}\right\} = 0. \quad (3)$$

By comparing equations (3) and (2,) we see that they will be the same if

$$\left.\begin{array}{l} p = -2(a^2 + b^2 + c^2), \\ q = -8abc, \\ r = (a^2 + b^2 + c^2)^2 \\ \quad - 4(a^2b^2 + a^2c^2 + b^2c^2) \end{array}\right\} \cdots\cdot (4)$$

Now, from the manner in which equation (3) was derived, it is evident that its roots are equal to $(a + b + c)$.

In order to determine the values of a b and c, let us regard a^2, b^2, c^2, as the roots of a cubic equation. The co-efficients of the different powers of the unknown quantity may then be found by the rule for the composition of equations, and if we denote the unknown quantity by z, we shall have the co-efficient of z^3 equal to 1; the co-efficient of z^2 equal to $-(a^2 + b^2 + c^2)$ which is equal from equations (4) to $\frac{p}{2}$; the co-efficient of z is equal to $a^2b^2 + a^2c^2 + b^2c^2$, or, from (4) equal to $\frac{p^2 - 4r}{16}$; and the absolute term is $-a^2b^2c^2$, or from (4), $-\frac{q^2}{64}$.

Hence, the auxiliary equation is,

$$z^3 + \frac{p}{2}z^2 + \frac{p^2 - 4r}{16}z - \frac{q^2}{64} = 0, \cdots (5)$$

in which the co-efficients are known from equation (2).

If now, we solve equation (5) by the rules for solving cubic equations, and denote its three roots by z', z'', and z''', we shall have

$$a = \pm\sqrt{z'}, \quad b = \pm\sqrt{z''}, \quad c = \pm\sqrt{z'''}.$$

In combining these terms to find the roots of the given equation, such signs must be given to a, b, and c, as to make their product negative, since from equation (4) $abc = -\frac{q}{8}$.

Only four such combinations can be made, and each combination corresponds to a root of the given equation. Denoting the four roots by x', x'', x''', and x'''', we have,

$$x' = +\sqrt{z'} + \sqrt{z''} - \sqrt{z'''},$$
$$x'' = \pm\sqrt{z'} - \sqrt{z''} + \sqrt{z'''},$$
$$x''' = -\sqrt{z'} + \sqrt{z''} + \sqrt{z'''},$$
and $$x'''' = -\sqrt{z'} - \sqrt{z''} - \sqrt{z'''}.$$

Analysts have been unable to solve, in a general manner, any equation of a higher degree than the fourth, except in some very particular cases, as in trinomial equations of a certain form, and binomial equations.

7. *Equations of a higher degree.*

Every equation of the m^{th} degree may be reduced to the form

$$x^m + Px^{m-1} + Qx^{m-1} + Rx^{m-2} + \cdots + Tx + U = 0,$$

in which P, Q, R, T, &c., are co-efficients in the most general sense of the term, that is, positive or negative, real or imaginary, entire or fractional, m being any positive whole number.

In speaking of equations of the m^{th} degree, we shall hereafter suppose them reduced to the preceding form.

When one or more of the co-efficients P,

Q, R, &c., is equal to 0, the equation is said to be *incomplete*, otherwise it is *complete*.

Although no rules have been deduced for solving equations which are of a higher degree than the fourth, many useful transformations have been discovered, which often lead to the solution of particular cases, and are otherwise of importance in analytical investigations. Rules have also been demonstrated for solving either exactly or approximately numerical equations of any degree.

We shall enumerate some of the most important principles employed in transforming equations, and then give some of the methods of solving numerical equations.

Transformations, Properties, &c.

1. If a is a root of an equation, $x - a$ will divide the first member; and conversely, if $x - a$ will exactly divide the first member, then is a a root of the equation.

2. Every equation has as many roots as there are units in the number which expresses the degree of the equation and no more.

As a consequence of these two properties, it follows that the first member of every equation has m different divisors of the first degree with respect to x, of the form

$$(x - a),\ (x - b),\ (x - c),\ \&c. \cdots$$

If these divisors be multiplied together in sets of two, three, &c., there will be formed as many different divisors of the second degree as there are different combinations of m letters taken in sets of 2, or

$$m \cdot \frac{m-1}{2};$$

there will be as many different divisors of the third degree as there are combinations of m letters taken in sets of 3, or

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}; \text{ and so on.}$$

On these principles also depends the rule for the composition of equations. See *Composition of Equations*.

3. If the co-efficients of the different powers of the unknown quantity are all whole numbers, all the commensurable roots of the equation are also whole numbers.

4. Every equation in which some of the co-efficients are fractions, can be transformed into one of the general form, in which the co-efficients are all whole numbers.

Assume the equation

$$x^m + Px^{m-1} + Qx^{m-2} + \cdots + Tx + U = 0,$$

some of the co-efficients being fractional, and for x substitute $\frac{y}{k}$, in which y is unknown and k arbitrary. After multiplying both members by k^m, there will result

$$y^m + Pky^{m-1} + Qk^2y^{m-2} + \cdots$$
$$+ Tk^{m-1}x + Uk^m = 0.$$

Assign to k such a value that its powers in the different terms shall contain the prime factors of the denominators in those terms, each raised to a power at least as great as that which enters the denominator. The resulting equation will be of the form required.

1. $x^4 - \frac{5}{6}x^3 + \frac{5}{12}x^2 - \frac{7}{150}x - \frac{13}{9000} = 0.$

The transformed equation is

$$y^4 - \frac{5}{6}ky^3 + \frac{5}{12}k^2y^2 - \frac{7}{150}k^3y - \frac{13}{9000}k^4 = 0.$$

Make $k = 2 \times 3 \times 5$, and reducing these results,

$$y^4 - 25y^3 + 375y^2 - 1260y - 1170 = 0.$$

5. Imaginary roots and surd roots enter by pairs; that is, if there is a root of the form

$$a + b\sqrt{-1}, \quad \text{or} \quad a + \sqrt{b},$$

there will be a second root of the form

$$a - b\sqrt{-1}, \quad \text{or} \quad a - \sqrt{b}.$$

Hence, every equation whose roots are all of the above forms, must be of an even degree.

A pair of roots of either of the above forms are called *conjugate roots*.

6. Every equation can be transformed into another, in which the second term shall be wanting, as follows:

Substitute for the unknown quantity, another unknown quantity minus the co-efficient of the second term divided by the number which expresses the degree of the equation.

The transformation may also be made by the method of synthetic division, as will be explained presently.

7. Every equation may be transformed into another, in which the roots are greater or less than those of the given equation by a constant quantity, by the aid of the derived polynomials of the first member.

The derived polynomial of a given polynomial, is the result obtained by multiplying every term of the given polynomial by the exponent of the unknown quantity in that term, and then diminishing the exponent of the unknown quantity by 1. The second derived polynomial is the derived polynomial of the first. The third derived polynomial is the derived polynomial of the second, and so on. These, taken in their order, are called *successive derived polynomials.*

Let it be required to transform the equation

$x^m + Px^{m-1} + Qx^{m-2} + \cdots + Tx + U = 0,$

into one whose roots shall be less than in the given equation by a. If we denote the unknown quantity by u, the transformed equation, will be

$$X' + \frac{Y'}{1}u + \frac{Z'}{1\cdot 2}u^2 + \frac{V'}{1\cdot 2\cdot 3}u^3$$
$$+ \cdots + U^m = 0:$$

in which X' is what the first member of the given equation becomes when a is substituted for x; Y' is what the first derived polynomial of the first member becomes under the same supposition; Z' is what the third derived polynomial becomes, &c.

This method of transformation is somewhat tedious, and it may be replaced by a method depending upon the following principle:

If it is required to transform a given equation to another, whose roots shall be less than those of the given equation by r, divide the first member by $x - r$; the remainder will be the absolute term of the transformed equation: divide the first quotient by $x - r$, and the remainder will be the co-efficient of the first power of the unknown quantity in the transformed equation: divide the second quotient by $x - r$, and the remainder will be the co-efficient of the second power of the unknown quantity, and so on. By continuing the operation, we may find all the co-efficients of the transformed equations in an inverse order.

By the method of synthetic division, this operation becomes a very simple one, as is shown by the following example: Let it be required to transform the equation

$$x^4 - 3x^3 - 15x^2 + 49x - 12 = 0,$$

into one whose roots shall be less by 3 than in the given equation.

OPERATION.

$$\begin{array}{l|l}
1-3-15+49-12 & 1,+3 \\
\quad +3+\ 0-45+12 & \\
\hline
\quad\ 0-15+\ 4,\ 0 & \\
\quad\ 3+\ 9-18 & \\
\hline
\quad\ 3-\ 6,-14 & \\
\quad\ 3+18 & \\
\hline
\quad\ 6,+12 & \\
\quad\ 3 & \\
\hline
1+9+12-14+0. &
\end{array}$$

Hence, the transformed equation is

$$y^4 + 9y^3 + 12y^2 - 14y = 0.$$

8. The following are some of the properties of numerical equations:

If two numbers p and q, substituted for x, in succession, in the first member, give results affected with contrary signs, the proposed equation has at least one real root comprised between these numbers.

9. When an uneven number of real roots is comprised between two numbers p and q, the results obtained by substituting them in succession for x in the first member, will be affected with contrary signs; but if they comprise an even number, the results will be affected with the same sign.

10. If the signs of the alternate terms of an equation be changed, the signs of the roots will be changed.

11. Every equation in which the signs of all the terms are plus, must have all its real roots negative.

12. Every complete equation having the signs of its terms alternately plus and minus, must have all of its real roots positive. The same principle holds in incomplete equations if we take care to supply the wanting terms by 0.

13. Every equation of an odd degree, whose co-efficients are real, has at least one root affected with a sign contrary to that of the last term.

14. Every equation of an even degree, the co-efficients being real, and the sign of the last term minus, has at least two real roots, one positive and the other negative.

15. When the last term of an equation is positive, the number of its real positive roots is even; and when it is negative, the number of such roots is uneven.

16. Descarte's Rule. When the roots

of an equation are all real, the number of positive roots is equal to the number of variations of sign, and the number of negative roots is equal to the number of permanences of sign.

A variation is a change of sign in passing along the equation; a permanence is when two consecutive terms have the same sign.

17. STURM'S RULE. Suppose an equation to have been freed of its equal roots, and then denote its first member by X, its derived polynomial by X_1, and then apply to X and X_1 the process of finding their greatest common divisor, differing only in this respect, that instead of using the successive remainders, as at first obtained, we change their signs, and take care also, in preparing for the operation, neither to introduce nor reject any factor except a positive one.

Denoting the successive remainders after their signs have been changed, by

$$X_2,\ X_3,\ \&c.,\ \text{to}\ X_n,$$

which will be independent of x, and writing the expressions in their order, we shall have

$$X,\ X_1,\ X_2,\ X_3 \qquad X_n.$$

Suppose a number p to be substituted for x in each of the expressions, and the signs of the resulting quantities, together with the sign of X_n, to be arranged in a line; also suppose another number q greater than p, to be substituted for x in the expressions, and the signs of the results to be arranged in like manner, then will the number of variations of signs in the first line, diminished by the number of variations of signs in the second, be equal to the number of real roots comprised between p and q. If $p = -\infty$, and $q = +\infty$, the rule will give the whole number of real roots of the equation, from which, and from the degree of the equation, the number of imaginary roots may be inferred.

By substituting for p and q successive numbers, we may determine the limits between which the individual roots are found. It is to be observed that if we find any expression as X, which retains the same sign for all values of x, the expressions after it may be neglected.

As an application of Sturm's rule, let it be required to find the number and places of the real roots of the equation

$$8x^3 - 6x - 1 = 0.$$

We shall find, in accordance with the rule, the following expressions:

$$X = 8x^3 - 6x - 1, \qquad X_1 = 4x^2 - 1,$$
$$X_2 = 4x + 1, \quad \text{and} \quad X_3 = +3,$$

Making $x = -\infty$, the resulting signs are $-\ +\ -\ +$, 3 *variations*.

Making $x = +\infty$, the resulting signs are $+\ +\ +\ +$, 0 *variations*.

There are then three real roots. To find their places:

For $x = -1$ the signs are $- + - +$, 3 variations.

For $x = 0$, the signs are $- - + +$, 1 variation.

For $x = +1$, the signs are $+ + + +$, 0 variation.

Hence, two of the roots are between 0 and -1, and the other root between 0 and $+1$.

18. HORNER'S METHOD. This rule only applies to finding the values of positive roots, but if negative roots are to be determined, we have only to change the signs of the alternate terms, when the corresponding roots of the transformed equation will be positive, and may thence be determined, and when taken with their signs changed will be the negative roots sought.

Horner's process consists in a succession of transformations of one equation into another, each transformed equation having its roots less than those of the given equation, by the difference between the true value of the root and that part of the value expressed by the figures already found, which are called the *initial figures*.

The transformations may be made by the method of synthetic division.

When the difference between the true value of the root and that part of it already found, is very small, the first figure of this difference is equal to the quotient obtained by dividing the absolute term by the co-efficient of the preceding term.

Horner's rule is as follows:

Find the number and places of the real roots by Sturm's rule, and set the negative roots aside. Consider only the positive roots. Transform the given equation into another whose roots shall be less than those of the given equation, by the initial figure or figures

already found; then by Sturm's rule find the places of the roots of this new equation, and the first figure of each will be the first decimal figure in the required root.

Divide the absolute term of the transformed equation by the co-efficient of the preceding term, and the first figure of the quotient will be the second decimal figure of the required root.

Transform the last equation into another whose roots shall be less than those of the previous equation by the figure last found, and operate as before, until the root is found to any desirable degree of accuracy.

This method is only one of approximation, and it may happen that the second decimal figure obtained may not be correct. It will, therefore, be well to find by Sturm's theorem, the first two decimal figures, after which the method of division may be resorted to without danger of error.

19. Newton's Method. The principle of Newton's method is, that after obtaining an approximate value of the root, the error is *nearly* equal to the quotient obtained by dividing the first member of the given equation by its derived polynomial, and in the result making x equal to the approximate value of the root found. The modification consists in combining with it Sturm's method and the method of transformation by synthetic division. The modified rule is as follows:

Find by Sturm's rule the number and places of the real roots, and set the negative roots aside.

Transform the given equation by the method of synthetic division, into another, in which the roots are diminished by the initial figures already found.

Form a fraction whose numerator is the first member of the given equation, and whose denominator is its first derived polynomial, and in it substitute for x the initial figures already found; the result will be the first correction.

Apply the correction found, and in the same fraction substitute for x the corrected value of the root; the result will be the second correction. Apply this as before, and continue the operation till the desired degree of accuracy is obtained.

This rule will in most cases give a rapidly approximating value for the root; but Fourier has shown that for the complete success of the rule in all cases, the following conditions must be satisfied:

1st. There must be no value of x between the limits within which the root is known to lie, that will make either the first member of the given equation, or its first or second derived polynomial, equal to 0.

2. The approximation must be commenced and continued from that limit which makes the first member and its second derived polynomial have the same sign.

We can ascertain whether these conditions are satisfied by means of Sturm's rule.

Equations, Differential. See *Differential Equations*.

Equations, Integral. See *Integration of Differential Equations*.

Equation of a Curve, is an Equation which expresses the relation between the co-ordinates of every point of the curve. If the curve is referred to rectilinear axes, the equation is called *rectilinear*; if to a polar system, it is called the polar equation. See *Analytical Geometry*.

Equation of Condition. An equation of condition is one which must be satisfied in order that a given condition may be fulfilled. In any given case, as many reasonable conditions may be imposed as there are disposable *arbitrary constants* entering the problem, and consequently as many equations of condition may be satisfied.

For example: the equation of condition that two straight lines shall be at right angles to each other is

$$1 + aa' = 0,$$

in which a and a' are the tangents of the angles which the lines make with the axis of X. If a and a' are both arbitrary constants, the equation is indeterminate, and may be satisfied in an infinite number of ways. If, however, a value for one of them be assumed, it is equivalent to assuming a second equation of condition, and the value of the other constant may be at once determined.

An equation of condition may also indicate a relation which must exist in order that some analytical operation may be performed. Thus, in order that a differential equation of the form

$$Pdx + Qdy = 0,$$

may be susceptible of direct integration, it is necessary that the equation of condition

$$\frac{dP}{dy} = \frac{dQ}{dx},$$

should be satisfied.

No device of analysis is more fruitful in results than the proper use and management of equations of condition.

EQUATION OF PAYMENTS. The name of a rule of arithmetic, the object of which is to find the mean time of payment of several sums due at different times. The rule is as follows:

Multiply each payment by the time before it becomes due, and divide the sum of these products by the sum of the payments: the quotient will be the mean time.

Let it be required to find the mean time of payment of a sum of \$200 due in two months, \$200 due in four months, and \$100 due in eight months.

Here,

$$\begin{array}{rr} 200 \times 2 = & 400 \\ 200 \times 4 = & 800 \\ 100 \times 8 = & 800 \\ \hline 500) & 2000 \end{array} \text{ (4 months.}$$

E-QUĀ'TOR. [L. *æquo*, to make equal]. A great circle of the sphere, whose plane is perpendicular to the axis of revolution. Longitude is reckoned upon the equator, and latitude upon meridians perpendicular to it. The equator is sometimes called the equinoctial.

E-QUI-AN″GU-LAR. [L. *æquus*, equal, and *angulus*, an angle]. Having equal angles. In Geometry, a polygon is *equiangular* when all its angles are equal to each other. Thus, a square is equiangular. Two polygons are *equiangular*, or *mutually equiangular*, when all the angles of the one are equal to all the angles of the other, each to each; that is, when taken in the same order, the first angle of the one is equal to the first angle of the other, the second angle of the one equal to the second angle of the other, and so on. A polyhedron is equiangular, when all its polyhedral angles are equal, each to each, as in the cube. Two polyhedrons are equiangular, or *mutually equiangular*, when the polyhedral angles of the one are equal to the polyhedral angles of the other, each to each. That *two* polygons or polyhedrons may be equiangular, it is not necessary that each be equiangular of itself; but the term simply implies that they are so, as compared with each other.

E-QUI-CRŬ'RAL. [L. *æquus*, equal, and *crus*, a leg]. Having equal legs or sides. Thus, an isosceles triangle is equicrural.

E-QUI-DIF'FER-ENT. Having equal differences. Thus, the terms of an arithmetical progression are equidifferent.

E-QUI-DIS'TANT. [L. *æquus*, equal, and *distans*, distant]. Two or more points are equidistant from a given point, when their distances from it are equal to each other. In a series, three or more terms are said to be equidistant, when there exists the same number of terms between each pair of consecutive terms. Thus, in the series of natural numbers, 1, 2, 3, 4, &c., the numbers, 1, 5, 9, 13, &c., are equidistant terms. These constitute a new series.

E-QUI-LAT'ER-AL. [L. *æquus*, equal, and *latus*, a side]. Having equal sides. In Geometry, a polygon is *equilateral*, when all its sides are equal to each other. Two polygons are *equilateral*, or *mutually equilateral*, when all the sides of the one are equal to all the sides of the other, each to each, and taken in the same order. Equilateral polygons, inscribed in circles, are necessarily equiangular; but the converse is not necessarily true. When the number of sides is odd, an inscribed equiangular polygon is also equilateral; but when the number of sides is even, the inscribed equiangular polygon may have all its sides equal, or the alternate sides may be respectively equal to each other, the two sets not being equal.

For example, an inscribed rectangle may have all its sides equal to each other, in which case it is a square, or the opposite sides may be equal to each other in pairs. Similarly, for any inscribed equiangular polygon, having an even number of sides.

EQUILATERAL HYPERBOLA. One whose axes are equal. Its equation, when referred to the centre and axes, is

$$y^2 - x^2 = -a^2,$$

in which a denotes the length of either semi-axis. In this curve, the asymptotes are at right angles to each other, and the curve is

equal to its conjugate. Any pair of conjugate diameters are equal to each other, and make equal angles with the asymptotes.

To cut the curve from the cone, requires that the cone be either rectangular or obtuse. Pass a plane through the vertex, cutting out two elements at right angles to each other; then will any plane, passed parallel to this, cut out an equilateral hyperbola. If the curve be referred to its asymptotes, and the abscissa of the principal vertex be regarded as 1; then will the area between this ordinate, the curve, the axis of X, and any other ordinate whatever, be equal to the Naperian logarithm of the abscissa. Hence, this curve is often called the *logarithmic hyperbola.*

E-QUI-MUL′TI-PLE. [L. *æquus*, equal, and *multiplico*, to increase]. The products obtained by multiplying two quantities by the same quantity, are equimultiples of the given quantities. Thus, ma and mb are equimultiples of a and b.

Equimultiples of two quantities are to each other as the quantities themselves: thus,

$$a \quad b :: ma \; . \; mb;$$

in which, m may be any quantity whatever.

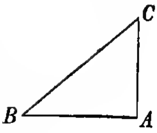
E-QUI-NOC′TIAL. [L. *æquus*, equal, and *nox*, night]. Belonging to the equinoxes: denoting an equal length of day and night.

Equinoctial Colure. The meridian which passes through the equinoctial points.

Equinoctial Points. The points in which the equator intersects the ecliptic. The equinoctial points in the heavens do not retain a constant position with respect to the fixed stars, but have a slow motion from west to east amounting to about 50 seconds yearly. This motion is called the precession of the equinoxes.

E-QUIV′A-LENT. [L. *æquus*, equal, and *valens*, to be worth]. Equal in value; in power, or force, or effect.

Equivalent Magnitudes, in Geometry, are those whose measures are equal but which do not admit of superposition. Thus, a rectangle is equivalent to a triangle having the same base and double the altitude of the rectangle. To express the idea of equivalency, or equality of measure, the symbol $\Leftrightarrow$ has been adopted. Thus, in the right-angled triangles ABC,



$$\overline{BC}^2 \Leftrightarrow \overline{AC}^2 + \overline{AB}^2:$$

this expression is read, the square of the side BC is *equivalent* to the sum of the squares of AC and AB.

E-RECT′. [L. *erigo*, to set upright]. To erect a perpendicular to a given line or plane at any point, is the same as to construct a straight line perpendicular to the line or plane at that point.

ER′ROR, [From *erro*, to wander]. The difference between the true result of any operation of arithmetic or algebra, and an approximate result. This term is particularly employed in the rule of double position. See *Position Double.*

ES-SEN′TIAL. [L. *essentia*, the essence, or substance]. Necessary to the existence of a thing.

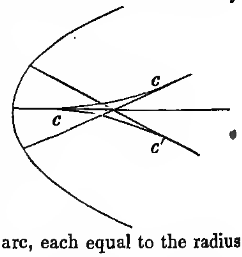
Essential Sign of a quantity, the sign resulting from the combination of the sign of the expression with the sign of operation; thus, in the expression

$$a - (b - c),$$

the essential sign of b is $-$, whilst that of c is $+$, resulting from the combination of the two negative signs, the one preceding the expression $(b - c)$, and the other preceding the quantity c.

E′VEN NUMBER. A number divisible by 2; as 6, 8, 10, &c.

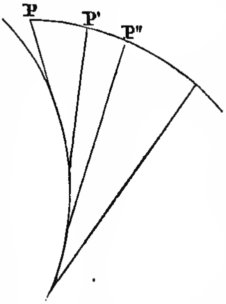
EV′O-LUTE OF A CURVE. [L. *e*, from, and *volvo*, to unfold]. The locus of the centres of all of the circles which are osculatory to the curve.



If at different points of a given curve, normals be drawn to the curve and distances be laid off upon these normals, in the concave side of the arc, each equal to the radius of curvature, then a curve drawn through the extremities c, c', &c., of these lines will be the evolute of the curve. The given curve, with respect to the evolute, is called the *involute*.

It is a property of the radius of curvature that it is normal to the involute, and tangent to the evolute. This property enables us to construct the evolute of a curve, approximately, as follows :

Draw any number of normals to the given curve at points P, P′, P″, &c.; draw a curve tangent to these several normals and it will be the evolute required, and will be more or less accurate according as the normals are more or less numerous.



This property, taken in connection with the property that any arc of the evolute is equal to the difference of the radii of curvature of the involute through its extremities, enables us to construct the involute when the evolute and one point are given. Let A be the given point. Wrap a string about the evolute and stretch it so that it will pass through A ; at A attach a pencil and unwind the string, keeping it stretched ; the pencil will describe the involute. It is plain, that since each point of the thread, as it unwinds, describes a curve, that the same evolute has an infinite number of involutes ; but any involute has only a single evolute.

To find the equation of the evolute of any curve, let us assume two of the equations of condition for osculatory circles, which are

$$y - \beta = -\frac{dx^2 + dy^2}{d^2y} \cdots\cdots (1), \quad \text{and}$$

$$x - a = -\frac{dy}{dx}(y - \beta) \,; \cdots (2).$$

In these, a and β are the co-ordinates of the osculatory circle drawn to the involute at a point whose co-ordinates are x and y. If, in addition to these two equations, we assume the equation of the involute and its differential equations of the first and second orders, we shall have, in all, five equations containing the six quantities, a, β, x, y, dy, and d^2y. By combining these equations, the quantities x, y, dy, d^2y, may be eliminated, leaving an equation between a and β. This resulting equation, expresses a relation between a, β, and constants, and is entirely independent of x and y. It is the *equation of the evolute.*

For example, the equation of the parabola and its differential equations of the first and second order are, respectively,

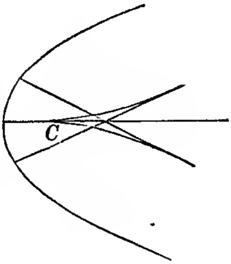
$y^2 = 2px \cdot\cdot (3)$, $ydy = pdx \cdot\cdot (4)$, $yd^2y + dy^2 = 0\,(5)$.

If now, equations (1), (2), (3), (4) and (5), be combined so as to eliminate x, dx, y, dy, and d^2y, the resulting equation will be

$$\beta^2 = \frac{8}{27p}(a - p)^3,$$

which is the equation of the evolute.

Hence, the evolute of the common parabola is a cubic parabola, having its origin at the point C, on the axis of the curve, and at a distance from the vertex equal to half the parameter of the curve.



In like manner, it may be shown, that the evolute of the cycloid is an equal cycloid.

The evolute of any algebraic curve is always rectifiable.

Evolute Imperfect. If tangents be drawn to a curve at every point, and at the points of contact instead of normals oblique lines be drawn, making a given angle with the tangents ; then will a curve tangent to all these oblique lines be what has been called an *imperfect evolute* It is of little practical importance in a mathematical point of view.

EV-O-LU′TION. [L. *evolutio*, unrolling]. In Arithmetic, is the same as the extraction of a root, and stands opposed to the term involution, which is the operation of raising a quantity to a power. See *Extraction of Roots.*

E-VOLV′ENT, The same as Involute, which see.

EX-AM′PLE. [L. *exemplum*, an example]. A particular application of a general principle

or rule, generally given to illustrate the nature of the rule or its mode of application.

EX-CEN'TRIC, EX-CEN-TRIC'I-TY. See ECCENTRIC, ECCENTRICITY.

EX-CESS'. [L. *excessus*, from *excedo*, to go beyond]. That which goes beyond. See *Property of* 9's.

EXCESS SPHERICAL. The excess of the sum of the three angles of a spherical triangle over two right angles, or 180°, is called the *spherical excess*. This element is of much importance in geodesical surveying, where extensive triangles upon the surface of the earth are considered. Any triangle employed in a geodesic survey is necessarily a very small portion of the surface of the entire globe, and consequently the spherical excess is small, but not so small as to be insensible to the accurate instruments now employed upon such surveys.

Legendre has shown that when the area of a spherical triangle is very small, compared with the entire surface of the sphere, it is sensibly equal in area to a plane triangle, whose sides are respectively equal in length to those of the spherical triangle, and whose angles are equal, respectively, to those of the spherical triangle, each diminished by one-third of the *spherical excess.*

If it is assumed that the three angles of a spherical triangle have been measured with equal accuracy, it is not necessary to know the spherical excess in order to compute the parts of the triangle ; but it is important to determine the excess for the purpose of estimating the accuracy of the observations.

The formula for computing the spherical excess, is

$$\mathrm{E} = \frac{S}{r^2 \sin 1''},$$

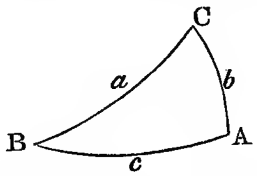
in which E denotes the excess, S the area of the spherical triangle, in square yards, and r the radius of the earth, in yards.

The area S is determined by taking the sum of the three angles, and subtracting from it 180°, and then taking one third of this from each of the measured angles ; with the angles thus obtained the area may be computed when the length of one side is known, in accordance with Legendre's principles, the triangle being regarded as plane.

The value of the logarithm of r, the mean radius of the earth, as adopted by the Topographical Bureau, is 6.8427917.

Having computed the spherical excess by the formula given, the difference between this and the excess found from the measured angles is due to error in observation, and ought to be distributed equally amongst the angles. When this error exceeds 3″ in any triangle, the measurements ought to be rejected, where great accuracy is required, particularly where there is a long succession of triangles dependent one upon another. This limit is the one adopted on the United States Coast Survey.

In order to illustrate the preceding principles, let us suppose that in a geodesic spherical triangle ABC, the side c is known,



and that the angles found by measurement are respectively

$$\begin{aligned} B &= 49° \, 17' \, 23''.24 \\ C &= 64° \, 08' \, 37''.78 \\ A &= \underline{66° \, 34' \, 04''.80} \\ & \qquad\qquad 5''.82 \end{aligned}$$

observed spherical excess.

Subtracting one third of this, or 1″.94, from each, we have the corresponding angles of the plane triangle,

$$\begin{aligned} B' &= 49° \, 17' \, 21''.30 \\ C' &= 64° \, 08' \, 35''.84 \\ A' &= 66° \, 34' \, 02''.86 \end{aligned}$$

The several parts may then be found from the formulas

$$b = c\,\frac{\sin B'}{\sin C'}, \quad a = c\,\frac{\sin A'}{\sin C'}, \quad S = \frac{bc \sin A'}{2}.$$

Substituting the value of S in the formula for the spherical excess,

$$E = \frac{S}{r^2 \sin 1''},$$

we find $E = 5''.67$. Hence the measured excess exceeds the true excess by 0″.15, and this is the measure of the error committed.

Subtracting one third of this error from each of the measured angles, the corrected angles are

$$\begin{aligned} B &= 49^\circ\ 17'\ 23''.19 \\ C &= 64^\circ\ 08'\ 37''.73 \\ A &= 66^\circ\ 34'\ 04''.75 \\ &\overline{\qquad\qquad 5''.67, \text{true excess.}} \end{aligned}$$

For places whose latitudes are between 25° and 45°, which includes a great portion of the United States, the spherical excess is about 1″ for every 75.5 square miles of area.

Hence, if the area of the triangle, in square miles, be known, a good approximation to the excess in seconds may be found by dividing this area by 75.5 square miles.

EX-CHĂNGE′. [Fr. *échanger*]. To barter. A rule in Arithmetic. The operation of findthe value of one commodity or denomination of money, in terms of another. When a unit of each denomination is fixed upon in terms of some assumed standard, the practical operations in this arithmetical rule become simple applications of the principles of proportion.

In commerce, *exchange* is a mercantile operation, by means of which debts due to individuals at a distance are paid without the transmission of money. This operation is effected by means of what are called bills of exchange.

A bill of exchange is an order addressed to some person, directing him to pay to a certain other person a specified sum. There are always three parties to a bill of exchange, and sometimes more.

1. He who writes or *draws* the order is called the *drawer* or *maker* of the bill.

2. The person to whom it is directed is called the *drawee.*

3. The person to whom the money is ordered to be paid is called the *payee.*

4. Any person who buys a bill is called the *buyer* or *remitter.*

5. The payee and other persons into whose hands the bill may pass previous to its being paid, and who write their names upon the back of it, are termed *indorsers.*

6. The person in whose possession the bill is at any given period, is termed the *holder* or *possessor.*

7. When the bill is presented to the *drawee*, if he agrees to pay it at the time specified, he signifies the same by writing *accepted*, and signing his name across the face of the bill. He is then called the *acceptor*, and the bill is said to be *accepted.*

The drawee does not become responsible till the bill is accepted. If on presentation he does not accept, the holder should give notice of the refusal to the drawer, and to all the indorsers. This notice is called a *protest*, and is given by a *notary* or *notary public*, an officer appointed for that purpose.

If the drawee accepts the bill, but fails to make payment at the time, the parties must be notified as before, and this is called *protesting for non-payment.* If the indorsers are not notified in proper time, they cease to be holden for the amount of the bill.

Days of Grace are a certain number of days granted to the person who pays the bill, after the time named in the bill has expired.

In ascertaining the time when a bill, which is payable so many days after sight, actually falls due, the day of presentment or the day of the date is not reckoned. When the time is reckoned in months, *calendar* months are always understood.

When the month in which a bill is due is shorter than the one in which it is dated, it is customary not to go on into the next month. Thus, a hill drawn on the 29th, 30th, or 31st of December, payable two months after date, would fall due on the last day of February, except for the days of grace, and would actually be due on the 3d of March.

Bills of exchange are the true money of commerce. When the drawer and drawee both reside in the same country, the bill is said to be *inland*, and when they reside in different countries, it is a *foreign bill.*

Par of Exchange. The par of exchange between two countries, is the equality in value of a certain amount of currency in one country with a certain amount in the other. Thus, according to the mint regulations of Great Britain and France, a pound sterling of the former country is equivalent to 25.20 francs of the latter, and the exchange between the two countries is said to be at par when a bill of 25.20 francs drawn in London on Paris is worth £1 in London, and when a bill of £1 drawn on London is worth 25.20 francs in Paris. When £1 in London buys a bill on Paris for more than 25.20 francs, the exchange is said to be *in favor* of London, and *against*

Paris. When it takes more than £1 to buy a bill of 25.20 francs, the exchange is against the former and in favor of the latter.

The COMMERCIAL PAR OF EXCHANGE depends upon the market value of the currency of the two countries, and may fluctuate from time to time. Thus the market value of an English sovereign in New York varies from \$4.83 to \$4.85, and it is this varying value which determines the *commercial par*.

The COURSE OF EXCHANGE is the variation of price paid at one place for bills drawn upon another. This variation may arise from two different circumstances. *First*, from a discrepancy between the intrinsic value of coins and their value as established by the mint regulations; and *secondly*, from any sudden increase or diminution in the amount of bills drawn in one place upon the other.

The exchange value of the pound sterling of Great Britain in the United States, is \$4.444, and it is upon this basis that bills of exchange are drawn in this country; but this is much below both the intrinsic and commercial values; hence, the amount of exchange on Great Britain has to be increased to make up the deficiency. The commercial value of the pound sterling exceeds the exchange value by nearly 9 per cent; thus,

The exchange value . . .	\$4.444
Add 9 per cent . . .	.399
Which gives	\$4.844,

and this is very near the average commercial value of the pound sterling. When, therefore, the exchange is at a premium of 9 per cent on the assumed unit, it is at *commercial par*, and it would stand at this rate between Great Britain and this country, were it not for fluctuations of trade and other accidental circumstances.

When the nominal exchange from this country to Great Britain is *more than* 9 per cent, it is above par; *when less*, it is below par

EX-HAUST'IONS. [L. *ex*, from, and *haurio*, to draw]. A method of demonstration much employed by the ancient geometers, nearly equivalent to the modern method of limits, and involving the principle of the *reductio ad absurdum*.

The principle of the method of exhaustions might be enunciated as follows: If a certain magnitude is less than a second, and greater than a third magnitude, whilst the difference between the second and third magnitudes may be made less than any assignable quantity, then will the three magnitudes ultimately become equal.

For example, if a regular polygon be inscribed within a circle, and a similar polygon be inscribed about the circle, the number of sides of these polygons may be taken so great that their difference will be less than any given area. By continually increasing the number of sides, this difference is continually diminished or exhausted, and as the two polygons approach each other in area, they both approach the circle in area. If, therefore, we commence the computation by finding the area of the circumscribed and inscribed square, and then from these the circumscribed and inscribed regular octagon, then figures of 16, 32, 64, &c., sides, as may easily be done, we shall ultimately arrive at two areas, one of the circumscribed, and the other of the inscribed polygon, which differ very little from each other, and as far as the expressions are the same, either one may be taken as the approximate expression for the area of the circle. In this manner, the area of the circle is arrived at in plane geometry.

In like manner, the ancients applied the method of exhaustions to a great variety of propositions appertaining to rectifications and quadratures, but which are of comparatively little importance since the introduction of the calculus.

EX-PAN'SION. [L. *expansio*, dilation, extension]. A term sometimes employed to denote the result of an indicated operation. Thus, the indicated cube of $a + b$ is $(a + b)^3$, and its expansion is

$$a^3 + 3a^2b + 3ab^2 + b^3.$$

Expansion in this sense is nearly synonymous with development.

EX-PECT-Ā'TION. [L. *expectatio*, an awaiting]. In the theory of chances, the value of any chance which depends upon some contingent event. Thus, if a person is to receive the sum of \$100 upon the occurrence of an event which has an equal chance of happening or failing, the expectation of the sum is worth \$50. In like manner, if there are three chances of the event's failing,

and only one of its happening, the expectation is worth only $25.

Expectation of Life. The average duration of life after any given age as determined by the tables of mortality. If it is found from a great number of recorded examples, that of all the individuals who reach the age of 25, the average remaining period of existence is 37.86 years, then is the expectation of life at that age 37.86 years.

The mathematical principles involved in computing the expectation of life are the following:

Let the probabilities that an individual of a given age will live,

1, 2, 3, 4, ... n years, be denoted by p' p'' p''' p^{iv} . . . . $p^{n'}$, respectively; then the probabilities that life will fail at the end of the n^{th} year, are $p^{n'-1} - p^{n'}$.

In computing the expectation of life for any future year, as the n^{th}, two contingencies are to be considered:

1st. He may live through that year, the value of which contingency is $p^{n'}$. 2d. He may die in the course of that year; and as he may die at any part of that year, we must regard his death as happening at the middle of the year; the value of this contingency is therefore

$$\tfrac{1}{2}(p^{n'-1} - p^{n'}).$$

Adding these contingencies, we find for the total value of the expectation of life, with respect to the n^{th} year from the time considered,

$$\tfrac{1}{2}(p^{n'} + p^{n'-1}).$$

If we denote the sum of the expectations by E, and substitute, in succession, for n', 1, 2, 3, 4, &c., in order to find the expectation for each succeeding year up to the last age in the table, we shall have

$$E = \tfrac{1}{2} + p' + p'' + p''' + \cdots \text{ \&c. for } p = 1.$$

Hence the true value of the expectation of life, is equal to $\frac{1}{2}$ plus the sum of the probabilities of the life enduring through 1, 2, 3, &c., years, up to the limiting age of the table of mortality.

The following table shows the expectation of life for every age, from 0 to 100, computed from the Carlisle tables of mortality.

Age.	Expectation.	Age.	Expectation.	Age.	Expectation.	Age.	Expectation.	Age.	Expectation.
1	38.72	21	40.75	41	26.97	61	13.82	81	5.21
2	44.68	22	40.04	42	26.34	62	13.31	82	4.93
3	47.55	23	39.31	43	25.71	63	12.81	83	4.65
4	49.82	24	38.59	44	25.09	64	12.30	84	4.39
5	50.76	25	37.86	45	24.46	65	11.79	85	4.12
6	51.17	26	37.14	46	23.82	66	11.27	86	3.90
7	50.80	27	36.41	47	23.17	67	10.75	87	3.71
8	50.24	28	35.69	48	22.50	68	10.23	88	3.59
9	49.57	29	35.00	49	21.81	69	9.70	89	3.47
10	48.82	30	34.34	50	21.11	70	9.18	90	3.28
11	48.04	31	33.68	51	20.39	71	8.65	91	3.26
12	47.27	32	33.03	52	19.68	72	8.16	92	3.37
13	46.51	33	32.36	53	18.97	73	7.72	93	3.48
14	45.75	34	31.68	54	18.28	74	7.33	94	3.53
15	45.00	35	31.00	55	17.58	75	7.01	95	3.53
16	44.27	36	30.32	56	16.89	76	6.69	96	3.46
17	43.57	37	29.64	57	16.21	77	6.40	97	3.28
18	42.87	38	28.96	58	15.55	78	6.12	98	3.07
19	42.17	39	28.28	59	14.92	79	5.80	99	2.77
20	41.46	40	27.61	60	14.34	80	5.51	100	2.28

EX-PLIC'IT FUNCTION. [L. *explicitus*, from *explico*. to unfold]. A function whose value is expressed directly in terms of the variable: thus, in the equation

$$y = ax^2 + bx^{\frac{1}{2}} + c,$$

y is an explicit function. The term stands opposed to *implicit* function, in which the relation between the function and variable is not directly expressed; as for example, in the equation

$$y^2 - 2px = 0,$$

in which y is an implicit function of x.

EX-PŌ'NENT. [L. *exponens*; *ex*, from, and *pono*, to expose]. In Algebra, a number

written to the right, and above a quantity, to show how many times it is to be taken as a factor: thus, in the expression a^3, the number 3 is an exponent, and shows that a is to be taken three times as a factor. The expression a^3 is equivalent to $a \times a \times a$, and is read, a cube. The exponent is properly the exponent of the power, but for simplicity, it is often called the exponent of the quantity a.

Such is the fundamental idea of the term *exponent;* but custom and the advance of algebraical science have generalized the idea, and the term exponent, is now applied to any quantity written on the right, and above another quantity, whether it be entire or fractional, positive or negative, constant or variable, real or imaginary: thus, in the expressions

$$a^3,\ a^{\frac{1}{3}},\ a^{-5},\ a^b,\ a^y \text{ and } a^{\sqrt{-1}},$$

3, $\frac{1}{3}$, -5, b, x, and $\sqrt{-1}$, are all called exponents. The reason for this generalization of the term lies in the fact that the result obtained by performing the operation indicated by an exponent, is entirely independent of the nature of the exponent. The quantity to which the exponent is annexed is called the base, and when the base is the same, two exponential quantities may be multiplied together by simply adding their exponents; thus,

$$a^{\frac{1}{3}} \times a^x = a^{x+\frac{1}{3}}.$$

One may be divided by the other by subtracting their exponents: thus,

$$\frac{a^y}{a^{\sqrt{-1}}} = a^{y-\sqrt{-1}}.$$

A quantity may be raised to any power by multiplying its exponent by the exponent of the power: thus,

$$(a^{-y})^{-x} = a^{xy}.$$

Any root of a quantity may be extracted by dividing the exponent by the index of the root: thus,

$$\sqrt[3]{a^5} = a^{\frac{5}{3}}.$$

In accordance with these principles, the rules for the transformation of radicals may be reduced to those for simple multiplication and division. The following examples of equivalent expressions show the application of these principles in some of the simplest cases.

$\sqrt[n]{a}$	equivalent to		$a^{\frac{1}{n}}$.
$\sqrt[n]{a^m}$, or $(\sqrt[n]{a})^m$	"	"	$a^{\frac{m}{n}}$.
$\frac{1}{a^n}$ - - - - -	"	"	a^{-n}.
$\frac{1}{\sqrt[n]{a}}$ or $\sqrt[n]{\frac{1}{a}}$	"	"	$a^{-\frac{1}{n}}$.
$\frac{1}{\sqrt[n]{a^m}}$ or $\sqrt[n]{\frac{1}{a^m}}$ or $\sqrt[n]{\left(\frac{1}{a}\right)^m}$			$a^{-\frac{m}{n}}$.

Fractional exponents denote that roots are to be extracted; negative exponents indicate reciprocals.

EX-PO-NEN'TIAL. Involving variable exponents. An exponential function is one in which the variable enters an exponent; thus, in the equations,

$$y = a^x \quad\text{and}\quad y^z - bx^{-x} + c^{x^2} = 0,$$

y is an exponential function of x.

EXPONENTIAL EQUATION. A name given to equations, in which the unknown quantity enters an exponent; thus, $a^x = b$ is an exponential equation. Every exponential equation of the simple form $a^x = b$, may be solved. There are two principle methods of solving exponential equations; *first,* by means of continued fractions; and, *secondly,* by the aid of logarithms.

1st. *By means of continued fractions:*

Let it be required to solve the equation,

$$2^x = 6 \ldots . (1).$$

We see by inspection that $2 < x < 3$; then make

$$x = 2 + \frac{1}{x'}.$$

Substituting this in the given equation, we have

$$2^{2+\frac{1}{x'}} = 6, \quad\text{or}\quad 2^2 \times 2^{\frac{1}{x'}} = 6;\ \text{whence,}$$

$$2^{\frac{1}{x'}} = \frac{3}{2}, \quad\text{or}\quad \left(\frac{3}{2}\right)^{x'} = 2 \ldots . (2).$$

We see by inspecting equation (2) that $1 < x' < 2$; making

$$x' = 1 + \frac{1}{x''},$$

and substituting in equation (2), we have,

$$\left(\frac{3}{2}\right)^{1+\frac{1}{x''}} = 2;\ \text{whence, by reduction,}$$

$$\left(\frac{4}{3}\right)^{x''} = \frac{3}{2} \cdots\cdots (3).$$

We see by inspecting equation (3) that $1 < x'' < 2$; making

$$x'' = 1 + \frac{1}{x'''},$$

and substituting in equation (3), we have, after reduction,

$$\left(\frac{9}{8}\right)^{x'''} = \frac{4}{3} \cdots\cdots (4),$$

from which $2 < x''' < 3$; making

$$x''' = 2 + \frac{1}{x^{iv}};$$

and, proceeding as before, we find

$$2 < x^{iv} < 3, \text{ \&c.}$$

If, now, we substitute these values successively in an inverse order in the preceding equations, we shall find

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \text{\&c.},}}}}$$

a continued fraction.

The first approximating fraction of the fractional part of x is $\frac{1}{1}$; the second, $\frac{1}{2}$; the third, $\frac{3}{5}$; the fourth, $\frac{7}{12}$, which is the true value to within less than $\frac{1}{144}$; hence, the value of x is $2\frac{7}{12}$, to within less than $\frac{1}{144}$. Other examples may be solved in the same manner.

2d. *By means of Logarithms:*

Let us take the same example $2^x = 6 \,.\, (1)$.

Taking the logarithms of both members,

$$x \log 2 = \log 6; \text{ whence,}$$

$$x = \frac{\log 6}{\log 2} = \frac{0.778151}{0.301030} = 2.584,$$

which differs from the value already found by $\frac{1}{1000}$. The latter value is more accurate and more easily obtained.

The advantage of the method by logarithms will be rendered still more apparent by considering a more complicated example.

Let it be required to solve the equation

$$(a)^{b^x} = c.$$

Make $b^x = y$; whence, $a^y = c$; and by the preceding method,

$$y = \frac{\log c}{\log a}, \quad \text{we have also} \quad x = \frac{\log y}{\log b};$$

and, by substitution, we have, finally,

$$x = \frac{\log\left(\frac{\log c}{\log a}\right)}{\log b},$$

which may be reduced by logarithms.

There are again other complicated cases which can only be solved by trial and approximation. The methods already given involve all principles necessary to such examples as arise in practice. There is, also, a method of approximating to the value of x by means of the rule of double position.

In exponential equations of the form $x^x = b$, we can only arrive at the value of x by the method of trial and error, or the method of position. In every equation of the form $x^x = a$, if a is greater than 1, there is but one real value of x, but if a is less than 1, there are at least two values of x, which are functions of each other. If x' is one root, and r be found from the equation

$$r^{\frac{r}{r-1}} = x';$$

then will rx' be another root of the equation. The minimum value of the expression is

$$x^x = \left(\tfrac{1}{e}\right)^{\frac{1}{e}},$$

in which e is the base of the Naperian system of logarithms, or 2.71828.

EX-PRES′SION. In Algebra, the representative of a quantity written in algebraic language; that is, by the aid of symbols. Thus, $9x^2 + 3y$, is the expression of the sum of the two quantities denoted by 9 times x^2, and 3 times y. In general, any quantity or relation denoted by algebraic symbols is an algebraic expression. We say that an equation expresses the relation existing between the quantities which enter it, by which we mean that this relation is written out in the algebraic language. If a rule is translated into algebraic language, the resulting expression is called a formula, and conversely if a formula is translated into common language, the result is a rule. Here the only difference is in the mode of expression, the method of algebra being usually more concise.

There is the same distinction between algebraic expressions that there is between expressions in ordinary language. Thus, a^2 may express the area of a square, one side of which is equal in length to some line denoted by a; this expression is called a *term*, or is the algebraic expression of a term.

Again, the combination of symbols,

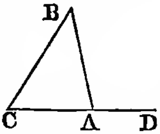
$$a^2 = bc, \quad a^2 > bc, \quad a^2 < bc,$$

may express the fact that a square is equal to, greater than, less than a rectangle, whose adjacent sides are respectively denoted by b and c. Such forms of expression are equivalent to propositions in ordinary language, and in algebraic language are termed equations or inequations. In these cases the terms are a^2 and bc, the copula being the symbol $=$ or $>$, or $<$.

EX-TEN'SION. [L. *extensio*, a stretching out]. In Geometry, that property of a body by virtue of which it occupies a limited portion of space. Extension has three attributes—length, breadth, and height, or thickness, and in order that anything may properly be termed a body, it must possess them all. These dimensions of extension are supposed to be estimated in lines, each of which is perpendicular to the plane of the other two. There are magnitudes which may be regarded as having but two or even one of the attributes of extension. Thus, a surface has length and breadth by no thickness; a line has length, but neither breadth or thickness. A line or surface cannot be called a body, but each has, nevertheless, *magnitude*, for it may be added to, subtracted from, and measured. We are accustomed to speak of them as extending to certain limits. They are, strictly speaking, the limits of bodies, and have no real existence except in the mind.

EX-TE'RIOR. [From *exterus*, foreign]. External, outward.

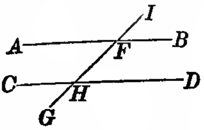
Exterior Angle of a Polygon. The angle included between any side of a polygon, and the prolongation of the adjacent one. Thus, in the triangle ABC, if the side AC be prolonged to D, then is the angle BAD an exterior angle. If all of the sides of a salient polygon be prolonged in the same direction, the sum of the exterior angles formed, is equal to four right angles.



In a triangle, an exterior angle is equal to the sum of the opposite interior angle; thus,

$$BAD = ABC + ACB.$$

If two parallel lines AB and CD are met by a third straight line IG, there will be formed eight angles. Those which lie without the parallel lines, and on the same side of the secant line, are called *exterior angles on the same side;* as IFB and DHG.



Those which lie without the parallels and on opposite sides of the secant, are called *alternate exterior angles*, as AFI and DHG.

EX-TERM-IN-A'TION. [From *ex* and *terminus*, literally to drive from within the limits]. The same as elimination. See *Elimination.*

EX-TERN'AL ANGLES. See *Exterior Angles.*

EX-TRAC'TION OF ROOTS. [L. *extraho*, to draw out]. The operation of finding a quantity, which being taken as a factor a certain number of times, will produce a given quantity. For the processes, see *Square Root, Cube Root, &c.*

Besides the processes laid down under the headings referred to, there is a general method of extracting roots of numbers of any degree depending upon the principles of logarithms. The rule is as follows:

To extract the n^{th} root of a number, find from a table the logarithm of the number and divide it by n, the index of the root; find from a table the number corresponding to the quotient, and this will be the root required.

The following method of extracting the n^{th} root of any number approximately, is due to Hutton. Let N designate any number, and R its nearest n^{th} root, already found: denote the true root by R′; then will the following formula express an approximate value of R′;

$$R' = \frac{(n+1)N + (n-1)R^n}{(n-1)N + (n+1)R^n} \times R.$$

Suppose it were required to extract the cube root of 2. Here, $N = 2$, $n = 3$, $R = 1$; substituting in the formula, we have

$$R' = \frac{8+2}{4+4} = 1.25,$$

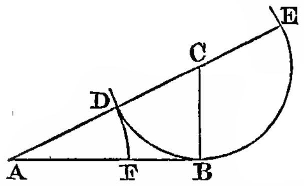
for the first approximation. Using this value and substituting in the formula, we again obtain

$$R = \frac{8 + 2\,(1.25)^3}{4 + 4\,(1.25)^3} \times 1.25 = 1.259921,$$

which is true to the very last figure.

EX-TRĒME' AND MEAN RATIO. In Geometry, a straight line is said to be divided in extreme and mean raatio, when the greater segment is a mean proportional between the whole line and the lesser segment.

The operation is effected geometrically, as follows ·



Let AB be the given line ; at B erect the perpendicular BC and make it equal to one-half of AB ; with C as a centre, and with a radius CB describe an arc BD cutting AC in D; with A as a centre, and with AD as a radius, describe the arc DF ; then will F be the point of division, and we shall have

$$AB : AF :: AF : FB.$$

EX-TRĒME'. [L. *extremus*, last]. The first and last terms of a proportion are called *extremes*, the remaining two being means. If the proportion has but three different terms, the middle one is a geometrical mean, or a *mean proportional* between the extremes. In the proportion

$$a : b :: c . d,$$

a and d are extremes, also in the proportion

$$a : b :: b : c$$

a and c are extremes, and b is a mean proportional between them. In a limited progression, either arithmetical or geometrical, the first and last terms are called *extremes*, and the remaining terms are means—arithmetical means in the first case, and geometrical means in the second.

In an arithmetical progression, the sum of the extremes is equal to the sum of any pair of terms which are situated at equal distances from the extremes. The sum of the progression is equal to the half sum of the extremes multiplied by the number of terms.

In a geometrical progression, the product of the two extremes is equal to the product of any two means equally distant from the extremes.

In a geometrical progression, any term is a mean proportional between the preceding and succeeding term, and if there is an odd number of terms, the middle one is a mean proportional between the extremes.

F, the sixth letter of the English alphabet. As a numeral, it has been employed to denote 40, with a dash over it, thus, $\bar{F}$, it denoted 40,000. In the calculus, it is employed as an abbreviation for the term function. For example, the symbols, $f(x)$, $F(x, y)$, are employed to denote functions of x, and of x and y respectively.

FĀCE. [L. *facies*, the face]. The plane surface of a solid.

FACE OF A POLYHEDRON. One of the bounding polygons of the solid. See *Polyhedron*.

FAC'TOR. [L. *factor*, from *facio*, a maker, or doer]. If two quantities are multiplied together, each is called a *factor*, and the result obtained is called a *product*. The term factor is used also in the same sense as divisor, so that any quantity which will divide another, is a factor of it. The entire factors of 12 are 1, 2, 3, 4, and 6. Taken in pairs, the factors are 1×12, 2×6, 3×4, &c. To resolve a quantity into its factors, is to find two or more quantities, which when multiplied together, will produce the given quantity. This may often be effected in a variety of ways, as in the example above given. When a quantity is given, and also one of its factors, the remaining one may be found by dividing the quantity by the given factor. The prime factors of a quantity, are those factors which cannot be exactly divided by any other quantity except 1. Every member has 1 for a prime factor. The prime factors of 12 are 1, 2, and 3. The operation of resolving a quantity into its factors, is called *factoring*. In the processes of the Diophantine analysis, the theory of numbers, the investigation of the nature and property of equations, in short, in almost every branch of analysis, the operation of factoring is of constant use. A single example in algebraic multiplication will serve to illustrate its utility. Let it be required to find the product of

$$\frac{x^4 - b^4}{x^2 - 2bx + b^2} \text{ and } \frac{x - b}{x^2 + bx};$$

by factoring and indicating the operation, we have

$$\frac{(x^2+b^2)(x+b)(x-b)(x-b)}{x\quad(x+b)(x-b)(x-b)}=\frac{x^2+b^2}{x},$$

by striking out the factors common to the two terms of the fraction. This result might have been obtained by performing the multiplication, and then reducing, but the process would have been much more tedious.

No definite rules can be laid down for factoring algebraic expressions. The following cases will be found useful in practice.

$a-b$ is a factor of a^m-b^m, for all entire values of m.

$a+b$ " a^m-b^m, when m is even.

$a+b$ " a^m+b^m, when m is odd.

$a-1$ " a^m-a^n, for all entire values of m and n.

$a+1$ " a^m-a^n, when $(m-n)$ is even.

$a+1$ " a^m+a^n, when $(m-n)$ is odd.

a is a numerical factor of r^a-r, for all entire numerical values of r, when a is a prime number.

a is a factor of $[1.2.3.4\ldots(a-1)]+1$ when a is a prime number.

a is a numerical factor of $r^{a-1}-1$, when a is prime with respect to r.

Imaginary factors of the second degree can always be reduced to the form

$$a+b\sqrt{-1},$$

and if any real quantity has a factor of the form

$$a+b\sqrt{-1},$$

it has also a factor of the form

$$a-b\sqrt{-1},$$

and these factors, from the fact of their being always united, are called *conjugate factors.*

FĀIL'ING CASE of a rule or formula. A case, in which the rule or formula does not apply, though included amongst those to which it is in general applicable. Thus, McLaurin's formula fails to develop a function of one variable into a series, when that function, or any of its successive differential coefficients becomes infinite, when the variable is made equal to 0.

FEL'LOW-SHIP. A rule of Arithmetic, so called from its being employed in adjusting the accounts of merchants and partners in trade. The object of the rule is, to explain the method of assigning to each partner his gain or loss, in accordance with any agreed conditions,—usually in proportion to the amount of capital contributed by each partner, and the time which it has been employed, or the risks to which it has been subjected.

The entire amount of money employed, is called the *capital stock.* The gain or loss to be shared, is called the *dividend.*

There may be two cases: 1st, when the stock of all the partners has been employed for the same length of time; and 2d, when the different partners put in stock for unequal periods of time. These two cases give rise to the two rules, *Single Fellowship,* and *Double* or *Compound Fellowship.*

Single Fellowship.

In this case, it is plain that the whole stock is to each man's share, as is the whole gain or loss to each man's share of the gain or loss.

1. A and B purchase goods to the amount of \$160, of which A contributes \$90, and B \$70; they gain by the purchase \$32: what is each one's share of the profit?

\$160 : \$90 :: 32 : x = \$18 for A's share.

\$160 : \$70 :: 32 : y = \$14 for B's share.

Double Fellowship.

In this case, each partner's share of the gain or loss is proportional to the product of his stock by the time which it is invested: hence,

Multiply each partner's stock by the time which it continues in trade, and take the sum of the products. Then will the sum of the products be to each product, as the whole loss or gain is to each partner's share of the loss or gain.

1. In an operation, A puts in \$840 for 4 months, and B puts in \$650 for 6 months, and they gain \$300: what is the share of each in the profit?

$840 \times 4 = 3360$
$650 \times 6 = 3900$

7260 : 3360 :: 300 : x
\$138.8418 A's share

7260 : 3900 :: 300 : y
\$161.1582 B's share.

FIĒLD. In Surveying, a tract of ground to be surveyed, usually bounded by some well defined limit, as a fence, hedge, stream, or other boundary.

FIELD-BOOK. In Surveying, a book in which are entered the notes of a survey as they are made on the field. The method of entering the field-notes in the field-book depends upon the nature of the survey, and also upon the surveyor himself, every surveyor having some peculiarity of his own. See *Field-Surveying* and *Leveling*.

FIELD'SUR-VEY-ING. The operation of finding the area of any portion of the earth's surface. The name is generally applied to those operations which are of limited extent, as in finding the number of acres in a farm or in a township. See *Surveying*.

FIG'U-RATE NUMBERS, or *Figurate Series*. [L. *figuratus*]. In Arithmetic, a series of numbers, the general term of each series being

$$\frac{n(n+1)(n+2)\cdots\cdots(n+m)}{1\cdot 2\cdot 3\cdot 4\cdots\cdots(m+1)};$$

in which m determines the nature of the series, and n is dependent upon the place of the required term of the series.

Figurate series are divided into orders: when $m = 0$, the series is of the 1st order; when $m = 1$, the series is of the 2d order; when $m = 2$, the series is of the 3d order; and so on.

The figurate series of the first order, is the series of natural numbers, 1, 2, 3, 4, 5, &c., n, &c.

The figurate series of the second order has, for its general term, $\frac{n(n+1)}{1\cdot 2}$; and the several terms are deduced from this, by making, in succession, n equal to 1, 2, 3, 4, 5, &c. The resulting series is

$$1, 3, 6, 10, 15, 21, 28, 36, \text{\&c.}, \frac{n(n+1)}{1\cdot 2} \text{ \&c.}$$

The numbers of this series are called triangular numbers, because they express the numbers of points which may be arranged in triangles, as in the annexed Figure.


, &c.

The figurate series of the third order has, for its general term,

$$\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3},$$

and the terms of the series may be deduced from it by making, in succession, n equal to 1, 2, 3, 4, &c. The series is

$$1, 4, 10, 20, \text{ \&c. } \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3} \cdot\cdot \text{ \&c.}$$

The figurate series of the fourth order is

$$1, 5, 15, 35, \text{\&c. } \frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4} \cdot\cdot \text{ \&c.}$$

&c. &c. &c.

One of the most remarkable properties of these series is that if the n^{th} term of a series of any given order be added to the $(n+1)^{th}$ term of the series of the preceding order, the sum will be equal to the $(n+1)^{th}$ term of the series of the given order.

Take the two series of the 3d and 4th orders,

1, 4, 10, 20, 35, &c.
1, 5, 15, 35, 70, &c.

Here, if we add to any term in the upper series, that term in the lower series which stands one place to the left, the sum is the next term in the lower series. Starting with a series of 1's, all of the series of figurate numbers may be deduced in succession by the aid of this principle:

ORDERS OF FIGURATE SERIES.

Series of 1's.

1,1, 1, 1, 1, 1, 1, 1, 1 1
1^{st}. 1,2, 3, 4, 5, 6, 7, 8, 9 &c.
2^{d}. 1,3, 6, 10, 15, 21, 28, 36, 45 &c.
3^{d}. 1,4.10, 20, 35, 56, 84, 120, 165 &c.
4^{th}. 1,5,15, 35, 70, 126, 210, 330, 495 &c.
5^{th}. 1,6,21, 56, 126, 252, 462
6^{th}. 1,7,28, 84, 210, 462
7^{th}. 1,8,36, 120, 330
8^{th}. 1,9,45, 165

It will be perceived, on inspecting the preceding table, that the numbers, read diagonally upwards, are the numerical co-efficients of the terms in the developments of $x + a$, with an exponent corresponding to the order of the series. It was this fact which first gave rise to a complete investigation of the subject of figurative numbers.

Other series may be deduced in a manner similar to that employed for deducing the figurate series, which in this respect resemble the latter.

For example: if we assume the arithmetical progression of odd numbers, the figurate series deduced will be the series of squares. Thus,

Assumed series	1, 3, 5, 7, 9, 11, 13, &c.
Deduced series	1, 4, 9, 16, 25, 36, 49, &c. (1)

In like manner,

Assumed series	1, 4, 7, 10, 13, 16, &c.
Deduced series	1, 5, 12, 22, 35. 51, &c. (2)

Also,

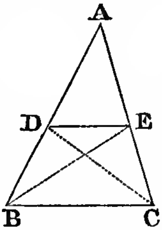
Assumed series	1, 5, 9, 13, 17, 21, &c.
Deduced series	1, 6, 15, 28, 45, 66, &c. (3)

The series (1) is the series of square numbers, (2) the series of pentagonal numbers, (3) of hexagonal numbers, so named on account of certain analogies existing between the relations of the numbers in these series, and the relations existing between the polygons whose names they bear. If points be arranged in squares, pentagons, hexagons, &c., their numbers will be expressed by terms of these series, as explained in triangular numbers.

FIG'URE. In Arithmetic, a character employed in representing numbers. The Arabic figures are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and by proper combination these are sufficient to represent every possible number.

When the scale of tens is adopted for expressing numbers, 10 separate figures are needed, but were we to adopt any other scale, a greater or less number would be required, according to the nature of the scale. In the duodecimal scale, in addition to the ordinary figures employed, the Greek characters ϕ and π are used, the former to denote 11 and the latter 12. See *Notation*.

FIGURE IN GEOMETRY. A diagram or drawing, made to represent a magnitude upon a plane surface. Thus, the diagram ABC is made to represent a triangle, and is called a figure. Sometimes the figure is accurately constructed, so that its parts bear to each other the same relative proportion as the parts of the magnitude represented, but more usually it is only a rough representation, intended to facilitate the reasoning of a proposition or to illustrate the application of mathematical principles in the solution of a problem. When the magnitude has three dimensions, the exact representation of it on a plane is impossible, and in this case several different conventional methods have been resorted to, each of which has its advantages in its own particular sphere of application. Some of the methods of representation are, by means of the principles of perspective, see *Perspective*; of isometrical perspective, see *Isometrical Perspective*; and of projections, see *Projections and Descriptive Geometry*.



Besides these methods, there is a rude conventional method of drawing figures, used generally in plane geometry, which cannot be referred to either of these methods, nor can it be reduced to any rules. A glance at any work on solid geometry will sufficiently show the nature of this kind of representation. It approaches more nearly to orthographic projection than to any other systematic method.

FIGURE OF THE EARTH. The result of numerous measurements of arcs of the meridian and experiments with the pendulum, together with the deductions of analytical mechanics, show that the figure of the earth is that of an oblate spheroid of revolution, the axis of which coincides with the axis of the earth. The terrestrial elements as adopted by the United States corps of Topographical Engineers are as follows:

Equatorial radius 6377397.15 metres, denoted by a.

Polar radius, 6356078.96 metres, denoted by b

Eccentricity of meridian, 0.0816967, denoted by e.

$$\text{Any radius} = a\left(1 - \frac{e^2(1-e^2)\sin^2 L}{1 - e^2 \sin^2 L}\right)^{\frac{1}{2}}$$

in which L denotes the latitude of the place.

The metre employed in these elements is taken equal to 39.36850154 American standard inches as determined by Hassler in 1832. These elements are those at present used upon the coast survey.

Reducing the elements to yards, and employing the same notation as before, we have,

$$a = 6974532.339 \text{ yards.}$$
$$b = 6951218.059 \text{ yards.}$$
$$e = 0.0816967$$
$$E = \frac{1}{299.66};$$

in which E denotes the ellipticity.

The following are the elements of the earth's figure, as given by Bessell and Airy:

Elements.	BESSELL.		AIRY.	
	feet	*miles*	*feet*	*miles*
Eq. radius, a	20923596	=3962.802	20923713	=3962.824
Polar " b	20853662	=3949.557	20853810	=3949.585
Diff. diam. $2a-2b$	139868	= 26.490	139806	= 26.478
Ellipt'y $\frac{a-b}{a}=E$	$\frac{1}{299.15}$		$\frac{1}{299.33}$	

The following terms are employed in the discussion of the figure of the earth:

1. The *axis* is that line about which the earth revolves.

2. The *equator* is a circle whose plane is perpendicular to the polar axis at its middle point. It is the largest plane curve that can be drawn upon the surface of the spheroid.

3. The *sensible horizon* of a place of the earth's surface is a tangent plane to the surface at the place. The rational horizon is a plane parallel to the sensible horizon, and passing through the centre of the earth.

4. The *meridian* of a place is a section cut out by a plane through the place and the axis of the earth. All normals to the surface of the earth intersect the axis. The rational meridian of a place is the intersection of the rational horizon with the meridian plane of the places.

5. The *geographic latitude* of a place is the angle included between the normal to the earth's surface at the place and the plane of the equator. This angle is expressed in degrees, and were the earth a perfect sphere, it would be equal to the angular distance of the place from the equator, measured in the meridian through the place.

The *geocentric latitude* of a place on the surface of the earth, is the angle included between the radius of the earth through the place and the plane of the equator.

The geocentric latitude is always a little less than the geographic latitude. The greatest difference between these at any place on the earth's surface, is at those places whose latitude is 45°, either north or south. Here it is about 11′ 30″ of arc.

Tables are constructed, by means of which either kind of latitude may be converted into the other by simple subtraction.

6. *Parallels of latitude* are small circles whose planes are perpendicular to the axis.

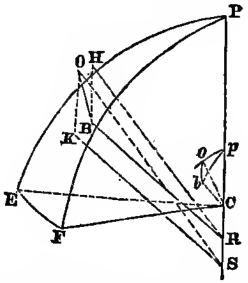
7. The *longitude* of a place is the angle included between the meridian of the place and the meridian of some other place agreed upon as the origin of longitudes. The places from which longitude is reckoned are different in different countries. In England, the longitude is reckoned from the observatory at Greenwich. In France, longitude is reckoned from Paris. In this country it is sometimes reckoned from Greenwich, and sometimes from Washington. It is probable that it will eventually be reckoned from the latter place altogether.

Longitude is reckoned in hours, minutes, and seconds (h., min., sec), or in degrees, minutes, and seconds (°, ′, ″). Each hour of longitude is equivalent to 15° of arc, each minute of longitude in time to 15′ in arc, each second in time to 15″ in arc; so that either may be readily converted into the other. This is usually effected by means of tables for the purpose.

The following principles are extensively used in investigating the figure of the earth, and also in the operations of geodesic surveys:

1. *Comparison of Azimuthal angles on a sphere and spheroid.*

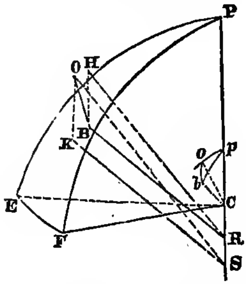
Let C be the centre of the spheroid, CP and CE its polar and equatorial radii, PE and PF any two meridians, B and O two stations,



having given latitudes on these meridians, and sufficiently elevated above the surface to be

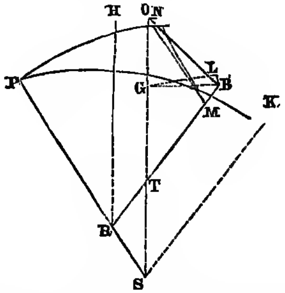
visible, each from the other; OB the straight line joining the two stations; draw the normals BR and OS, meeting the axis in R and S, and from thence draw RH and SK respectively parallel to SO and RB.

Let C be the centre of a sphere on which the points b and o have the same latitudes and the same difference of longitudes, as the points B and O on the spheroid; draw the radii Cb, Co, and the arc bo.



Now the azimuth of the point O as seen from B, is the angle included between the planes OBR and PBR; the azimuth of B, as seen from O, is the angle included between the planes POS and BOS. Let it be required to ascertain in what respect these azimuths differ from the azimuths on the surface of the sphere formed by the inclination of the planes obC, pbC, and poC, boC. Since each of the planes HBR, OKS, is parallel to obC, PBR and pbC are in the same plane, and POS and PoC are also in the same plane; it follows that the inclination of the planes POS, KOS, is equal to the angle pob, and the inclination of the planes PBR, HBR, is equal to the angle pbo; and therefore, the azimuthal angle at O is less than that at o, by the inclination of the planes BOS, KOS, and the azimuthal angle at B greater than that at b by the inclination of the planes OBR, HBR.

If, therefore, the inclination of BOS, KOS, be equal to the inclination of OBR, HBR, the sum of the azimuthal angles at B and O will be equal to the sum of those at b and o. This happens when R coincides with S, that is, when the surface is spherical, or when the latitudes of the two points on the spheroid are equal. It also happens when the reciprocals of the tangents of the depressions at the two stations are equal: for from B draw BG perpendicular to OS, and BL perpendicular to the plane KOS; from O draw OM perpendicular to BR, and ON perpendicular to the plane HBR; join GL and MN.



Then the tangent of the inclination of OBR, HBR, is equal to

$$\frac{ON}{MN},$$

and the tangent of the inclination of BOS, KOS, is

$$\frac{BL}{GL};$$

now, the numerators ON and BL are equal to each other, being the distances between the same parallel planes; let us ascertain under what circumstances the denominators will also be equal. Project the figure on the plane HRB, then the lines MN, GL, will be equal if OT and BT are equal, or when the angle BOT is equal to the angle OBT. If the stations are unequally elevated, the error in the assertion that the inclinations of OBR, HBR, and BOS, KOS are equal, arises from the fact that a perpendicular to the horizon at O does not appear vertical when viewed from B, but the effect is quite insensible, so that for all practical purposes of Geodesy, the sum of the azimuthal angles on the spheroid, may be considered equal to the sum of the azimuthal angles on a sphere, even when the stations are not equally elevated above the surface of the sea.

It has been shown analytically that when m and n represent the number of degrees in the differences of latitude and longitude of the two stations, the excess of the sum of the spheroidal over the sum of the spherical azimuths is less than $.000012'' \times m^2n$, when the stations are equally elevated; and that

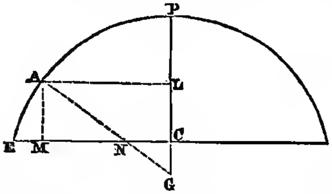
when these are unequal, and p represents the difference of elevation in miles, this excess is less than $.173'' \times \sin^2 A \times \cos^2 L \times p$, in which A denotes the azimuth, and L the middle latitude.

The sums of these differences will be less than the probable errors of observation, and not large enough to be taken into account.

Hence, we conclude, that the spheroidal excess in a spheroidal triangle, is the same as the spherical excess in a spherical triangle whose vertices have the same geographic latitudes, and the same difference of longitude.

Radius of Parallel.

Let CP, CE be the semi-axes of a meridian section of the earth, AN a normal at A, AM and AL perpendiculars from A on the axes.



Denote the semi-axes by a and b, and make $a = b(1+e)$. Then, from the expressions for a sub-normal in an ellipse,

$$MN = CM \times \frac{b^2}{a^2}.$$

Denote the geographic latitude of A by L, equal to the angle ANE. We have

$$AM = NM \tan L = \frac{b^2}{a^2} CM \tan L, \quad \text{and}$$

$$AM^2 = \frac{b^4}{a^4} CM^2 \cdot \tan^2 L; \quad \text{but}$$

$$AM^2 = \frac{b^2}{a^2}(a^2 - CM^2); \text{ hence,}$$

$$\frac{b^2}{a^2}(a^2 - CM^2) = \frac{b^4}{a^4} CM^2 \tan^2 L; \quad \text{whence,}$$

$$a^4 = (a^2 + b^2 \tan^2 L)\, CM^2, \quad \text{or}$$

$$CM^2 = \frac{a^4}{a^2 + b^2 \tan^2 L} = \frac{a^4 \cos^2 L}{a^2 \cos^2 L + b^2 \sin^2 L}.$$

And, by reduction,

$$CM^2 = \frac{a^4 \cos^2 L}{a^2 \cos^2 L + b^2(1 - \cos^2 L)}$$

$$= \frac{a^4 \cos^2 L}{b^2 + (a^2 - b^2)\cos^2 L}; \quad \text{whence,}$$

$$CM = \frac{a^2 \cos L}{\sqrt{b^2 + (a^2 - b^2)\cos^2 L}}.$$

Substituting for a its value $b(1+e)$, we have

$$CM = \frac{b^2(1 + 2e + e^2)\cos^2 L}{b\sqrt{1 + [(1 + 2e + e^2) - 1]\cos^2 L}}$$

$$= b\frac{(1 + 2e + e^2)\cos L}{\sqrt{1 + (2e + e^2)\cos^2 L}}.$$

But e being a small fraction, its square may be neglected in comparison with $2e$. Neglecting e^2, and denoting the radius of the parallel by R, we have

$$R = \frac{b(1+2e)\cos L}{\sqrt{1+2e\cos^2 L}} = b\cos L(1+e+e\sin^2 L)$$

If P denotes the difference of longitude of two places on the same parallel of latitude, and λ denote the length of the parallel between the two places, we shall have

$$\lambda = b\cos L(1+e+e\sin^2 L)\frac{\pi P}{180}.$$

Radius of Curvature of the Meridian.

Retain the same notation as before, and denote the radius of curvature at the point A by ρ. The general formula for the radius of curvature for a plane curve, is

$$R = \frac{-\left[1 + \left(\frac{dy''}{dx''}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y''}{dx''^2}} \quad \ldots . \ (1).$$

The equation of the curve is $a^2y^2 + b^2x^2 = a^2b^2$; whence, by differentiation, we have

$$\frac{dy''}{dx''} = -\frac{b^2x''}{a^2y''}; \quad \text{and} \quad \frac{d^2y''}{dx''^2} = -\frac{b^4}{a^2y''^3}.$$

Substituting and reducing, we have, after omitting the accents,

$$\rho = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4} \quad \ldots (2).$$

But, from what has just been shown, we have

$$CM^2 = x^2 = \frac{a^4\cos^2 L}{b^2\sin^2 L + a^2\cos^2 L}, \quad \text{and}$$

$$AM^2 = y^2 = \frac{b^4}{a^4} CM^2 \tan^2 L$$

$$= \frac{b^4}{a^4} \cdot \frac{a^4\cos^2 L\tan^2 L}{b^2\sin^2 L + a^2\cos^2 L}$$

$$= \frac{b^4\sin^2 L}{b^2\sin^2 L + a^2\cos^2 L}.$$

Substituting these in (2), and reducing, we find

$$\rho = \frac{a^2b^2}{(b^2 \sin^2 L + a^2 \cos^2 L)^{\frac{3}{2}}} \qquad (2').$$

$$= \frac{b(1 + 2e + \varepsilon^2)}{[1 + (2\varepsilon + \varepsilon^2)\cos^2 L]^{\frac{3}{2}}} \cdots (3).$$

If we neglect the square of ε in comparison with 2ε, (3) reduces to

$$\rho = b(1 + 2e - 3\varepsilon \cos^2 L)$$
$$= b(1 - e + 3\varepsilon \sin^2 L). \qquad (4).$$

Length of Normal on Transverse Axis.

Denoting the normal by N, we have

$$N = y\sqrt{1 + \frac{dy''^2}{dx''^2}} \cdots (5).$$

Substituting for $\frac{dy''}{dx''}$ its value deduced above, and in that result the values of x''^2 and y''^2, as just found, and reducing, we have, after reduction,

$$N = \frac{b^2}{\sqrt{b^2 \sin^2 L + a^2 \cos^2 L}} \cdots (6).$$

Normal on Conjugate Axis, or the Radius of Curvature of a Section perpendicular to the Meridian.

If we take the normal on the conjugate axis, and denote its length by ρ', we have

$$\rho' = \frac{CM}{MN} \times N = \frac{a^2}{\sqrt{b^2 \sin^2 L + a^2 \cos^2 L}} \cdot (7).$$

If we denote the eccentricity by ε, that is, make

$$\varepsilon = \frac{\sqrt{a^2 - b^2}}{a}; \quad \text{whence} \quad b^2 = a^2(1 - \varepsilon^2),$$

the preceding formulas may be simplified.

Substituting and reducing, we have from equation (2′), for the radius of curvature of the meridian,

$$\rho = \frac{a(1 - \varepsilon^2)}{(1 - \varepsilon^2 \sin^2 L)^{\frac{3}{2}}} \cdots (8).$$

From equation (6) for the normal on transverse axis,

$$N = \frac{a(1 - \varepsilon^2)}{\sqrt{1 - \varepsilon^2 \sin^2 L}} \cdots\cdots (9)$$

From equation (7), for the normal on conjugate axis,

$$\rho' = \frac{a}{\sqrt{1 - \varepsilon^2 \sin^2 L}} \cdots\cdots (10)$$

To find the Radius of a Spheroid.

Denote the radius through the point A by r; then we shall have

$$r^2 = \overline{AM}^2 + \overline{CM}^2 = \frac{a^4 \cos^2 L + b^4 \sin^2 L}{a^2 \cos^2 L + b^2 \sin^2 L} \quad (11).$$

Substituting for b its value in terms of the eccentricity, we have

$$r^2 = \frac{a^4(\cos^2 L + \sin^2 L) - a^4 \sin^2 L(2\varepsilon^2 - \varepsilon^4)}{a^2(1 - \varepsilon^2 \sin^2 L)}$$

whence, by reduction,

$$r = a\sqrt{\frac{1 - (2\varepsilon^2 - \varepsilon^4)\sin^2 L}{1 - \varepsilon^2 \sin^2 L}} \cdot (12).$$

or,

$$r = a\,[(1 - \tfrac{1}{2}\varepsilon^2 \sin^2 L)$$
$$+ \tfrac{1}{8}\varepsilon^4 \sin^2 L\,(4 - 5 \sin^2 \mathrm{L})\ \&c.] \cdot (13).$$

If we assume

$$\sin\phi = \varepsilon \sin L,$$

whence,

$$\cos\phi = \sqrt{1 - \varepsilon^2 \sin^2 L},$$

we shall have for the radius of the parallel,

$$R = \frac{a \cos L}{\cos\phi} \cdots\cdots\cdots (13)$$

For normal ending at transverse axis,

$$N = \frac{b^2}{a} \cdot \frac{a}{\cos\phi} = \rho'(1 - \varepsilon^2) \cdots (14).$$

For normal ending at conjugate axis, or radius of curvature of a section perpendicular to the meridian,

$$\rho' = \frac{a}{\cos\phi} \cdots\cdots\cdots (15).$$

For radius of curvature of meridian,

$$\rho = \frac{b^2}{a} \cdot \frac{1}{\cos^3\phi} = \frac{\rho'}{a^2}(1 - \varepsilon^2) \cdot (16).$$

For the radius of the spheroid,

$$r = \frac{a}{\cos\phi}\sqrt{1 - (2\varepsilon^2 - \varepsilon^4)\sin^2 L}$$
$$= \rho'\sqrt{1 - (2\varepsilon^2 - \varepsilon^4)\sin^2 L} \qquad (17).$$

For the normal ending at transverse axis,

$$N = \frac{b^2}{a} \cdot \frac{a}{\cos\phi} = \rho'(1 - \varepsilon^2) \cdots (18).$$

In like manner the following formulas may be deduced:

For the tangent to meridian, ending at conjugate axis,

$$t = \rho' \cot L = \frac{a}{\cos\phi} \cot L \cdots (19).$$

For the tangent ending at transverse axis,

$$T = \rho'(1-\varepsilon^2)\tan L = \frac{b^2}{a}\cdot\frac{1}{\cos\phi}\tan L \cdot (20).$$

For the distance on the conjugate axis between the points of intersection of two normals ρ', ρ'', at points whose latitudes are L and L′,

$$\left.\begin{aligned} D &= (\rho''\sin L' - \rho'\sin L)\,\varepsilon^2 \\ &= \left(\frac{\sin L'}{\cos\phi'} - \frac{\sin L}{\cos\phi}\right)a\varepsilon^2 \ldots \end{aligned}\right\} (21)$$

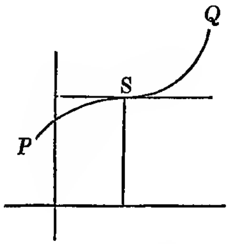
If ρ''' denote the radius of curvature of any section making an angle θ with the meridian, we shall have

$$\rho''' = \frac{\rho\rho'}{\rho\sin^2\theta + \rho'\cos^2\theta} \cdot\cdot\cdot\cdot (22)$$

FI′NĬTE. [L. *finitus*, from *finio*, to finish, and *finis*, a limit]. Having a limit or bound, in contradistinction to infinite, which has no limit. See *Infinite*.

FLEX′URE [L. *flexura*, a bending]. Flexure of a curve, its bending towards or from a straight line.

Flexure, Point of Contrary, or Point of Inflexion. In the analysis of curved lines, a point at which a curve ceases to be concave and becomes convex, or the reverse, with respect to a given straight line not passing through the point.



Thus S is a point of contrary flexure, or point of inflexion, in the curve PSQ. The analytical characteristic of a point of inflexion is, that the second differential co-efficient changes its sign at the point. This affords a means of ascertaining whether a given curve has any points of inflexion.

Differentiate the equation of the curve twice, and from the differential equation and the equation of the curve, find an expression for the second differential co-efficient of the ordinate in terms of the abscissa and constants: place this equal to 0 and ∞, and solve the resulting equations; the roots found, will be all the values of the abscissa that can possibly correspond to points of inflexion. To see which of these roots correspond to points of inflexion, each must be tested separately, as follows: Substitute first, one of the roots plus an infinitely small quantity, then minus an infinitely small quantity for x in the second differential co-efficient of the ordinate; if the results thus obtained have contrary signs, the point corresponding to this value of x will be a point of contrary flexure, or point of inflexion. If the results obtained have the same signs, there will be no corresponding point of inflexion.

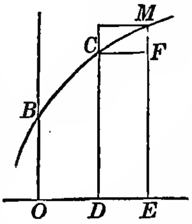
At a point of inflexion, the radius of curvature changes its sign by passing through infinity, as a general thing, though there are cases in which it changes sign by passing through 0.

FLU′ENTS AND FLUX′IONS. [L. *fluo*, to flow]. These two terms are so connected that they can best be defined together. The fluent, or *flowing quantity* as the term signifies, is the same as that which, in the modern calculus, is called the function, and the fluxion is its differential. In this connection, the fluxion indicates the law of increase of the fluent, under the supposition, that the motion which generates the fluent is uniform.

The idea of fluids and fluxions was first presented by Newton, and was based upon the idea of motion. According to his view, a plane curve or line may be conceived as generated by a point moving uniformly in the direction of some fixed line, and having at the same time a lateral motion with respect to this line, which is governed by some law dependent upon the nature of the curve generated. The part of the curve generated at any instant of time, is called the fluent, and that infinitely small element generated during the next infinitely small and constant period of time, is called its fluxion.

Let us suppose that C is the position of the generating point at any instant of time, that DE is the line in the direction of which the motion of C is uniform, and suppose that DE is the distance over which the point C moves in the direction of the line DE, in an infinitely small portion of time; suppose also, during the same space of time, that the

point has a transverse motion at right angles to DE, measured by FM, then will M be the position of the generating point, and CM is the fluxion of the line BM. For an infinitely short space, the line CM may be regarded as a straight line, and is equal to



$$\sqrt{MF^2 + CF^2}.$$

Although the spaces passed over in the direction of DE, in equal small intervals of time are equal, the spaces passed over in lateral direction are unequal in case of curves, and will always depend upon the nature of the curve: conversely, if we know the law of variation of these last spaces, the nature of the curve may be determined. It is upon this fact that the science of fluents and fluxions rest.

If in addition to the two motions already explained, we conceive the point to have a third motion at right angles to the plane of the other two, and also regulated by a law which must depend upon the nature of the curve, the generating point will describe a curve in space which may be either a plane curve, or one of double curvature.

It is easy to conceive, that by suitably regulating the laws of motion of the generating point, any curve whatever may be described; it is also plain, that from the law of relation between the fluxions of the elements, the nature of the curve may be made known.

Recurring to the figure already employed, the line OD is the abscissa of the point C, CD its ordinate, and during the motion each of these varies by infinitely small increments, which are the fluxions of these elements. DE is the fluxion of the abscissa, and is constant, under the supposition made, and MF is the fluxion of the ordinate.

If we suppose the ordinate of the curve to move with the generating point, a plane area will be generated bounded by the curve, the axis of X, and by two ordinates perpendicular to this axis. The infinitely small area included between the two ordinates CD and EM, is the fluxion of this plane area. If we suppose the plane YAX to turn around the axis of X as an axis, the line BM will generate a surface, and the area BCD a volume of revolution; at the same time, the fluxion CM will generate the fluxion of the surface of revolution, and the area DCME will generate the fluxion of the solid of revolution.

In general, if we suppose any line which may vary according to some law, to move according to any fixed law, it will generate a surface, and the portion generated during an infinitely small portion of time, will be the fluxion of the surface, and the portion generated will be the fluent.

In like manner, if a plane arc move in any direction, the area being supposed to vary by a fixed law, then will the infinitely small volume generated in an infinitely small portion of time be the fluxion of the solid, and the portion generated will be the fluent.

In this system, any magnitude may be regarded as flowing ultimately from a point, for the point in its motion may generate any line, a line in its motion may be made to generate any surface, and a surface may be made to generate any volume.

The system of fluents and fluxions was exceedingly ingenious, and for conveying an idea of the nature of the operations of calculus, has had no superior. It has, however, been superseded by the method of integrals and differentials; which method, on account of its wider range, and of its dispensing with the foreign idea of time, has been found more convenient in practice.

The principal advantage of the system of differentials, consists in its more convenient and elegant system of notation.

As the notation of fluxions is often met with in works of science, a short account of it is here appended.

The variables are denoted by the final letters of the alphabet; as, x, y, z, &c., and their fluxions are indicated by the same letters with a dot over them. Thus, $\dot{x}$, $\dot{y}$, and $\dot{z}$, are symbols for the fluxions of x, y, and z. If the fluxions are variable, they may be regarded as fluents, whose fluxions may be taken, and then are denoted by the same letters with two or more dots over them, according to the order of the fluxion. Thus, $\ddot{y}$; $\dddot{y}$, $\ddddot{y}$, &c., denote fluxions of y of the *second, third, fourth, &c., orders.*

If the fluent is a radical, as $\sqrt{x-y}$, its fluxion is denoted by placing the radical in a parenthesis, and writing a dot over it and to the right, as $(\sqrt{x-y})^{\cdot}$. Also, the fluent of a fraction is written in a similar manner; thus, the fluent of $\frac{x}{y}$, is written $\left(\frac{x}{y}\right)^{\cdot}$.

Sometimes the fluxion is indicated by the letter F, and the fluent by the letter f. Thus, $F(\sqrt{x-y})$, is the same as

$$(\sqrt{x-y})^{\cdot} \quad \text{and} \quad F\left(\frac{x^2}{2y}\right), \quad \text{as} \quad \left(\frac{x^2}{2y}\right)^{\cdot}.$$

Also, the expression,

$$f(\dot{x}\sqrt{a+bx^2}) \quad \text{and} \quad f\left(\frac{b\dot{x}}{a+x^2}\right),$$

denote the fluents of

$$\dot{x}\sqrt{a+bx^2} \quad \text{and} \quad \frac{b\dot{x}}{a+x^2}, \quad \text{respectively.}$$

This notation is exceedingly cumbrous, particularly in the higher branches of analysis, and for this reason, principally, the method of fluxions has gone into disuse.

FLUX'ION-A-RY. Pertaining to fluxions, as the fluxionary calculus, or analysis.

Fluxionary Analysis. All operations involving fluxions and fluents. See *Fluents and Fluxions*.

FLUX'ION-IST. One skilled in the Fluxionary Analysis.

FŌ'CAL. Pertaining to the focus. A focal tangent to a conic section is the tangent drawn to the curve at the extremity of the ordinate through the focus.

FŌ'CUS. [L. *focus*, fire, the hearth]. A point in which rays of light meet after deviation by a lens or mirror.

Focus of a Conic Section. A point on the principal axis, such that the double ordinate to the axis through the point shall be equal to the parameter of the curve. The most important property of the focus has been adopted as a suitable definition, though its name might indicate a very different definition. This method of defining elements of a curve by describing some characteristic property of them, is very common in mathematics. Such are the definitions of the parabola, ellipse and hyperbola.

The equation of the conic sections, referred to the principal vertex, and the principal axis, is

$$y^2 = 2px + r^2x^2,$$

in which $2p$ is the parameter of the curve. In the ellipse r^2 is negative, in the hyperbola it is positive, and in the parabola it is 0. In order to find the number and positions of the foci in the conic sections, we have only to substitute p for y in the equation of the curve, and deduce the corresponding values of x; these will be the abscissas of the foci.

Making the substitution, and reducing, we have

$$x^2 + \frac{2px}{r^2} = \frac{p^2}{r^2}; \text{ whence,}$$

$$x = -\frac{p}{r^2} \pm\sqrt{\frac{p^2}{r^2} + \frac{p^2}{r^4}};$$

or $$x = \frac{-p(1 \pm \sqrt{r^2+1})}{r^2}.$$

1. In the parabola, $r^2 = 0$: this reduces the first value of x to ∞, and the second one to $\frac{0}{0}$. The last value is, for $r^2 = 0$, a vanishing fraction. Finding its value by the rule for finding the value of a vanishing fraction, which we do by taking the differentials of both terms, and then making $r^2 = 0$ in th results, we have

$$(x)_{r^2=0} = \left(\frac{-p(1-\sqrt{r^2+1})}{r^2}\right)_{r^2=0}$$

$$= \left(\frac{p\dfrac{rdr}{\sqrt{r^2+1}}}{2rdr}\right)_{r^2=0} = \frac{p}{2}.$$

This shows that the parabola has but one focus at a finite distance, and that it is situated on the axis at a distance from the vertex equal to one-fourth of the parameter.

2. In the ellipse, $r^2 < 0$, and is always numerically less than 1. Both values of x are, in this case, real, and both positive. The half sum of the two values of x is equal to $-\frac{p}{2r^2}$, which is the abscissa of the centre. Hence, in the ellipse, there are two foci, which are equally distant from the centre.

The distance from the centre to either focus is

$$\pm\frac{p}{r^2}\sqrt{1+r^2}.$$

In the ellipse

$$p = +\frac{b^2}{a} \quad \text{and} \quad r^2 = -\frac{b^2}{a^2},$$

which reduces this distance to

$$\pm\sqrt{a^2 - b^2}.$$

3. In the hyperbola $r^2 > 0$; hence, the two values of x are always real, the one being positive, and the other negative. This shows that the hyperbola has two foci on opposite sides of the principal vertex. If we take the half sum of the two values of x, we shall find $-\frac{p}{2r^2}$, which is the abscissa of the centre. Hence, in this case, the foci lie at equal distances from the centre. The distance from the centre to either focus is

$$\pm\frac{p}{r^2}\sqrt{1+r^2};$$

in the hyperbola,

$$p = \frac{b^2}{a} \quad \text{and} \quad r^2 = \frac{b^2}{a^2};$$

hence, this distance becomes

$$\pm\sqrt{a^2 + b^2}.$$

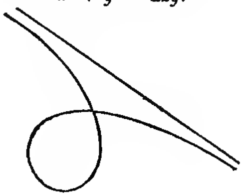
In all the conic sections, the foci possess the remarkable property of being the only points in the plane of the curves, from which the distances to every point of the curve can be expressed rationally in terms of the abscissas of the points.

The name focus was originally given to the points just discussed, on account of the property, that when rays of light proceeding from one focus are reflected from the curve, the reflected rays all pass through the other focus.

In the ellipse, rays proceeding from one focus, and reflected at the curve, pass directly to the other focus. In the parabola, rays proceeding parallel to the axis, and reflected at the curve, pass directly to the focus. In the hyperbola, rays proceeding towards one focus, and reflected at the curve, go to the other focus.

FŌ'LI-ATE CURVE. [L. *foliatus*, leaf-shaped]. A species of curve of the third order, whose general equation is

$$x^3 + y^3 = axy.$$



It consists of two infinite branches which have a common asymptote, and which intersect each other, forming a leaf-shaped branch, —whence the name of the curve.

FOOT. A linear measure whose length differs in different countries. The English foot, and the foot of the United States, is twelve inches in length, the inch being determined as follows:

The length of a simple pendulum, which beats seconds in the Tower of London, is taken as the unit, and an inch is $\frac{1}{39.13908}$ of this; so that an English foot is $\frac{12}{39.13908}$ of the length of a simple seconds pendulum in the Tower of London. This is equal to $\frac{12}{39.10120}$ of a simple seconds pendulum in the City Hall of New York. The length of the seconds pendulum serves in Great Britain and the United States, as the basis of a system of weights and measures. See *Weights and Measures*.

FOOT OF A PERPENDICULAR. In Geometry, the point in which a perpendicular meets the line or plane to which it is drawn.

FŌRE'SĪGHT. In Leveling, any reading of the leveling-rod, after the first, taken at a given station.

The first reading is called a *back-sight*, and serves to connect the observations at the new station with those of the former station.

Any number of foresights may be taken at a single station. In Topographical Surveying, it is customary to take as many in every direction as can be reached by the instruments employed.

The excess of the back-sight, taken at a given station, over any foresight, is equal to the height of the point at which the foresight is taken, over that of the point at which the back-sight is taken. If the back-sight is less than the foresight, the first point is lower than the second.

FORE'STAFF. An instrument formerly used for taking altitudes of the heavenly bodies. It is so named, because the observer turned his face to the object, instead of turning his back, as in the case of the back-staff. Neither of these instruments is now in use.

FORM. [L. *forma*, a form]. In Geometry, the shape of an object. In Analysis, the mode of algebraic expression. Two expressions are said to be of the same form, when

they indicate the same relation between the quantities which enter them. Thus, $(a+b)^n$ and $(c+d)^n$ are of the same form. The form of an expression can often be changed without altering its value; thus,

$$(x+a)^2 = x^2 + 2ax + a^2.$$

In the first member, the form indicates an operation to be performed; in the second, the operation is performed, but the value of the expression is unchanged. A large portion of the operations of analysis consists in changing the form of expressions, without altering their value.

FORM'U-LA. [L. *formula*, a form, or model]. The algebraic expression of a general rule or principle. If a rule or principle be translated into algebraic language, the result is a formula; conversely, if a formula is translated into ordinary language, the result is a rule, or principle.

For example, the equation

$$(a+b)(a-b) = a^2 - b^2,$$

is a formula, being the algebraic expression of the fact, that *the sum of two quantities multiplied by their difference, is equal to the difference of their squares.* This is true, whatever may be the nature of the quantities; that is, the form of the expression does not depend at all upon the nature of the quantities which enter it; hence, the name *formula*.

In Algebra, the binomial formula holds a prominent place on account of its frequent use in all branches of analysis.

In the Differential Calculus, there are three formulas which, from the universality of their application, require particular notice. Indeed, by the aid of one or the other of these formulas, most of the developments of analysis are effected.

1. *McLaurin's Formula.*

The object of McLaurin's formula is, to develop a function of a single variable into a series, arranged according to the ascending powers of that variable, with constant co-efficients. The formula is as follows:

$$f(x) = A + A'x + \frac{A''}{1\cdot 2}x^2 + \frac{A'''}{1\cdot 2\cdot 3}x^3 + \cdots + \frac{A^{n\prime}}{1\cdot 2\cdot 3\cdot n}x^n + \cdots$$

In which the first member is any function of x; A is what that function becomes when x is made equal to 0; A', A'', A''', $\cdots$ &c., $A^{n\prime}$, &c., are respectively what the first, second, third, &c, n^{th}, &c. differential co-efficients of the function become, when x is made equal to 0 in them.

If the function to be developed, or if any of its successive differential co-efficients becomes infinite, when the variable is made equal to 0, the formula fails and the function cannot be developed according to the proposed law. It is to be observed that the formula holds good until the term is reached which is infinite and here the law of the series changes.

2. *Taylor's Formula.*

The object of Taylor's formula is to develop any function of the Algebraic sum of two variables into a series arranged according to the ascending powers of one of the variables with co-efficients which are functions of the other. It is

$$f(x+y) = u + \frac{du}{dx}y + \frac{d^2u}{dx^2\,1\cdot 2}y^2 + \frac{d^3u}{dx^3\cdot 1\cdot 2\cdot 3}y^3 + \cdots + \frac{d^nu}{dx^n\,1\cdot 2\cdot 3\cdots n}x^3 +$$

In which the first member is any function of the sum of two variables; u is what the function becomes when the leading variable is made equal to 0;

$$\frac{du}{dx},\ \frac{d^2u}{dx^2},\ \text{\&c.},\ \frac{d^nu}{dx^n},$$

are the successive differential co-efficients of the first term of the development. If the first term of the development, or any of its successive differential co-efficients become infinite for any particular value of the variable which enters them, the formula fails for that particular value from the term which becomes infinite. The law of the series changes at this term, as explained under McLaurin's formula.

3. *Lagrange's Formula.*

The object of this formula is to develop any function of two variables into a series arranged according to the ascending powers of the two variables. It is as follows:

$$f(x,y) = A + (Bx + B'y) + \frac{1}{1.2}(Cx^2 + 2C'xy + C''y^2) + \frac{1}{1.2.3}(Dx^3 + 3D'x^2y + 3D''xy^2 + D'''y^3) + \text{\&c.}$$

The first member is any function of two variables; A is what that function becomes when both variables are made equal to 0; B and B′ what the partial differential co-efficients of the first order become under the same supposition; C, C', C'' are what the partial differential co-efficients of the second order become under the same supposition; D, D', D'', D''' are what the partial differential co-efficients of the third order become, under the same supposition, and so on.

If the given function, or any of its partial differential co-efficients become infinite, when both variables are made equal to 0, the formula fails; that is, the function cannot, in such a case, be developed according to the proposed law.

In the Integral Calculus there are several formulas which aid in performing the integration of differentials, some of which will be given. We have already, under the head of *Calculus Integral*, given some of the most useful formulas for integration; those now to be given are only used as auxiliary means of reducing differentials to more simple forms.

1. Formula for integrating by parts.

$$\int udv = uv - \int vdu.$$

Wherever $\int vdu$ is more simple than $\int udv$, this can be used with advantage. To apply it we resolve the given differential into two factors, one of which is integral, and which we denote by u; the other is differential, and is denoted by dv. We then differentiate the first and integrate the second, and substitute the results in the formula. The same formula may be again employed to simplify the second term of the second member, and so on.

From this formula result several formulas for simplifying binomial differentials which it will be sufficient to write out, their form indicating sufficiently the manner and circumstances under which they are to be applied.

2. Formula A.

$$\int x^{m-1}\,dx\,(a+bx^n)^p = \frac{x^{m-n}(a+bx^n)^{p+1} - a(m-n)\int x^{m-n-1}dx(a+bx^n)^p}{b\,(pn+m)}$$

3. Formula B.

$$\int x^{m-1}\,dx\,(a+bx^n)^p = \frac{x^m(a+bx^n)^p + pna\int x^{m-1}dx\,(a+bx^n)^{p-1}}{pn+m}$$

4. Formula C.

$$\int x^{-m-1}\,dx\,(a+bx^n)^p = \frac{x^{-m}(a+bx^n)^{p+1} - b(n-m+np)\int x^{-m+n-1}dx(a+bx^n)^p}{-\,am}$$

5. Formula D.

$$\int x^{m-1}\,dx\,(a+bx^n)^{-p} = \frac{x^m(a+bx^n)^{-p+1} - (m+n-np)\int x^{m-1}dx(a+bx^n)^{-p+1}}{an\,(p-1)}$$

6. Formula E.

$$\int\frac{x^q dx}{\sqrt{2cx-x^2}} = -\frac{x^{q-1}\sqrt{2cx-x^2}}{q} + \frac{(2q-1)\,c}{q}\int\frac{x^{q-1}\,dx}{\sqrt{2cx-x^2}}.$$

The following formulas are of use in rectifications and quadratures:

7. Differential of plane arc,

$$dz = \sqrt{dx^2+dy^2}.$$

8. Differential of plane area,

$$ds = ydx.$$

9. Differential of surface of revolution,

$$du = 2\pi y\sqrt{dx^2+dy^2}.$$

10. Differential of solid of revolution,

$$dv = \pi y^2\,dx.$$

See *Calculus*, *Radius of Curvature*, &c., &c.

FRAC′TION. [L. *fractio*, from *frango*, *fractus*, to break]. A collection of equal parts of 1. The term *collection* is technical, and embraces the case in which there is but one part considered. One of the equal parts of the collection is called the fractional unit. For example,

$$\frac{3}{4},\quad \frac{8}{9},\quad \frac{a}{b},\quad .05,\quad \&c.,$$

are fractions. In the first case, the collection consists of 3 parts of 1, each equal to $\frac{1}{4}$; and the fractional unit is $\frac{1}{4}$. In the fraction $\frac{a}{b}$, each of the equal parts collected is equal to $\frac{1}{b}$, and there are a of them collected or taken. In the fraction .05, the fractional unit is $\frac{1}{100}$, or .01.

Fractions are usually divided into two kinds, *Vulgar* and *Decimal*. Vulgar fractions are those in which the denominator is

expressed, and may be any quantity. Decimal fractions are those in which the denominator is not expressed, and is always some power of ten.

I. Vulgar Fractions.

A vulgar fraction is generally written under the form $\frac{a}{b}$, which denotes that the quantity $\frac{1}{b}$ is taken a times. In this case, the fractional unit is $\frac{1}{b}$. The parts a and b are called terms of the fraction; the one lying above the horizontal line is the *numerator*, the one below is the *denominator*. Hence the denominator shows into how many equal parts the unit 1 is divided to form the fractional unit, and the numerator shows how many of these are collected or taken. Vulgar fractions are *proper* and *improper*. A *proper* fraction is one in which the numerator is less than the denominator; an *improper* fraction is one in which the numerator is greater than the denominator. Thus, $\frac{3}{4}$ is a proper fraction, and $\frac{5}{4}$ is an improper fraction.

A *mixed* fraction is an expression composed of two parts, one of which is entire and the other fractional. Thus, $a + \frac{c}{d}$, or $5\frac{1}{3}$, are mixed fractions. Every entire quantity can be reduced to a fractional form, having a given fractional unit, by multiplying it by the denominator of the fractional unit and then writing the result over the denominator. Thus, c may be reduced to the form $\frac{bc}{b}$, $\frac{fc}{f}$, and so on. Conversely, if there is a common factor in both terms of the fraction, it may be stricken out without changing the value of the fraction, since $\frac{bc}{b} = c$.

Upon the following principles depend all the rules for transforming fractions; that is, changing their forms without altering their values:

1. If the numerator a of any fraction $\frac{a}{b}$, be multiplied by any quantity q, the resulting fraction will be q times as great as the given fraction. Hence, multiplying the numerator of a fraction is equivalent to multiplying the fraction by the same quantity. Also, dividing the numerator of a fraction is equivalent to dividing the fraction by the same quantity.

2. Multiplying the denominator of any fraction by a given quantity, is equivalent to dividing the fraction by the same quantity. Also dividing the denominator of a fraction by any quantity, is equivalent to multiplying the fraction by the same quantity.

3. If both terms of a fraction be multiplied or divided by the same quantity, the value of the fraction will be unchanged.

Transformation of Fractions.

The *transformation* of a fraction, is the operation of changing its form, without altering its value.

1. *To reduce a fraction to its simplest form.*

Resolve both terms of the fraction into their simplest factors; then suppress all the factors common to the numerator and denominator, snd the fraction will be in its simplest form. Thus,

$$\frac{3ab + 6ac}{3ad + 12a} = \frac{\cancel{3a}(b + 2c)}{\cancel{3a}(d + 4a)} = \frac{b + 2c}{d + 4a}.$$

Also,

$$\frac{36}{48} = \frac{\cancel{2}.\cancel{2}.\cancel{3}.3}{\cancel{2}.\cancel{2}.\cancel{3}.2.2} = \frac{3}{4}.$$

2. *To reduce a mixed fraction to an equivalent vulgar fraction.*

Multiply the entire part by the denominator of the fractional part, and add to the product the numerator of the fractional part; write the sum over the denominator of the fractional part, and the result will be the equivalent fraction required. Thus,

$$a + \frac{b}{c} = \frac{ac + b}{c}; \text{ also, } 2\tfrac{1}{4} = \frac{9}{4}.$$

The result in the last case is an improper fraction.

3. *To reduce an improper fraction to an equivalent mixed fraction.*

Apply the rule for dividing the numerator by the denominator, and continue the operation till a remainder is found less than the denominator; write this over the denominator and add the fraction thus formed to the quotient obtained; the result will be the equivalent mixed fraction. Thus,

$$\frac{25}{4} = 6\tfrac{1}{4}.$$

When the fraction is expressed in algebraic

language, continue the division as long as possible, and proceed as before. Thus,

$$\frac{a-b}{a} = 1 - \frac{b}{a}.$$

4. *To reduce fractions having different fractional units to equivalent ones, having a common fractional unit.*

Find the least common dividend of all the denominators, this will be the common denominator of the required fractions; divide this dividend by the denominator of each fraction, and multiply the results by the numerators of the several fractions; these products will be the respective numerators of the required fractions. Thus,

$\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{10}$, are respectively equal to $\frac{10}{20}$, $\frac{5}{20}$, and $\frac{2}{20}$.

This operation is called, reducing the fractions to a *common unit*, or *common denominator*.

5. *To convert a decimal fraction into an equivalent vulgar fraction.*

Omit the decimal point, and the resulting number is the numerator of the required fraction, under which write the number 1, followed by as many 0's as there are decimal places, and the result is the denominator of the required fraction. Thus,

$$.0056 = \frac{56}{10000}.$$

The preceding transformations apply to decimal fractions, after they have been converted into vulgar fractions.

6. *To add fractions.*

Reduce them to a common fractional unit (4); add the numerators together, the sum is the numerator of the required sum; write this over the common denominator, and the result is the sum required. Thus,

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}.$$

7. *To subtract one fraction from another.*

Reduce the fractions to a common unit, subtract the numerator of the subtrahend from that of the minuend; the difference is the numerator of the remainder; write this over the common denominator and the result is the difference required. Thus,

$$\frac{5}{7} - \frac{2}{5} = \frac{25}{35} - \frac{14}{35} = \frac{11}{35}, \text{ also,}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd}.$$

8. *To multiply one fraction by another.*

Multiply their numerators together, and the product is the numerator of the required product; multiply the denominators together, and the product is the denominator of the required product. Thus,

$$\frac{5}{7} \times \frac{2}{3} = \frac{10}{21}, \text{ also, } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

9. *To divide one fraction by another.*

Invert the terms of the divisor, and multiply the dividend by the resulting fraction, the product is the required quotient. Thus,

$$\frac{3}{6} \div \frac{2}{9} = \frac{3}{6} \times \frac{9}{2} = \frac{27}{12} = \frac{9}{4} = 2\tfrac{1}{4},$$

$$\text{also, } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

Whole numbers may be regarded as fractions whose denominations are equal to 1, and the above transformations then become applicable to them.

In applying the above rules, it will be found useful to reduce the fractions to their simplest form. Further simplifications may be made by the operation of factoring in certain cases: that is, by striking out common factors which would otherwise enter both terms of the result.

Thus, let it be required to find the continued product of $\frac{2}{9}$, $\frac{3}{8}$, $\frac{3}{10}$: indicating the operation and resolving into factors, we have,

$$\frac{\cancel{2}}{\cancel{3} \times \cancel{3}} \times \frac{\cancel{3}}{\cancel{2} \times 4} \times \frac{\cancel{3}}{10} = \frac{1}{40},$$

and striking out the common factors, 3, 3, and 2, and performing the indicated operation, there results $\frac{1}{40}$.

II. Decimal Fractions.

Decimal fractions may be transformed into equivalent vulgar fractions, and then be treated by the rules already laid down, but it is found more convenient, in most cases, to operate upon them by separate rules. The following are the principles upon which the transformation of decimal fractions depend.

1st. Annexing 0's to a decimal fraction does not alter its value.

2d. Prefixing 0's to a decimal, removes the point to the left, and is equivalent to divid-

ing the fraction by that power of ten whose index is equal to the number of 0's prefixed.

The rules for addition and subtraction, are the same as in whole numbers. The rules for multiplication and division differ only in the method of pointing off the result.

1. *To multiply decimal fractions together.*

Neglecting the consideration of the decimal points, multiply as in whole numbers; point off from the right of the product as many decimal places as there are in both factors together, prefixing 0's if necessary, the result is the product required. Thus,

$$2.5 \times 4.16 = 10.400.$$

2. *To divide one decimal fraction by another.*

Make the number of places of figures after the decimal point the same in both, by annexing 0's to one, if necessary; neglecting the decimal points, divide as in whole numbers, the quotient will be the quotient sought. By this rule the quotient will often appear as a vulgar fraction, which may be converted into a decimal by the usual process, or after having obtained the entire part of the quotient, the decimal part may be found by annexing 0's to the last remainder, and continuing the operation.

The above rule is equivalent to the following:

Annex as many 0's to the dividend as may be necessary; divide as in whole numbers, and point off from the right hand of the result, as many places of decimals as the number of decimal places in the dividend exceeds the number of decimal places in the divisor, prefixing 0's, if necessary, to make the requisite number. Thus,

$$1.38483 \div 60.21 = .023.$$

3. *To convert a vulgar, into an equivalent decimal fraction.*

Annex a sufficient number of 0's to the numerator, considering them decimal places, and divide the result by the denominator; point off in the quotient as in the division of one decimal fraction by another; the result is the equivalent decimal fraction required. Thus,

$$\frac{3}{4} = .75; \text{ also, } \frac{1}{24} = .041\dot{6}.$$

There are two kinds of decimal fractions resulting from the conversion of a vulgar fraction; first, those which can be expressed by means of a limited number of places of decimals, and second, those whose expression would require an infinite number of places.

1. *In the first place*, if the denominator contains, as prime factors, the numbers 2 and 5, and does not contain any other factors, the resulting decimal fraction will be expressed by a finite number of places of decimals. For, let $\frac{a}{b}$ be a vulgar fraction, which we suppose reduced to its simplest form; suppose also $a < b$. Now, if b contains only the powers of 2 and 5, the fraction may be written $\frac{b}{2^m \times 5^n}$. If $m > n$, multiply the numerator by 10^m; or, if $n > m$, multiply the numerator by 10^n, then will the resulting product necessarily be divisible by $2^m \cdot 5^n$, and the resulting decimal will, in the first case, contain m places of figures, and in the second place, it will contain n places of figures; since, multiplying by 10^m or by 10^n, is equivalent to annexing m or n, 0's. Hence, in order to determine the number of places of decimals in a fraction of the kind considered, resolve the denominator into its prime factors 2 and 5; then will the number of decimal places be indicated by the highest exponent of the factors 2 or 5. Either m or n may be 0.

2. *In the second place*, suppose that the denominator of a proper vulgar fraction, reduced to its lowest terms, does not contain either of the factors 2 or 5, or that it contains any other factors:

It is a property of numbers, that if a number divides the product of two given numbers, and does not divide one of them, it must divide the other. Now, by hypothesis, in the given fraction $\frac{a}{b}$, a is prime with respect to b, and is therefore not divisible by it. Now, to annex 0's to a, is equivalent to multiplying it by some power of 10.

Since the only prime factors of 10 are 5 and 2, it follows that no power of 10 can be divided by b; hence, in accordance with the principle above enunciated, the fraction can not be expressed decimally by a finite number of places of figures.

Every vulgar fraction, that cannot be expressed by a finite number of places of figures, gives rise to a circulating decimal.

Let $\frac{a}{b}$ be a vulgar fraction in its lowest terms, in which b is prime with respect to 10, or which does not contain either of the factors 2 or 5. By the rule for converting it into an equivalent decimal fraction, we multiply a by some power of 10, or annex a certain number of 0's, and then divide by the denominator. If we denote the first digit by d', the second by d'', the third by d''', and so on, we shall eventually reach some digit $d^{n\prime}$, exactly equal to some preceding one; and then, since the remainders will be the same as in the preceding case, the following digits will be the same as before, and be repeated in the same order; consequently, the decimal will be *circulating*, as ennunciated.

Those decimal fractions which are expressed by a finite number of places of figures, are called *terminating decimals*.

A *terminating decimal* may be transformed into an equivalent vulgar fraction by the following rule:

Omit the decimal point, and write the decimal fraction for the numerator of the vulgar fraction, and the denominator will be equal to 1, followed by as many 0's as there are places of figures in the decimal fraction. Thus,

$$.25 = \frac{25}{100}; \quad \text{also,} \quad .099 = \frac{99}{1000}; \text{ and so on.}$$

For the method of converting circulating decimals into equivalent vulgar fractions, and also for performing other transformations upon them, see *Circulating Decimals*.

FRACTIONS CONTINUED. See *Continued Fractions*.

FRACTION RATIONAL. A rational fraction, in analysis, is one in which the variable is not affected with any fractional exponents. The co-efficients may be irrational, but that does not prevent the fraction's being considered as a rational fraction. Every rational fraction, which is a function of one variable, may be reduced to the form of

$$\frac{Ax^m + Bx^{m-1} + Cx^{m-2} + \cdots + K}{A'x^n + B'x^{n-1} + \cdots + L'}.$$

If $m > n$, the operation of division may be applied and continued till the highest power of x in the remainder is, at least, one less than in the denominator, when the fraction will take the form

$$X + \frac{A''x^{n-1} + B''x^{n-2} + \cdots + M'}{A'x^n + B'x^{n-1} + \cdots + L'};$$

in which the entire part is a rational function of x, and the remaining part a rational fraction, having its numerator of a lower degree, with respect to x, than the denominator. Every such fraction can be separated into partial fractions; that is, into parts, so that their sum will equal the given fraction. This separation is of great use, in the Integral Calculus, for integrating rational fractions which are differentials of a single variable. We shall point out some of the methods of separating fractions into their component parts.

The theory of separation depends upon the possibility of resolving the denominator into factors of the first degree. We shall suppose this always possible. There will be two cases: *first*, when the factors of the denominators are all real; *second*, when some of them are imaginary.

1. When the factors of the denominators are all real.

Write the given fraction equal to the sum of as many partial fractions as there are units in the highest exponent of the variable in the denominator, the numerators of which are constants to be determined, and the denominators the different powers of the factors of the first degree from the m^{th} to the 1^{st} inclusive, m being the number of times any factor enters; then clear the equation of denominators, and equate the co-efficients of the like powers of the variable in the two members; from these equations find the values of the constants, and substitute them for the constants in the partial fractions: the resulting fractions will be the fractions required. Thus, let it be required to separate the fraction

$$\frac{x^3 + x^2 + 2}{x^5 - 2x^3 + x}$$

into partial fractions. The factors of the denominator are

$$x, \ (x+1)^2 \quad \text{and} \quad (x-1)^2,$$

hence by the rule,

$$\frac{x^3+x^2+2}{x^5-2x^3+x} = \frac{A}{x} + \frac{B}{(x+1)^2} + \frac{C}{x+1} + \frac{D}{(x-1)^2} + \frac{E}{x-1},$$

which cleared of denominators gives the equation,

$$x^3 + x^2 + 2 = A(x^2-1)^2 + Bx(x-1)^2 + Cx(x+1)(x-1)^2 + Dx(x+1)^2 + Ex(x+1)^2(x-1).$$

Developing and equating the co-efficients of like powers of x, we obtain a system of equations from which we find,

$A=2$, $B=-\frac{1}{2}$, $C=-\frac{5}{4}$, $D=1$ and $E=-\frac{3}{4}$, hence,

$$\frac{x^3+x^2+2}{x^5-2x^3+x}=\frac{2}{x}-\frac{1}{2(x+1)^2}-\frac{5}{4(x+1)}$$
$$+\frac{1}{(x-1)^2}-\frac{3}{4(x-1)}.$$

2. When the factors of the denominator are all imaginary, we suppose the denominator to be resolved into factors of the second degree, each of which set equal to 0, will give two imaginary roots. Then write the fraction equal to the sum of as many partial fractions as there are single factors of the second degree in the denominator, the numerators being of the form $Mx+N$, M and N being constants to be determined, and the denominators being the different powers of the factors of the second degree from the m^{th} to the 1st inclusive, m being the number of times any factor is taken. Then proceed as before. Thus,

$$\frac{f(x)}{(x^2+2ax+a^2+b^2)^n(x^2+2cx^2+d^2+f^2}$$
$$=\frac{M+Nx}{(x^2+2ax+a^2+b^2)^n}+\frac{M'+N'x}{(x^2+2ax+a^2+b^2)^{n-1}}$$
$$+\ \&c.+\frac{M^{m'}+N^{m'}x}{x^2+2cx+c^2+f^2}.$$

For the method of integrating rational fractions, see *Integral Calculus.*

The following simple rule serves to separate a given rational fraction into partial fractions, when the denominator can be separated into real factors of the first degree, no two of which are alike.

Let the fraction be $\frac{U}{V}$, in which U and V are entire functions of x.

1st. Let

$$\frac{U}{V}=\frac{N}{x-a}+\frac{P}{Q}$$

to find N; P and Q being entire functions, reducing to common denominator, we have,

$$U=NQ+P(x-a)\ \therefore\ P=\frac{U-NQ}{x-a};$$

since P is an entire function of x, $U-NQ$ must be divisible by $(x-a)$; hence

$$(U)_{x=a}-N(Q)_{x=a}=0,\ \text{or}\ N=\frac{(U)_{x=a}}{(Q)_{x=a}}.$$

Hence, to find the partial fraction corresponding to any factor of the 1st degree of the denominator:

Substitute the corresponding value of x in the numerator of the given fraction, and also in the continued product of all the remaining factors of the denominator. Divide the first result by the second, and this quotient will be the numerator of the required partial fraction, and its denominator will be the binomial factor of the 1st degree.

Example. $\dfrac{2x+3}{x^3-x^2-2x}$;

$x=0,\quad x=2,\quad x=-1$;

$(U)_{x=0}=(2x+3)_{x=0}=3,$

$(Q)_{x=0}=(x^2-x-2)_{x=0}=-2\ \therefore\ N=-\frac{3}{2},$

$(U)_{x=2}=7,\ (Q)_{x=2}=6\ \therefore\ N'=\frac{7}{6}$;

$(U)_{x=-1}=1,\ (Q)_{x=-1}=3\ \therefore\ N=\frac{1}{3}$;

$$\frac{2x+3}{x^3-x^2-2x}=-\frac{3}{2x}+\frac{7}{6(x-2)}+\frac{1}{3(x+1)}.$$

Fractions, Vanishing. A vanishing fraction is one that reduces to the form of $\frac{0}{0}$ for a particular value of the arbitrary quantity which enters it, in consequence of the existence of a common factor in both terms of the fraction, which factor reduces to 0 under the particular supposition made upon the arbitrary quantity. Thus,

$$\frac{x^2-a^2}{x^3-a^3}$$

is a vanishing fraction for $x=a$, in consequence of the existence of the factor $(x-a)$ in both terms of the fraction which reduces to 0 when $x=a$. The fraction may be written

$$\frac{x-a}{x-a}\times\frac{x+a}{x^2+ax+a^2},$$

which, on suppressing or canceling the common factor, and then making $x=a$, reduces to

$$\frac{2a}{3a^2}=\frac{2}{3a};$$

this is the true value of the fraction under the particular hypothesis.

Every vanishing fraction may be considered a particular case of the fraction

$$\frac{M(x-a)^n}{N(x-a)^m},$$

in which M and N are functions of x, and which always takes the form of $\frac{0}{0}$ when $x = a$. Now there may be three cases.

1st. When $m > n$; in this case the fraction takes the form of

$$\frac{M}{N(x-a)^{m-n}},$$

which for $x = a$ becomes ∞.

2d. When $m = n$; in this case the fraction takes the form of $\frac{M}{N}$, which for $x = a$ reduces to

$$\left(\frac{M}{N}\right)_{x=a} = \frac{A}{B}, \text{ a finite quantity.}$$

3d. When $m < n$; in this case the fraction takes the form

$$\frac{M(x-a)^{n-m}}{N},$$

which for $x = a$ reduces to 0.

These are the only cases that can ever arise; hence, the true value of a vanishing fraction for that value of the variable which causes it to reduce to the form $\frac{0}{0}$, is always *infinite*, *finite*, or *zero*.

If a fraction reduces to $\frac{0}{0}$ for a particular value of the arbitrary quantity which enters it, we should first examine carefully whether the result is due to the existence of a common factor in the terms of the fraction which becomes 0, for the particular supposition made, if not, the expression is truly indeterminate; but if so, then the true value may be found by either of the following rules:

1. Seek the common factor which reduces to 0, under the particular supposition, and strike out the highest power of it which is common to both terms of the fraction, after which. make the particular supposition, and the result obtained will he the required value.

2. Substitute for the arbitrary quantity that value of it which reduces the common factor to 0, plus a variable quantity: reduce the result to its simplest form and then make the variable equal to 0; the result will be the true value of the fraction under the particular supposition.

3. Differentiate both numerator and denominator of the fraction with respect to the arbitrary quantity, and in these results make the particular supposition; if both do not reduce to 0 or ∞, what the first becomes divided by what the last becomes, is the true value; if both reduce to 0, find the second differentials of the terms and make the same supposition: continue this operation till two differentials are found of the same order both of which do not reduce to 0 or ∞, for the particular value of the arbitrary quantity in question; then what the first becomes divided by what the last becomes, is the true value of the fraction for the particular value of the arbitrary quantity.

The first and second rules are perfectly general, but cannot always be so easily applied as the third one, which fails in a certain case, viz.: when the exponents of the common factors in both terms are fractional, and lie between two consecutive whole numbers. The reason of this failure is, that by a continued application of the rule, we must at length arrive at two differentials of the same order which both become ∞, for the particular value of the arbitrary quantity, and in all subsequent applications the same results are obtained. In this case, we have to fall back upon the preceding rules.

Let it be required to find the value of

$$\frac{x - x^5}{1 - x}$$

when $x = 1$. By the first rule, we have

$$\frac{x(1-x)(1+x+x^2+x^3)}{1-x},$$

which after striking out the common factor gives

$$x(1+x+x^2+x^3)$$

and this, for $x = 1$, becomes 4.

By the second rule making $x = 1 + h$, we have

$$\frac{1+h-(1+5h+10h^2+10h^3+5h^4+h^5)}{1-(1+h)},$$

and reducing

$$4 + 10h + 10h^2 + 5h^3 + h^4;$$

this, by making $h = 0$, reduces to 4.

By the third rule,

$$d(x-x^5) = dx - 5x^4dx \quad \text{and} \quad d(1-x) = -dx;$$

making $x = 1$, the first becomes $-4dx$, and the second $-dx$, and the quotient obtained is 4. The results of these three rules agree, as they ought.

There are several other expressions, which for particular values of the arbitrary quantity, may be reduced to the form of vanishing

fractions, and treated accordingly. The principle ones are the following:

1st. Let $\frac{p}{q} - \frac{r}{s}$, be an expression in which p, q, r and s, are functions of x, such that for $x = a$ both q and s reduce to 0: then will the expression become $\infty - \infty$. To reduce this to the form of a vanishing fraction, let the fractions be reduced to a common denominator, then will the expression have the form $\frac{ps - rq}{qs}$, which, for $x = a$, becomes $\frac{0}{0}$, a vanishing fraction.

Again, let there be the fraction $\frac{p}{q}$, which for $x = a$, reduces to $\frac{\infty}{\infty}$. To reduce this to the form of a vanishing fraction, let it be placed under the form $\frac{\left(\frac{1}{q}\right)}{\left(\frac{1}{p}\right)}$ which for $x = a$ reduces to $\frac{0}{0}$, a vanishing fraction.

Also, let there be the expression, pq, in which for $x = a$, p reduces to 0, and q to infinity, then to reduce it to the form of a vanishing fraction, let it be written $\frac{p}{\left(\frac{1}{q}\right)}$ which, for $x = a$, becomes $\frac{0}{0}$, a vanishing fraction.

FRAC'TION-AL. Pertaining to fractions; as, fractional numbers, fractional exponents, &c.

FRUS'TUM. [From *frusto*, to break]. A piece or part of a solid separated from the rest.

FRUSTUM OF A PYRAMID OR CONE. In Geometry, the part contained between the base and a plane parallel to the base between it and the vertex. In general, the frustum of any solid body is that portion of the body lying between any two parallel planes which intersect the body. A frustum of a sphere is often called a *segment*. A middle *frustum*, or *segment* of a sphere, is that frustum whose bases, or plane sections, are equal circles.

To find the volume of the frustum of a cone or pyramid, add to the sum of the areas of the upper and lower bases a mean proportional between them, and multiply the sum by one-third of the altitude; the product will be the volume required. Or, denoting the upper base by A, the lower base by B, the altitude by h, and the volume by v,

$$v = \tfrac{1}{3}h(A + B + \sqrt{AB})\ldots$$

The volume of a spherical frustum may be found by adding together the squares of the radii of the parallel bases, $\frac{1}{3}$ of the square of its altitude, and multiplying the sum by $\frac{1}{2}\pi$, the formula is

$$v = (r^2 + r'^2 + \tfrac{1}{3}h^2)\tfrac{1}{2}\pi.$$

In general, the volume of any frustum of a solid of revolution may be found by adding together the area of its two parallel bases, and four times the area of the middle section, and multiplying this sum by one-sixth of the altitude of the frustum. The formula is

$$v = (A + B + 4C) \times \tfrac{1}{6}h,$$

in which A and B denote the areas of the two bases, C the area of the middle section, and h the altitude of the frustum.

FUNC'TION. [L. *functio*, from, *fungor*, to perform]. One quantity is said to be a function of another, when it is so connected with it, that no change can be made in the latter without producing a corresponding change in the former. Thus, in the equation

$$y^2 = R^2 - x^2,$$

y is a function of x, and x is also a function of y, a general relation of dependency of value, that may be expressed by the symbols

$$y = f(x), \text{ or } x = f'(y), \text{ or } \phi(x, y) = 0.$$

These symbols are called functional, the first indicating that y is a function of x, the second that x is a function of y, and the third that x and y are functions of each other.

A quantity is a function of two or more quantities, when it is so connected with them, that no change can be made in either of the latter without producing a corresponding change in the former; thus, in the equation,

$$y^2 = 2x + 3z + b,$$

y is a function of x and z, z is also a function of x and y, and x is a function of z and y, relations that may be expressed thus,

$$y = f(x, z), \quad z = f'(y, x),$$
$$x = f'''(y, z), \quad \text{or} \quad \phi(x, y, z,) = 0.$$

The symbols used to denote functions, are generally the letter f, with dashes, if necessary, and the Greek letters, ϕ, ψ, π, &c.

The term function implies variability, or that two or more quantities vary together in accordance with some mathematical law. All the quantities in an equation, except one,

may vary in any arbitrary manner; the former are called *independent variables*, whilst the term *function* is reserved for the latter one only. In a curve, for example, we generally suppose that the abscissa varies arbitrarily, whilst the ordinate varies with it to correspond with the law expressed by the equation of the curve. In a curved surface, we generally suppose the horizontal co-ordinates to vary arbitrarily, whilst the vertical one varies with them to correspond to the law expressed by the equation of the surface, and so on for other functions.

DIVISION OF FUNCTIONS. Functions are divided into *Algebraic* and *Transcendental.*

ALGEBRAIC FUNCTIONS are those in which the relation between the function and the independent variables can be expressed by the six ordinary operations of algebra; that is, *Addition, Subtraction, Multiplication, Division, raising to powers denoted by constant exponents, and the extraction of roots, indicated by constant indices.*

TRANSCENDENTAL FUNCTIONS are those in which the relation between the function and independent variables cannot be thus expressed. In the expressions

$$y^2 = 2px + \sqrt{yz}, \quad \text{and} \quad y^2 - x^2 = Rz^2,$$

y is an algebraic function of x and z. In the expressions

$$y = a^x, \quad y = \sin^{-1} x, \text{ \&c.},$$

y is a transcendental function of x. Transcendental functions are differently named from the manner of expressing the relation between the function and variable.

LOGARITHMIC FUNCTIONS are those in which the relation is expressed by the aid of logarithms, as

$$y = \log x.$$

EXPONENTIAL FUNCTIONS are those in which the variable enters an exponent, as $y = a^x$.

CIRCULAR FUNCTIONS are those in which the variable enters some trigonometrical element, as

$$y = \sin z, \quad x = \cos^{-1} y, \text{ \&c.}$$

Functions are *explicit* and *implicit*. *Explicit*, when the value of the function is directly expressed in terms of the independent variable, as

$$y = \sqrt{a^2 - x^2}, \quad y = \sin z, \text{ \&c.}$$

Implicit, when the relation is expressed implicitly, or when the functional equation requires solution, in order to show the value of the function in terms of the variables; thus, y is an implicit function of x, in the expressions

$$y^2 + 2xy + x^2 + b = 0, \quad x^2 \sin y = 2ax \cos y.$$

Functions are *direct* and *inverse.* These terms are correlative. Thus, the functions

$$y = a^x \quad \text{and} \quad x = \log y,$$

are direct and inverse with respect to each other. It is customary to consider the former as the direct function, and the latter as its inverse. The following are instances of direct circular functions:

$$y = \sin x, \quad y = \cos x, \quad y = \tan x,$$
$$y = \cot x, \quad y = \text{ver-sin}\, x, \text{ \&c.}$$

The inverse circular functions are

$$x = \sin^{-1} y, \quad x = \cos^{-1} y, \quad x = \tan^{-1} y,$$
$$x = \cot^{-1} y, \quad x = \text{ver-sin}^{-1} y, \text{ \&c.}$$

The entire number of functions is extremely small, the following table comprising all at present known. They are arranged in pairs, each pair being correlative, so that if one be regarded as direct, the other is inverse with respect to it.

$$\text{1st. pair} \begin{cases} u = x + a \ldots\ldots \text{Sum.} \\ x = a - u \ldots\ldots \text{Difference.} \end{cases}$$

$$\text{2d. pair} \begin{cases} u = ax \ldots\ldots\ldots \text{Product.} \\ x = \dfrac{u}{a} \ldots\ldots\ldots \text{Quotient.} \end{cases}$$

$$\text{3d. pair} \begin{cases} u = x^m \ldots\ldots\ldots \text{Algeb. power.} \\ x = \sqrt[m]{u} \ldots\ldots \text{Algebraic root.} \end{cases}$$

$$\text{4th. pair} \begin{cases} u = a^x \ldots\ldots\ldots \text{Exponential.} \\ x = \log u \ldots\ldots \text{Logarithmic.} \end{cases}$$

$$\text{5th. pair} \begin{cases} u = \sin x \ldots\ldots \text{Direct circular.} \\ x = \sin^{-1} u \ldots\ldots \text{Inverse ``} \end{cases}$$

There are certain definite integrals, which, from their constant use, are getting to be considered as elementary functions.

FUNCTION DERIVED. Same as *differential co-efficient*, which see. The correlative term of *derived function*, is *primitive function.* If we regard $2ax^2 + x^3$ as a primitive function of x, then is $4ax + 3x^2$ its derived function, or its first derived function, and $4a + 6x$ is its *second derived function*, and so on. If any function is regarded as a derived function, then the primitive function may be found by integration. In the theory of equations containing but one unknown quantity,

after all of the terms have been transposed to one member, that member taken by itself is a function of the unknown quantity, which then becomes variable, and the successive derived polynomials are nothing else than successive derived functions of it, regarded as a primitive function. Thus, in the general equation

$$x^m + Px^{m-1} + Qx^{m-2} + \cdots + U = 0.$$
$$x^m + Px^{m-1} + Qx^{m-2} + \cdots + U,$$

is the primitive function, and

$$mx^{m-1} + (m-1)\, Px^{m-2} + \&c.,$$
$$m(m-1)x^{m-2} + (m-1)(m-2)Px^{m-3}, \&c. \&c.,$$

are the successive derived functions.

FUNCTION SYMMETRICAL. A symmetrical function of the roots of an equation, is an algebraic expression which contains these roots combined in the same manner, either with each other, or with other quantities. Thus, the sum

$$a + b + c + \cdots + i + l,$$

of the roots of an equation, the sum

$$ab + ac + ad + \cdots + il$$

of their products taken in sets of 2, the sum

$$abc + abd + \&c.,$$

of their products taken in sets of three, are called symmetrical functions of the roots. It is a characteristic property of these functions that any two roots whatever may change places throughout, without changing the value of the function.

Since, in the most general equation containing but a single unknown quantity, we have

$$P = -a - b - c - \&c. \quad Q = ab + ac + \&c.$$
$$R = -abc - abd - \&c. \&c., \quad u = \pm\, abcd\,..$$

it follows that these co-efficients are symmetrical functions of the roots. It is susceptible of demonstration, that every symmetrical function of the roots of an equation can be expressed in terms of these co-efficients without even knowing the roots themselves.

The most simple of the symmetric functions of the roots of an equation, and those from which all others may be formed, are the sums of the like powers of the roots; thus,

$$a + b + c + \cdots + l,$$
$$a^2 + b^2 + c^2 + \cdots + l^2,$$
$$\cdots\; a^n + b^n + c^n + \cdots l^n.$$

To show that these functions may be expressed in terms of the co-efficients P, Q, R, &c., without knowing the value of the roots, let us take the general equation

$$x^m + Px^{m-1} + Qx^{m-2} + \cdots + Tx + U = 0,$$

the roots of which are $a, b, c, d, \cdots k, l$.

If now the first member of the proposed equation be successively divided by the quantities $x - a$, $x - b$, $x - c$, &c., the quotients will take the forms

$$\begin{array}{ll|ll|ll}
x^{m-1} & +a & x^{m-2} & +a^2 & x^{m-3}+\ldots & +a^{m-1} \\
 & +P & & +Pa & & +Pa^{m-2} \\
 & & & +Q & & +Qa^{m-3} \\
 & & & & & +\,. \\
 & & & & & \cdot \\
 & & & & & \cdot \\
x^{m-1} & +b & x^{m-2} & +b^2 & x^{m-3}+\ldots & +b^{m-1} \\
 & +P & & +Pb & & +Pb^{m-2} \\
 & & & +Q & & +Qb^{m-3} \\
 & & & & & +\,. \\
 & & & & & +\,. \\
 & & & & & +\,. \\
\cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & & \cdot \quad \&c. \\
\cdot & \cdot & & \cdot & & \cdot
\end{array}$$

and so on for each factor. If now we take the sum of these quotients, and for simplicity make

$$a + b + c + \cdots + l = S_1;$$
$$a^2 + b^2 + c^2 + \ldots + l^2 = S_2;$$
$$a^3 + b^3 + c^3 + \ldots + l^3 = S_3; \quad \&c.$$
$$a^n + b^n + c^n + \ldots + l^n = S_n;$$

we shall evidently obtain for the sum,

$$\begin{array}{ll|ll|ll|ll}
mx^{m-1} & +S_1 & x^{m-2} & +S_2 & x^{m-3} & +S_3 & x^{m-4}+\ldots & +S_{m-1} \\
 & +mP & & +PS_1 & & +PS_2 & \cdot & +PS_{m-2} \\
 & & & +Q & & +QS_1 & \cdot & +QS_{m-3} \\
 & & & & & +R & \cdot & +RS_{m-4} \\
 & & & & & & & \cdot \\
 & & & & & & & \cdot \\
 & & & & & & & +mT
\end{array}$$

Now the first derived polynomial of the first member of the given equation is equal to the sum of the quotients first obtained, and since this derived polynomial is equal to

$mx^{m-1}+(m-1)Px^{m-2}+(m-2)Qx^{m-3}+\ldots+T$

we shall have, by equating the co-efficients of the like powers of x, in these identical expressions,

$$\left.\begin{array}{lll} S_1+mP=(m-1)P & \text{or} & S_1+P=0 \\ S_2+PS_1+mQ=(m-2)Q & \text{or} & S_2+PS_1+2Q=0 \\ S_3+PS_2+QS_1+mR=(m-3)R & \text{or} & S_3+PS_2+QS_1+3R=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ S_{m-1}+PS_{m-2}+QS_{m-3}+\ldots mT=T & \text{or} & S_{m-1}+PS_{m-2}+QS_{m-3}+\ldots+(m-1)T=0 \end{array}\right\}(1)$$

The first of these formulas makes known the value of S_1 in terms of P; the second makes known the value of S_2 in terms of P, Q and S; the third makes known the value of S_3 in terms of P, Q, R, S_1 and S_2, and so on in succession the values of S_4, S_5, &c., may be determined up to S_{m-1}. To extend these formulas to cover the case of the sum of any powers whatever of the roots, let us substitute a, b, c, &c., for x, in the general equation, giving the results

$$a^m+Pa^{m-1}+Qa^{m-2}+\text{\&c.}+Ta+U=0.$$
$$b^m+Pb^{m-1}+Qb^{m-2}+\text{\&c.}+Tb+U=0.$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Multiplying both members of these equations respectively by a^n, b^n, c^n, &c., and adding them member to member, we find

$$S_{m+n}+PS_{m+n-1}+QS_{m+n-2}+\ldots+TS_{n+1}+US_n=0.$$

In this result make $n=0$; since

$$S_n=a^0+b^0+c^0\ \text{\&c.}=m,$$

$$\left.\begin{array}{lllll} \text{it reduces to} & S_m+PS_{m-1}+QS_{m-2}+ & \ldots & +TS_1+mU=0 \\ \text{for } n=1\ , & S_{m+1}+PS_m+QS_{m-1}+ & \ldots & +TS_2+US_1=0 \\ \text{for } n=2\ , & S_{m+2}+PS_{m+1}+QS_m+ & \ldots & +TS_3+US_2=0 \\ \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot & & \cdot\quad\cdot\quad\cdot \end{array}\right\}(2)$$

This group of formulas is immediately connected with the group already deduced, and their use is the same. By inspecting these formulas, we see that the sums of the first m powers being known, the sums of the following powers are consecutive terms of a recurring series, whose scale is the m co-efficients P, Q, R, &c., of the proposed equation, with their signs changed.

The same formulas are applicable to the summation of negative powers; for making $n=-1$, we have

$$S_{m-1}+PS_{m-2}+QS_{m-3}+.+TS_0+US_{-1}=0,$$

from which we can find the value of S_{-1} in terms of S_{m-1}, S_{m-2}, &c., S_1, S.

In like manner, making $n=-2$, $n=-3$, $n=-4$, &c., we can find formulas from which the values of S_{-2}, S_{-3}, S_{-4}, &c., may be determined in succession.

From the preceding discussion we conclude that any equation being given, we may, without knowing its roots, always find the sum of their similar powers of any degree, either positive or negative.

As an example, let us take the equation

$$x^4+x^3-7x^2-x+6=0,$$

in which

$$P=1,\quad Q=-7,\quad R=-1,\quad U=6.$$

By applying the preceding formulas, we find

$$S_1=-P=-1$$
$$S_2=-PS_1-2Q=1+14=15$$
$$S_3=-PS_2-QS_1-3R=-15-7+3=-19$$
$$S_4=-PS_3-QS_2-RS_1-4U=19+105-1-24=99$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$S_{-1}=\frac{-S_3-PS_2-QS_1-RS_0}{U}=\frac{19-15-7+4}{6}=\frac{1}{6}$$
$$S_{-2}=\frac{-S_2-PS_1-QS_0-RS_{-1}}{U}=\frac{-15+1+28+\frac{1}{6}}{6}=\frac{85}{36}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

and so on. The symmetrical functions considered have been supposed rational and entire, and no others will be discussed.

Symmetrical functions are distinguished as those of *one*, *two*, *three*, &c., letters. Those already discussed, in which but a single root enters each term, are called symmetrical functions of one letter. Those in which two roots enter each term, are called symmetrical functions of two letters, as

$$a^n b^p + a^n c^p + a^n d^p + \&c.$$

Those in which three roots enter each term, are called symmetrical functions of three letters, as

$$a^n b^p c^q + a^n b^p d^q + a^n c^p d^q + \&c.$$

and so on. The method of forming symmetrical functions of two letters is to take all the arrangements of the roots in sets of two, and affect the letters of each product with the respective exponents, n and p. To form the symmetrical functions of three letters, form all the arrangements of the m roots in sets of three, and give to the letters, respectively, the exponents n, p, and q, and so on. The number of terms in a symmetrical function of two letters is $m(m-1)$; the number of terms in one of three letters is $m(m-1)(m-2)$; the number of terms in one of four letters is $m(m-1)(m-2)(m-3)$; and so on, the law being apparent.

For the purpose of simplifying the notation, we represent a symmetrical function by writing a single term preceded by the symbol Σ, which stands for *algebraic sum*.

Thus, $a^n + b^n + c^n + \&c.$, would be written $\Sigma(a^n)$; $a^nb^p + a^nc^p + a^nd^p + \&c.$, would be written $\Sigma(a^nb^p)$; $a^nb^pc^q + a^nb^pd^q + b^nc^pd^q + \&c.$, would be written $\Sigma(a^nb^pc^q)$; and, in like manner, other functions are expressed. To show how to find the values of symmetrical functions of two or more letters in terms of the co-efficients of the equation, let us take the equations

$$a^n + b^n + c^n + \&c. = \Sigma(a^n),$$

and $$a^p + b^p + c^p + \&c. = \Sigma(a^p).$$

Multiplying them, member by member, the second member becomes, $\Sigma(a^n) \times \Sigma(a^p)$. As to the first, two cases may arise: 1st, the terms of the multiplicand and multiplier may have the same letter; in which case, the partial product is one of the terms of the function $\Sigma(a^{n+p})$; 2d, the factors of a partial product may have different letters; in which case, the product is a term of the function $\Sigma(a^nb^p)$. The first member then becomes, $\Sigma(a^{n+p}) + \Sigma(a^nb^p)$.

Hence,

$$\Sigma(a^{n+p}) + \Sigma(a^nb^p) = \Sigma(a^n) \times \Sigma(b^p);$$

whence,

$$\Sigma(a^nb^p) = \Sigma(a^n) \times \Sigma(b^p) - \Sigma(a^{n+p}) \quad (3);$$

or, returning to the notation first used,

$$\Sigma(a^nb^p) = S_nS_p - S_{n+p} \quad (4).$$

The functions, S_n, S_p, and S_{n+p}, being made known by formulas (1) and (2), we are enabled to find the value of $\Sigma(a^nb^p)$ in terms of P, Q, R, &c.

If, in formula (4), we suppose $n = p$, the second member reduces to $(S_n)^2 - S_{2n}$. To find what becomes of the first member, we must recollect that

$$\Sigma(a^nb^p) = a^nb^p + a^nc^p + a^nd^p + \cdots$$
$$+ b^na^p + \cdots + c^na^p + \cdots;$$

and when $n = p$, the terms of this expression become equal, two and two; that is, $\Sigma(a^nb^p) = 2\Sigma(a^nb^n)$; in which the number of terms in the function, instead of being equal to the number of arrangements of m letters, two in a set, is equal to the number of combinations of m letters, taken two in a set, or to $\dfrac{m(m-1)}{2}$.

Substituting, we have finally, in the case where $n = p$,

$$\Sigma(a^nb^n) = \frac{(S_n)^2 - S_{2n}}{2} \quad (5).$$

Again, let us seek to find the value of $\Sigma(a^nb^pc^q)$ in terms of P, Q, R, &c. We have the equations

$$a^nb^p + a^nc^p + a^nd^p + \cdots + b^na^p + \cdots$$
$$= \Sigma(a^nb^p); \quad \text{and}$$

$$a^q + b^q + c^q + \cdots = \Sigma(a^q).$$

Multiplying these, member by member, we find, for the second member,

$$\Sigma(a^nb^p) \times \Sigma(a^q);$$

and, for the first member, we get three species of terms, viz.: terms of the form $a^{n+q}b$, a^nb^{p+q}, and $a^nb^pc^q$; hence, we have

$$\Sigma(a^{n+q}b) + \Sigma(a^nb^{p+q}) + \Sigma(a^nb^pc^q)$$
$$= \Sigma(a^nb^p) \times \Sigma(a^q);$$

or transposing, substituting the values of $\Sigma(a^nb^p)$, $\Sigma(a^{n+q}b^p)$, and $\Sigma(a^nb^{p+q})$, taken

from formula (4), and returning to the primitive notation, we have

$$\Sigma(a^n b^p c^q) = S_n S_p S_q - S_{n+p} S_q - S_{n+q} S_p - S_{p+q} S_n + 2S_{n+p+q} \qquad (6).$$

If $p = q$, we shall have, by the same course of reasoning as before,

$$\Sigma(a^n b^p c^p) = \frac{S_n (S_p)^2 - 2S_{n+p} S_p - S_n S_{2p} + 2S_{n+2p}}{2} \quad (7).$$

If $p = q = n$, we shall have, by making the necessary reductions,

$$\Sigma(a^n b^n c^n) = \frac{(S_n)^3 - 3S_{2n} S_n + 2S_{3n}}{2 \cdot 3} \quad (8).$$

This operation of seeking the values of symmetrical functions of any number of letters, may be continued as already indicated, to any extent; hence, we may affirm, that it is always possible to express the value of any symmetrical function of the roots of an equation in terms of the co-efficients alone, without knowing the values of the roots.

The principle of symmetrical functions has been extensively applied in analysis, both in developing the general properties of equations, and in elimination. We shall only indicate the method of application in each case.

Let it be proposed to find an equation, whose roots are equal to the sum of those of the given equation, taken two and two.

Let a, b, c, &c., be the roots of the given equation; then will $a + b$, $a + c \ldots b + c$, $b + d$, &c., be those of the required equation, and their number will be the number of different combinations of m letters, taken in sets of two; that is, the degree of the resulting equation will be indicated by the number

$$\frac{m(m-1)}{2}.$$

From the rule for the composition of equations we shall have, for the co-efficient of the second term,

$$-(a + b) - (a + c) - (a + d) \ldots ;$$

an expression in which a, b, c, &c., all enter in the same manner, and which is consequently a symmetrical function of these roots, and may be computed by the preceding formulas. In like manner, the co-efficient of the third term,

$$(a + b)(a + c) + (a + b)(b + c) + \text{\&c.},$$

is a symmetrical function, and may be computed by previous formulas. The same conclusion may be arrived at in regard to the several remaining co-efficients of the new equation. Hence we may, from the formulas already given, find the co-efficients of the new equation in terms of those of the given equation, and consequently may find the new equation.

Again, let it be required to find an equation such that its roots shall be equal to the squares of the differences of the roots of a given equation. If the degree of the proposed equation is indicated by m, that of the required equation will be indicated by

$$m(m - 1);$$

moreover, it contains only the even powers of x; if, therefore, we make

$$x^2 = z \quad \text{and} \quad \frac{m(m-1)}{2} = n,$$

the form of the required equation will be

$$z^n + P'z^{n-1} + Q'z^{n-2} + \cdots T'z + U' = 0 \ldots (1).$$

The roots of this equation are

$$(a - b)^2, \ (a - c)^2, \ (b - c)^2, \ \text{\&c.}$$

From the properties of equations, we deduce

$$P' = -(a-b)^2 - (a-c)^2 \ldots\ldots (b-c)^2 -$$
$$Q' = +(a-b)^2 \times (a-c)^2 + (a-b)^2 (b-c)^2 +$$
$$R' = -(a-b)^2 \times (a-c)^2 \times (b-c)^2 - \text{\&c.}$$

All the expressions being symmetrical functions of a, b, c, &c., the values of P', Q', R', &c., may be found by the aid of the formulas already given; consequently, the equation required becomes known.

In like manner, equations may be deduced whose roots are of the form

$a + b + kab$, $a + c + kac$, $a + d + kad$, &c.

k being any quantity.

The following method of applying the principle of symmetrical functions to the operation of elimination, we translate from Bourdon's great work on Algebra.

The following theorem is due to Bezout: "The degree of the final equation resulting from the elimination of one of the unknown quantities from two equations containing two unknown quantities of any degree whatever, can never be greater than the product of the numbers expressing the degrees of the two equations; and it is just equal to that product, when the proposed equations are the most general of their respective degrees."

Before developing the demonstration, it is necessary to make known the form of a complete equation of the m^{th} degree between two unknown quantities.

Every such equation is of the form

$$Px^m + Qx^{m-1} + Rx^{m-2} + \cdots + Tx + U = 0,$$

in which P is a known quantity.

Q	is a polynomial of the first degree in y;	as,	$by + b'$
R	" " " second " " "		$cy^2 + c'y + c''$
&c.	&c.		&c.
T	" " " $(m-1)^{\text{th}}$ " " "		$ty^{m-1} + t'y^{m-2} + \cdots$
U	" " " m^{th} " " "		$uy^m + uy^{m-1} +$ &c.

Let there be two equations of this form,

$$\left.\begin{array}{l} Px^m + Qx^{m-1} + Rx^{m-2} + \cdot + Tx + U = 0 \; (1) \\ P'x^n + Q'x^{n-1} + R'x^{n-2} + \cdot + T'x + U' = 0 \; (2) \end{array}\right\}.$$

Let us consider the first of these equations, solved with respect to x, although we have not the means of effecting the operation, and suppose that it gives the m roots a, b, c, &c.; a, b, c, &c., being functions of y. Each of these m values of x substituted for x in equation (1) ought to satisfy it, whatever the value attributed to y after the substitution. If, now, we substitute these values for x in the first member of equation (2), we obtain the expressions,

$$P'a^n + Q'a^{n-1} + \text{\&c.},\; P'b^n + Q'b^{n-1}, \text{\&c.},$$
$$P'c^n + Q'c^{n-1} + \text{\&c. \&c. \&c.},$$

which are, in general, irrational functions of y. Now any value of y, as $y = \beta$, which reduces one of these expressions to 0, is a compatible value. Suppose, for example, that $y = \beta$ renders the expression

$$P'a^n + Q'a^{n-1} + \text{\&c.}, 0.$$

Denote by a what a becomes, when for y in it we substitute β; the set of values a and β evidently satisfy equation (1). Since, as we have already seen, $x = a$ verifies that equation, whatever value may be attributed to y. In the second place, $y = \beta$ makes the function,

$$P'a^n + Q'a^{n-1} + \text{\&c.}$$

equal to 0, or, what is the same thing, satisfies equation (2), after having substituted a for x. Hence, $x = a$ and $y = \beta$ are a set of compatible values. *Conversely*, every compatible value of y ought to render one of the above functions zero, for in order that it may be compatible, it must satisfy the two equations, at the same time with a certain value of x. Now, all compatible values of x are comprised amongst the values

$$x = a, \quad x = b, \quad x = c, \text{ \&c.},$$

and when

$$x = a, \quad x = b, \text{ \&c.}$$

the first member of equation (2) reduces to one of the above functions, and ought consequently to be satisfied for the value $y = \beta$. We conclude, therefore, that if we equate with 0, the product of all the above functions, the resulting equation will be the same as the final equation that would be obtained by eliminating x between the two given equations. This final equation is, therefore,

$$(P'a^n + Q'a^{n-1} + \text{\&c.}) \times$$
$$(P'b^n + Q'b^{n-1} + \text{\&c.})\,(\ldots)\,.. = 0 \ldots (3).$$

The formation of this equation seems to depend upon the complete solution of equation (1), but in reality that operation is not necessary. We remark that equation (3) does not change, whatever changes we may make between the quantities a, b, c, &c.; hence, the first member is a symmetric function of the roots of equation (1), when solved with reference to x. This first member may therefore be expressed by means of the coefficients P, Q, R, &c., of equation (1), and it is possible to form equation (3) without solving equation (1).

The operation is simple enough when the equations are of the second degree, but when of a higher degree, it becomes tedious. In the latter case, however, it has the advantage of conducting to the true final equations, without introducing any foreign factor.

In order to determine the degree of the final equation, it is sufficient to consider that of any term whatever. Now, every term of the product is formed, by multiplying a term of the first factor by a term of the second by a term of the third. &c. Let

$$Ka^h,\; K'b^{h'},\; K''c^{h''}, \text{ \&c.},$$

be terms taken at random, one from each of the m factors; the corresponding term in the product is

$$KK'K'' \ldots \times a^h b^{h'} c^{h''} \ldots,$$

and the total product is symmetrical with reference to a, b, c, &c.; therefore this term

makes a part of the symmetrical functions which enter into the composition of the first member of equation (3), and this partial function may itself be represented by

$$KK'K'' \ldots \times \Sigma(a^h b^{h'} c^{h''} \ldots).$$

It is sufficient to determine the highest power of y in this function. If we recall the composition of the formulas, which give S_1, S_2, S_3, &c., and if we regard the exponents of y, in the co-efficients P, Q, R, &c., of equation (1), we shall see that S_1, S_2, S_3, are of the 1st., 2d., 3d., &c., degrees, with reference to y. Hence, the product

$$S_1 \times S_2 \times S_3 \ldots$$

is of a degree denoted by

$$h + h' + h'' + \cdots,$$

and from the formulas which give the value of any symmetrical function whatever,

$$\Sigma(a^h b^{h'} c^{h''} \ldots),$$

is also of a degree expressed by

$$h + h' + h'' + \cdots,$$

and cannot exceed it. On the other hand, let k, k', k'', ..., &c., be the exponents of y in the co-efficients K, K', K'', &c., the sum of the exponents of the product

$$KK'K'' \ldots \text{ is } k + k' + k'', \text{ \&c.}$$

Thus, in the function,

$$KK'K'' \ldots \Sigma(a^h b^{h'} c^{h''} \ldots),$$

the sum of the exponent is

$$k + k' + k'' + \cdots + h + h' + h'' + \cdots;$$

but from the composition of equation (2), we have, at most,

$$k + h = n, \quad k' + h' = n,$$
$$k'' + h'' = n, \text{ \&c.};$$

hence, the symmetrical function considered, is at most of the degree expressed by

$$n + n + n + \cdot \cdot \quad = mn.$$

FUR'LONG. A unit of linear measure, equal to the eighth part of a mile, or 660 English feet.

G. The seventh letter of the English alphabet. As a numeral, it has been used to represent 400, and with a dash over it, $\bar{G}$, it represented 400,000.

GĀUG'ING. In Mensuration, the operation of finding the contents of casks, barrels, vats, &c. The operation depends upon the same rules as the other operations of mensuration, but on account of the great frequency with which these measurements have to be made, a set of technical rules and instruments have been arranged for determining the volumes approximately. The instrument generally used is called a gauging rod, by means of which the contents of a cask are inferred from the diagonal distance from the bung to the extremity of the opposite stave at the head. The gauging rod has a square section, and on one face is marked a scale of inches for measuring the diagonal distance, and on the opposite face, a scale expressing the corresponding contents in gallons. The results obtained by the gauging rod are very rough, but sufficiently accurate for the ordinary liquids of commerce.

The following rule for obtaining an approximate expression for the contents of any cask is given by Hutton.

Add together 39 times the square of the bung diameter, 25 times the square of the head diameter, and 26 times the product of these diameters; multiply this sum by the length of the cask, and divide the product by 114: this quotient divided by 231 will give the contents in wine gallons, and by 282 will give it in ale gallons.

The following are the formulas for the two kinds of gallons.

$$wg = (39b^2 + 25h^2 + 26bh)\frac{l}{114 \times 231};$$

$$ag = (39b^2 + 25h^2 + 26bh)\frac{l}{114 \times 282}.$$

GEN'E-RANT. [L. *genero*, to produce]. Anything generated by a generatrix, moved according to a mathematical law.

GEN-ER-Ā'-TION. [L. from *genero*, to produce]. The formation of any magnitude by the motion of a point, or a magnitude of an inferior order. Thus, if a point move in accordance with any mathematical law, the path which it traces out is said to be *generated* by the point, and is a mathematical line. If a mathematical line be moved in accordance with a mathematical law, the surface in which it is always found is said to be generated by the line, and is always a mathematical surface. If a mathematical surface be moved according to a mathematical law, the volume swept over by it, in its motion, is said to be generated by it, and is a mathematical

solid or volume. If, of two straight lines coinciding with each other, the one be revolved about some fixed point of the other, remaining in the same plane, the indefinite space swept over by that part of the moving line, lying on either side of the fixed point, is an angle, and is said to be generated by the revolving line. Thus we have a complete idea of the mathematical theory of the origin of all mathematical magnitudes.

The moving point or magnitude, is called the *generatrix*, the magnitude generated, is called the *generant*, and the law according to which the motion takes place, is called the *law of generation*. We subjoin an instance of each of the principal modes of generation.

If a point move in the same plane, in such a manner that the sum of its distances from two fixed points of that plane shall be equal to a given line, the curve generated will be an *ellipse*.

If a straight line be moved in such a manner as to touch a given curve, not in its own plane, and continue parallel to a given straight line, the surface generated will be that of a *cylinder*.

If a semicircle be revolved about its diameter, as an axis, the solid or volume generated, is a *sphere*.

In general, the *generant* is measured by the generatrix, multiplied by the path described or generated by its centre of gravity.

GEN'ER-Ā-TRIX. That which generates a line, surface or solid. See *Generation.*

GEN'E-SIS. A term formerly used, meaning the same as generation (see *Generation*). In the genesis of figures, the moving magnitude or point, is called the *describent;* the guiding line of the motion is called the *dirigent.*

Generation and Genesis, are terms sometimes used in fluxions, signifying the production of the fluent according to a particular law.

GEN'ER-AL TERM of a series. That term from which any term whatever may be deduced, by assigning proper values to the arbitrary constants which enter it. Thus, in the binomial formula, the general term is,

$$\frac{m\,.\,(m-1)\,(m-2)\,.\,.\,.\,(m-n+1)}{1\ .\ 2\ \ \ 3\ .\,.\,.\,.\ \ \ n}\,a^{m-n}x^{n},$$

and may be made to represent the 4th term by making $n = 3$, which gives,

$$\frac{m\,(m-1)\,(m-2)}{1\ .\ 2\ .\ 3}\,a^{m-3}x^{3},$$

and, so for any other term that might be required.

It will be perceived, that the general term is the one which has n terms preceding it. This is true in nearly every series.

GE-O-CEN'TRIC. [Gr. γη, the earth, and κεντρον, centre]. Having the same centre as the earth. The geocentric latitude of a place, is the angle made by the radius of the earth through the place and the plane of the equator.

GE-O-DES'IC, OR, GE-O-DET'IC. Pertaining to geodesy. See *Geodesy.*

GE-OD'E-SY. [Gr. γη, the earth, and δαιω, to divide]. That branch of Surveying in which the curvature of the earth is taken into account. This becomes necessary in all extensive operations, such as the survey of a state, or of a long line of coast, as in the United States Coast Survey.

The general operations of geodesic surveying are conducted on the supposition that the form of the earth's surface is that of an oblate spheroid of revolution, the shorter axis coinciding with the polar diameter: setting aside the comparatively minute irregularities of surface, repeated measurements have shown that this supposition is sufficiently accurate for all practical purposes.

The multiplicity of observations to be taken, the numerous corrections to be made, the nice calculations to be performed, together with the great practical sagacity required in the observer, serve to render a geodesic survey, the most difficult, as well as the most delicate, of all the operations of applied mathematics.

To attempt anything more than a mere outline of the operations of such a work, would far exceed the limits of a single article; we shall therefore confine our attention to a brief synopsis of the principles employed, referring the reader, who desires a more detailed account, to the voluminous works relating to the subject, by AIRY, KATER, PUISSANT, FRANCOUER, FISCHER, &c.

Preliminary Reconnoissance.—The first step in a geodesic survey consists in making a *preliminary reconnoissance* of the country to

be surveyed. The object of this reconnoissance is to acquire a general knowledge of the great natural features of the country, the direction and extent of its coasts, its mountain ranges and its water courses, to select proper stations for trigonometrical points, and to decide upon the most suitable locality for the measurement of a *base line.* Upon the proper location of the base line, and the judicious selection of trigonometrical points, depends, in a great measure, the success of subsequent operations. The base line is generally selected on a smooth surface, as near the level of the sea as possible, and so that its two extremities shall conform to the general conditions imposed upon trigonometrical points. The triangulation points are to be chosen in conformity with the following considerations: when united by straight lines, the triangles formed should be *well conditioned*, that is, their angle should be neither too acute, nor too obtuse; a rigorous analysis has shown that there is less probability of error, when all of the angles of each triangle are equal, and it may be laid down as a rule, that no triangle should, except under extraordinary circumstances, be admitted, any of whose angles are less than 30°; the successive triangles formed, starting from the base line, should gradually increase in size until the lengths of their sides reach the maximum limit, fixed by the extent of the survey, or by the distance of distinct vision; the several triangulation points should be selected so that from each, as many of the remaining ones as possible may be distinctly visible, and that without the necessity of raising artificial structures.

Signals.—The triangulation points having been selected, are to be *marked by signals.* To the variety, character, and form of signals, there is no limit; they depend upon the nature of the country, the distance between stations, the accuracy of the proposed survey, and a great variety of other circumstances. The object, in all cases, is to mark the exact locality of the point at which they are erected, and at the same time to admit of the exact placing of the instrument over the centre of the station when required.

The simplest signal consists of a simple staff, often painted of different colors, and surmounted by a flag, or more commonly by a frustum of a tin cone. Such signals are visible at considerable distances.

When their height is considerable, they may be braced by pieces driven obliquely into the ground and nailed to the body of the mast.

When larger signals are required, they are to be constructed of frame-work of timber, or scantling, thoroughly braced so as to stand firm against the winds and storms; these usually terminate at the top in an apex, which is directly over the centre of the station, and are sometimes arranged in such manner, that the instrument for measuring angles can be planted upon a platform several feet above the ground. The manner of forming signals must depend, in a great measure, upon the locality, the facilities for obtaining materials, and upon the peculiar views of the surveyor. No definite rules can be given.

Various expedients have been resorted to for rendering distant signals visible, some of which are as follows:

The first and most common is that already alluded to, of fastening to the top of the signal a frustum of a tin cone; this serves to reflect the rays of light in all directions, and experience has shown that they answer a useful purpose. Another method consists in reflecting the light from a mirror, so that the reflected rays shall proceed directly from the station observed to the observer. The particular apparatus by which this is effected, is called a *heliotrope;* this obviously requires an assistant at the station to be observed, who is called a heliotroper. The heliotroper, on a given signal from the observer, so directs his instrument, that the rays of light falling upon the heliotrope, may enter the telescope of the observer.

Another contrivance, by means of which extremely distant stations are rendered visible, consists in heating a piece of quick lime before the oxy-hydrogen blowpipe. This affords a brilliant light, which has been seen 60 to 90 miles, and even at considerably greater distances.

Parabolic reflectors, similar to those used in light-houses, have also been employed with advantage under certain circumstances.

Measurement of base line. The base line is the principal line of the survey, to which all the sides of the triangles are referred,

and upon the accurate measurement of which much of the final accuracy of the work is dependent. Different instruments have been employed at different times for measuring base lines. *Deal rods* were early employed, but were soon laid aside, in consequence of their great liability to sudden changes in length, due principally to thermometric and hygrometric changes of the atmosphere. However, for ordinary purposes, deal rods saturated with boiling oil, and thickly coated with varnish, are found to answer very well. When they are employed they should be capped at their ends with metallic cases, to prevent wearing, and to assume a more perfect contact.

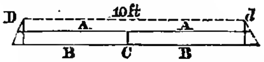
In some of the early English surveys, rods of glass, furnished with caps of bell metal, were used. It was found that hollow glass tubes were less likely to sudden contraction and expansion than solid glass rods.

Steel chains have also been used, and are preferable to glass rods. The chains are of a peculiar construction, and when used have to be stretched, by means of weights, to a certain degree of tension. The chain, with the stretching weights attached, is made to rest upon deal trestles till a perfect level is insured, and coincidence of the ends of the measures is only made when the detached thermometers disposed along its length indicate a uniform temperature throughout. The reading of the thermometer in each case gives the means of correcting the measured length by reducing it to a given standard at a given temperature.

Some French engineers made use of a combination of rods of platina and brass, in accordance with a suggestion made by Borda. In this combination the mercurial thermometer is dispensed with, the combination itself acting as a sort of metallic thermometer. It was found by experiment, that for every degree of the centigrade thermometer, the expansion of platina was 0.000008565 of the entire length, whilst that of brass was 0.000017843 of the length. A rod of platina 12.78 feet in length, was overlaid by a rod of brass about 6 inches shorter, and both were firmly riveted together at one extremity only. The other extremities being left free, the different changes of temperature were rendered manifest by the difference of expansion in the two metals. This difference was measured by a delicate vernier. It was found that changes of temperature considerably less than a degree could be detected and noted.

More recently, compensating rods have been introduced, and the state of perfection to which an apparatus of this kind has been carried by Prof. BACHE, the able head of the United States' Coast Survey, will probably do away with all other methods, where great accuracy is required. The principle of the compensating base apparatus consists in combining rods of different metals so that by their expansion the length of the combined rod shall be as much increased by the expansion of one metallic rod as it is diminished by that of another.

The following description of a compensating rod used by Col. Colby in measuring a base of between 7 and 8 miles in Ireland, will serve to illustrate the principle which has been more completely developed in this country. It may not be amiss to state that in the long base of more than 7 miles, above referred to, "the greatest possible error is supposed not to exceed 2 inches." The apparatus used is constructed as follows: "Two bars, one of iron, the other of brass, 10 feet long, were placed parallel to one another, and riveted together at their centres, it having been previously ascertained by numerous experiments, that they expanded or contracted in their transition from heat to cold, and the reverse, in the proportion of 3 to 5. The brass bar was coated with some non-conducting substance, to equalize the susceptibility of the two metals to change of temperature. Across each extremity of these combined bars was fixed a tongue of iron, with a minute dot of platina, almost invisible to the naked eye, and so situated on this tongue, that under every change of contraction or expansion, the dots at each extremity always remained at the constant distance of 10 feet.



Let A be the iron bar, the expansion of which is represented by 3 ; B the brass bar, the expansion of which is 5, the two being riveted together at C ; D and *d* are two iron tongues pinned on the bar, so as to admit of their expansion, with the platina dots D and

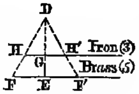
d. The tongues are, by construction, made perpendicular to the rods at a mean temperature of 60° Fahr., and the expansion taking place from their common centre, when A expands any quantity which may be expressed by 3; B expands at the same time a quantity equal to 5, and the inclination of the tongue is changed, the dots D and *d* remaining unalterably fixed at the distance of 10 feet."

The distance between the bars being given, the position of the dots D and *d* is found by the following proportion:

$$DG : GH :: DE - DG : FE - HG.$$

If GE is equal to 4 inches, we shall have

$$DG : 3 :: 4 : 2 \therefore DG = 6 \text{ in.}$$



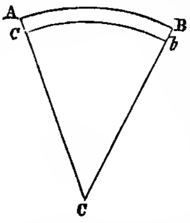
From the construction of these bars, the dots on two bars cannot be brought to coincidence. The distance between them, when the rods are placed, is measured by a micrometrical arrangement.

Enough has been said to indicate the general methods of measuring base lines. The tendency towards accumulation of error, is so great, even in trigonometrical operations of limited extent, that too much care cannot be taken in the measurement of the fundamental line of a survey, and particularly when that survey is to extend over hundreds of miles of territory.

The base line is first carefully ranged, and the measurements made as above indicated, the various corrections for want of contact, &c., are next applied, and we have the horizontal distance between the stations at the two extremes of the base line.

If the base is not on the level of the sea, it must be reduced to what it would have been, had it been measured on that level, as follows:

Let R denote the radius of the earth at the level of the sea, that is, the radius of curvature of the earth at the middle latitude of the base line; R + *h* the radius at the elevation of the measured base; A the length of the measured base regarded as an arc of a circle; *a* the length of the reduced base *ab*. From the figure,



$$R + h : R :: A : a \therefore a = \frac{R \cdot A}{R + h};$$

whence the reduction,

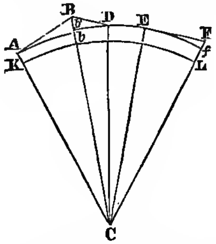
$$A - a = A - \frac{R \cdot A}{R + h};$$

or by reduction,

$$A - a = \frac{A \cdot h}{R + h}.$$

The reduction to the level of the sea should be made to coincide with that of mean low water mark.

It may happen that a base line cannot be measured on ground that is continuously horizontal; in which case the line may be measured along the inclined line of the surface, and then reduced to the horizontal by computation.



Let C be the centre of the earth, and ABDEF the measured base, of which different portions are at different elevations, and variously inclined to the horizon. The lines A*b*, *b*D, DE, E*f*, are horizontal. To reduce one of the lines, as AB, for example, to the sea level, let the difference of level between A and B be ascertained by the ordinary process of leveling; then we shall have

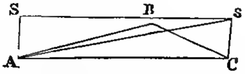
$$Ab = \sqrt{AB^2 - Bb^2},$$

and each result may be reduced to the sea level by the process already explained. Having reduced each section in this manner, the sum of the reduced levels will make up the required length, K*b*L, of the base. If the angles which the several sections make with the horizon are known, the first reduction may be made by the trigonometrical formula for finding the base of a right angled triangle from the hypothenuse and angle at the base; thus,

$$Ab = AB \cos BAb.$$

When parts of the measured base are in different vertical planes, as may sometimes

occur when obstacles intervene between the two extreme points, or when the extremities are unfitted for stations.



Let ABC be the measured base, S and s stations near its extremities; the angles ABC, BAS, or CAS and ACs, or BCs, are measured with extreme care. In the triangle ABC we have two sides, and the included angle, hence we may compute AC and the remaining angles. In the triangle ACs, we have two sides and their included angle, hence we may compute As. In the triangle ASs we have two sides, and their included angle, hence we may compute Ss, the required base, by the ordinary formulas.

These latter corrections should be avoided, except in cases of extreme necessity.

Test bases, or bases of verification, are measured at a part of the survey, remote from the beginning, for the purpose of testing the accuracy of the work. After a long succession of triangles has been completed, the last side of the last triangle may be measured with all the accuracy of the original base line; its length is also computed, and by comparing the measured and computed lengths, we may decide as to the accuracy of the entire work. See *Base* and *Base* Apparatus.

Triangulation.

The next step after measuring the base, is the triangulation. This consists in measuring with great care each angle of the triangles formed by joining the several stations by lines.

The angles are generally measured with a theodolite, and, where great accuracy is desired, instruments are constructed for the purpose much more accurate than ordinary theodolites. The instrument used in the primary triangulation of the coast survey, has a limb of 30 inches diameter, and is graduated with the nicest care. Each station is occupied for a sufficient length of time to ensure a complete set of observations upon all of the visible signals, which in extensive works often requires many weeks. The measurement of each angle is repeated a great number of times, and upon all parts of the graduated limbs, in order to neutralize both the errors incident to the construction of the instruments employed, and those due to observation.

The first class of errors arise from want of accuracy in graduating the limb, from defect of centering, from the impossibility of making the axis of the limb exactly at right angles to the plane of the limb, together with the constantly varying changes in the masses of metal of which the instrument is composed, due to the ever-changing fluctuations of heat and cold. No human skill can provide instruments which are not more or less subject to these errors, and it is therefore left to the observer so to combine his observations, choose his opportunities, and so to familiarize himself with all the causes of instrumental derangement. as to eliminate that which is erroneous, and retain only that which is true. The perfection of modern instruments has reduced the instrumental errors to very narrow limits. The second class of errors, or those of observation, arise from inexpertness, defective vision, atmospheric indistinctness, and irregular refraction, to which may be added momentary instrumental derangement, slips in clamping, looseness of screws, &c. In all cases, the greatest vigilance is requisite in the detection and correction of errors, and it is only from a long succession of observations that anything like perfection in the measurement of an angle can be arrived at.

Besides the measurement of the angles of the chain of triangles thrown over the country, observations are made at the principal stations for determining their latitude and longitude, and also for determining the azimuth or bearing of the sides of the triangles. These data serve to correct the work. and aid in making the plots of the whole on paper.

When permanent objects, as towers, spires, or trees. are employed for signals, as is sometimes the case, it is not convenient to place the axis of the instrument exactly over the centre of the station; in this case, an auxiliary station is assumed as near as practicable to the main point, and such angles measured as will enable the observer to determine what the angles would be, were the instrument properly placed.

After the principal triangulation is com-

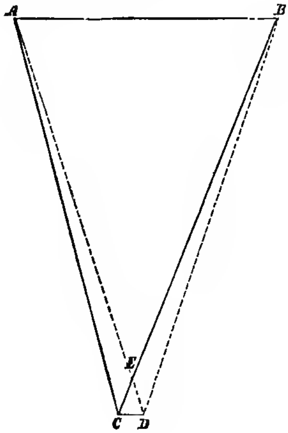
pleted, the main triangles, whose sides are often from 30 to 80 or 90 miles in length, are broken up into smaller triangles, and these again may be subdivided into still smaller ones till the final filling in is brought within the limits of plane surveying, by the aid of the plane table or the compass. These complete what are called the field operations of a geodesic survey.

We come next to the corrections, computations, and final reduction of the whole to a map or draft. Before leaving the subject of field operations, however, it may not be inappropriate to make a single remark, as to the preservation of monuments to mark the principal points, so that they may be found should they be needed at any subsequent time.

Under each extremity of the base line, and under each principal station, a permanent monument of stone, or pottery, should be buried, at a sufficient depth to escape accident. Upon this monument a plate of gold or silver should be fixed, and the exact axis of the station marked upon this plate by a small point of platina. The bearings of surrounding stations should be taken and carefully noted on the records of the survey.

Reduction to the Centre.

The first correction to be made upon the observation is called the reduction to the centre. This reduction is necessary when the instrument cannot be exactly placed over the station, as in the case where a tower is employed as a signal.



Let C be a triangulation point, from which it is required to measure the angle ACB, and suppose D to be the nearest convenient place for the instrument. We measure carefully the distance CD, the angle CDA, and the angle CDB. If the angles CAB and CBA have been measured, we may compute the approximate lengths of AC and BC, but if both of these angles are not known, we may use the measured angle ADB for the angle ACB, and with one of the angles A or B, and the side AB, we may compute AC and BC. It is plain, that since CD is very small in comparison with the sides of the triangles that no great error can arise in making that change for the purpose required.

Now, from the figure we have the following relations between the angles:

$$CED = CAD + ACB \ldots (1),$$
$$CED = CBD + ADB \ldots (2);$$

whence, by equating the second members, and transposing,

$$ACB = ADB + (CBD - CAD) \ldots (3).$$

We have also from the property of sines of angles

$$\sin CBD : \sin BDC :: CD : BC \ldots (4),$$
$$\therefore \sin CBD = \sin BDC \frac{CD}{BC},$$
$$\sin CAD : \sin ADC :: CD : AC \ldots (5),$$
$$\therefore \sin CAD = \sin ADC \frac{CD}{AC};$$

and, because the angles CBD and CAD are exceedingly small, never being over a few seconds in magnitude, it follows that the angles may be taken for their sines, and the reverse; hence, substituting in (3), we have

$$ACB = ADB + \left(\sin BDC \cdot \frac{CD}{BC} - \sin ADC \frac{CD}{AC}\right).$$

The quantity within the parenthesis is called the correction, and may be computed, as all the elements entering the expression are known. The correction will be expressed in linear units, and denoting it by c, it may be reduced to seconds of arc by the proportion,

$$3.1416 : 648000 :: c : n,$$

n being the number of seconds in the correction. The correction will be positive when the first term in the parenthesis is greater than the second, and will be negative when the second term is the greater. If greater

accuracy is required, the operation may be repeated, using the angle just found for the angle at C.

Correction for Oblique Illumination.

The next correction is for oblique illumination. When the light falls obliquely upon the observed signal, the signal appears to be differently situated from what it does when the light is direct. This idea is better illustrated by considering the case in which a tin cone is used. If the light falls upon the cone obliquely, the brilliant element seen by the observer is not in the line from him to the axis of the signal, and a correction has to be made in the observed angle, which can be effected by the following formula:

$$C = \pm \frac{r \cos^2 \frac{1}{2} Z}{D \sin 1''},$$

in which C denotes the correction, r the radius of the signal, Z the angle at the point of observation, and D the distance to the signal.

The circumstances of the case will make known which of the two signs of the second member is to be used.

Correction for Spherical Excess.

The measured angles are really spherical angles, or rather spheroidal angles, and the triangles of the survey are spheroidal; hence, it generally follows that the sum of the three measured angles of a triangle exceeds 180°: this excess is called *the spherical excess*. The principle object in determining the spherical excess in any case, is to arrive at an idea of the accuracy of the measured angles. The method of making the application is as follows:

Legendre has demonstrated that the area of a spherical triangle, which is small in comparison with the whole sphere, is equivalent to a plane triangle, whose sides are equal to the sides of the spherical triangle, and whose angles are equal to those of the spherical triangle, each diminished by one-third of the spherical excess. Taking, then, the sum of the measured angles of a triangle, and deducting 180° from it, we have the measured excess. Subtracting one-third of this from each measured angle, we have the approximate angles of the equivalent plane triangle, whose area may then be computed. Knowing the approximate area of the spherical triangle, its true spherical excess may be computed by the formula,

$$E = \frac{S}{r^2 \sin 1''} = \frac{ab \sin C}{2r^2 \sin 1''} \cdots$$

In which S denotes the area of the triangle, r the radius of the earth, a and b two sides of the triangle, and C their included angle. See *Excess Spherical.*

Having the true spherical excess, it may be compared with the measured excess, and their difference is the error due to measurement. When this error exceeds 3″, it is customary in the coast survey to reject the work and repeat the observations. To such a state of accuracy has that great work been brought, that few cases of re-measurement ever occur.

Besides these corrections, some others have to be made, depending upon local circumstances, which need not be described. The correction being made, the lengths of the sides of the several triangles may be accurately computed, beginning at those having one side coinciding with the base line, and so on to the most remote, the whole being checked by suitable test bases.

The latitudes and longitudes of stations are computed as well as the azimuths. The secondary and tertiary triangles are in like manner computed, and the whole is then ready for projection upon the maps.

The method of projecting the map, plotting in the triangulation, and filling in the details of the map, will be found under the several heads of *Plotting*, *Projections*, *Plane Surveying*, &c.

GE-O-DET′IC LINE, on the surface of an ellipsoid, is the shortest line that can be drawn between two points on the surface. It is a characteristic property of this line that at every point of the curve, its curvature is less than that of any other curve of the surface through that point; that is, its radius of curvature at every point is greater than the radius of curvature of any other curve of the surface through that point.

The geodetic line, then, has no curvature in the direction of a tangent plane to the surface at any point, except in the direction of the surface. The shortest line that can be drawn on any surface, whatever, is of the same general character. On the surface of

the cone, cylinder, or other developable surface, the curve is such that if the surface be developed, the curve will develop into a straight line.

GE-O-GRAPH′IC LATITUDE of a place on the earth's surface. The angle included between the normal to the surface at the point, and the plane of the equator. See *Figure of the Earth.*

GE-OM′E-TER. [Gr. *γεωμετρης;* from *γη*, the earth, and *μετρον*, measure]. One skilled in Geometry, a geometrician.

GE-OM′E-TRAL. Pertaining to Geometry. See *Geometrical.*

GE-O-MET′RIC-AL. Something relating to Geometry. Thus, a geometrical construction is the operation of drawing a figure, by means of right lines and circles. The geometrical construction of an algebraic expression consists in drawing a figure such, that each of its parts shall have its representative in the expression, and that the relation between them shall be the same as that between their representatives in the given expression.

Geometrical Curve. Same as *Algebraic Curve,*—which see. It is so called, because its ordinates can, in general, be constructed by the aid of the right line and circle.

Geometrical Locus. The curve or surface in which a point or line is always found moving, in accordance with an algebraic law. See *Locus.*

Geometrical Progression. A progression, or series, in which each term is derived from the preceding, by multiplying it by a constant quantity, called the ratio. See *Progression.*

Geometrical Solution. A solution of a problem effected geometrically; that is, by the aid of the right line and circle. This rejects all solutions made by aid of the higher curves, or by approximation.

GE-OM′E-TRY. [Gr. *γεωμετρια;* from *γη*, the earth, and *μετρον*, measure]. That branch of Mathematics which has for its object the investigation of the relation, properties, and measurement of solids, surfaces, lines, and angles.

A *solid* or *volume* is a portion of space limited in all directions. The term volume, is the preferable one of the two; because the idea of a solid carries with it that of matter, which, in fact, has nothing, and should have nothing to do with geometrical considerations.

Every solid or volume occupies a portion of space; the boundary of this is common to both the volume and the surrounding portion of space, and is called *a surface;* hence, we define a surface as having length and breadth, but no thickness.

If we conceive a surface to be made up of two parts, that which is common to both is called a *line.* Hence, we define a line to have length, without breadth or thickness. If a line is conceived as made up of two parts, that which is common to both is a point; hence, a point has neither length, breadth, nor thickness, but position only.

A *plane angle* is a portion of a plane included between two straight lines meeting in a common point, called the *vertex.*

A *polyhedral angle* is a portion of space included between several plane angles having a common vertex. The four magnitudes, viz.: lines, surfaces, solids, and angles, are called *geometrical magnitudes,* and taken together, they constitute the only things with which geometry, as a science, is conversant.

Geometry is divided into two parts:

I. *Elementary Geometry,* which treats of those magnitudes whose elements are the straight line and the circle.

It embraces:

1. All propositions relating to plane figures bounded by right lines, or by the circumference of a circle, or by a circular arc and a straight line.

2. All propositions relating to the surfaces of the cone, cylinder, and sphere,—which are called the three round bodies.

3. All propositions relating to solids bounded by planes, or to the solidities of the three round bodies.

An immediate application of this part of geometry is found in plane and spherical trigonometry, which treat of the relations of the sides and angles of triangles. It also embraces all constructions that can be made by the aid of the straight line and circle.

II. *Higher Geometry* embraces those branches, in which the elements are more complex lines, such as the Conic Sections, &c. It includes the higher investigations of

the ancients, which are now more elegantly treated of in Analytical Geometry, and by the aid of the Calculus.

It also embraces the treatment of the famous isoperimetrical problems, from which originated the Calculus of Variations, as well as the great problems of the duplication of the cube, and the trisection of an angle. It includes also the solution of all geometrical problems which cannot be effected by the aid of the circle and straight line alone.

The direct *applications of Geometry, in general,* are

1st. *Descriptive Geometry,* which has for its object the graphic solution of all problems involving three dimensions. In this branch of construction, lines are given by their projections upon two planes of reference, generally taken at right angles to each other. Planes are given by their traces upon these planes, and surfaces by the projections of certain of their elements.

An extensive and useful part of Descriptive Geometry is found in its application to problems of Shades and Shadows.

2d. *Perspective,* in which objects, as they would appear to the eye, taken in a certain position, are represented upon a plane or other surface. The plane employed is called the perspective plane, and the position of the eye is the *point of sight.*

A modification of perspective is employed in projecting the circles of the sphere on a plane. called spherical projections. It is used in constructing maps of the earth, or heavens, or any portion of them.

For a more full account of these several applications of geometry, see *Descriptive Geometry, Shades and Shadows, Perspective, Spherical Projections, and Isometrical Projections.*

For more detailed information on the subject of the higher geometry, see *Analysis. Analytical Geometry, Calculus, Calculus of Variations, Isoperimetrical Problems, Duplication of the Cube.* and *Trisection of an Angle.* under their respective heads.

We shall now enter a little more into detail with respect to the nature of *Elementary Geometry,* and a good portion of the explanation given will be found applicable to *higher Geometry,* and its applications.

Objects to which the reasoning is applicable.

1st. Lines.—The only lines considered, are the straight line and the circumference of the circle. The *straight line* is a line which does not change its direction between any two of its points. Straight lines are, in general, supposed to extend indefinitely in both directions, but in Elementary Geometry, they are often supposed to be limited, that is, terminated by points. The length of a straight line is the shortest distance between its limiting points, or extremities. No property of the straight line is either assumed or proved, in addition to those already mentioned, except the additional fact. that two straight lines cannot include a space.

A *circle* is a portion of a plane bounded by a curved line, every point of which is equally distant from a point within, called the centre. The curve is called the circumference, or in common language, the *circle.* So that in speaking of the circle, we sometimes mean the surface within the circumference, and sometimes the circumference itself, but the connection in which the term is used serves to prevent any ambiguity in the meaning.

2d. Surfaces.—The surfaces considered are of two kinds, *plane* and *curved* surfaces.

A *Plane Surface.* is a surface in which, if any two points be taken at pleasure and joined by a straight line, that line will be wholly in the surface. As in the case of right lines, only limited portions of planes are generally considered in Elementary Geometry. These limited portions may be bounded by straight lines, by curved lines, or by both. Those bounded wholly by straight lines, are called *polygons;* those by curves are circles, and those by both, form parts of circles, as sectors, segments, &c.

A *polygon* is a part of a plane bounded by straight lines, called sides.

The simplest polygon is the *triangle* bounded by *three* sides; then the *quadrilateral,* bounded by *four;* the *pentagon,* by *five;* the *hexagon,* by *six;* the *heptagon,* by *seven;* the *octagon,* by *eight;* the *nonagon,* by *nine;* the *decagon,* by *ten;* the *undecagon,* by *eleven;* the *dodecagon,* by *twelve;* and so on.

A *Curved Surface* is any surface not a *plane surface,* or made up of plane surfaces. The only curved surfaces treated of in Elementary Geometry, are those of the three

round bodies, the cone, the cylinder, and the sphere

3d. SOLIDS.—The solids, or volumes considered, are either bounded by polygons, by curved surfaces, or by both.

Those bounded by polygons are called polyhedrons. Amongst these, are found the *pyramid*, the *prism*, the *parallelopipedon*, the *octahedron*, the *dodecahedron* and the *icosahedron*.

The only solid of Elementary Geometry, bounded entirely by a curved surface, is the sphere.

Of those bounded in part by curved, and in part by plane surfaces, may be mentioned, cones, cylinders and their frustums or segments, and segments of the sphere.

4th. ANGLES.—Angles are *plane* or *polyhedral*, both of which have already been defined.

The magnitudes above enumerated, are the only ones considered in Elementary Geometry.

Object of Elementary Geometry.

The object of Elementary Geometry is to investigate the *properties*, and *relations* of the magnitudes above named.

A *property* of a geometrical magnitude is an attribute common to all of the class to which the magnitude belongs. For example, it is the property of a plane triangle that the sum of its three angles is equal to two right angles; because, this is one of the attributes common to all plane triangles.

A property may be *characteristic* or *secondary*.

A *characteristic* property is one without which the magnitude could not exist, and it is one not possessed by any other class of magnitudes. Thus, that every triangle has but three angles is a characteristic property.

Secondary properties are those upon which our conception of the existence of the magnitude is not dependent, and which may be shared by magnitudes of other classes. Thus, that the area of a square is equal to the product of the perimeter, and one-half of the radius of the inscribed circle is a secondary property of the square. It is secondary, because we can conceive the existence of the square as independent of this property, and further, the same property holds true of every regular polygon. It is a property, then, rather of regular polygons, to which class the square belongs, and only secondarily, a property of the square.

The enunciation of a characteristic property is a sufficient definition of a magnitude. In general, a definition is nothing else than an enumeration of one or more characteristic properties of the magnitude defined. Since the same magnitude may have several characteristic properties, it follows that it may be defined, and correctly too, by several different definitions. An investigation of the properties of a magnitude enables us to select the best definition, that is, the one most calculated to aid us in the ultimate object of comparing different magnitudes.

The *relations* investigated in Elementary Geometry are of two kinds. Those of *equality* or *inequality*, and those of proportionality. As an example of the first species of relation, we may instance the following. The sum of any two sides of a plane triangle is greater than the third, and their difference is less than the third. The square described upon the hypothenuse of a right angled triangle, is equivalent to the sum of the squares described upon the other two sides.

The second kind of relation is that of proportionality, and it is reached by the process of comparison of the things between which a relation is sought, with some known or assumed thing of the same kind, regarded as a standard; the standard is called the *unit of measure*.

The unit of measure for lines, is a straight line of known length, as a *foot*, a *yard*, a *mile*, &c. The unit of measure for surfaces, is a square described upon the lineal unit as a side. The unit of measure for volumes, is a cube described upon the lineal unit as an edge.

It is sometimes possible to compare one magnitude with another directly, but in general, this comparison is effected through the instrumentality of a unit of measure. Two figures or magnitudes are equal, when one may be placed upon the other so that they will coincide throughout their whole extent. Two magnitudes are equivalent, when they contain the same unit of measure the same number of times. Equality refers to possibility of coincidence; equivalency, to equality of measure only. Equal quantities are necessarily equivalent, but equivalent quantities may not be equal.

Methods of Investigation.

The truths of Geometry form a chain of dependent propositions, which may be separated into three classes.

1st. Truths implied in the definitions, viz.: that things do or may exist corresponding to the things defined. For example, when we say, "a quadrilateral is a polygon of four sides," we imply that such a figure may exist.

2d. Self-evident, or intuitive truths, which are contained in the *axioms*.

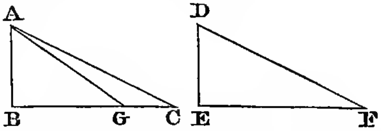
3d. Truths inferred from the definitions and axioms, called demonstrative truths. We say that a truth, or proposition, is demonstrated, when by a course of reasoning it is shown to be included under some other truth or proposition previously known, and from which it is said to follow. A demonstration is a train of logical arguments brought to a conclusion, in which the premises are definitions, axioms, hypotheses, and propositions already established. The arguments are the links that connect the premises, logically, with the conclusion, or ultimate truth to be proved.

Two methods of demonstration are employed; the *direct* and the *indirect*, or the *reductio ad absurdum*.

In the *direct* method the premises are definitions, axioms, and previous propositions, and by a process of logical argumentation, the magnitudes of which something is to be proved, are shown to bear the mark by which that something may always be inferred; or, in other words, they are shown to fall under some definition, axiom, or proposition previously laid down. Direct demonstrations are divided into two classes. 1st. Where the argument depends upon superposition; that is, on the coincidence of magnitudes when applied one to the other. 2dly. When it depends on addition, subtraction, or immediately on principles previously laid down.

The *indirect* method rests upon an hypothesis. This hypothesis is combined in a process of logical argumentation with definitions, axioms, and previous propositions, until a conclusion is obtained, which agrees or disagrees with some known truth. Now, if the conclusion arrived at agrees with some previously known truth, the hypothesis is said to be proved; if it disagrees with some known truth the hypothesis is false, and its contrary is said to be proved. In the indirect demonstration, therefore, the conclusion is compared with the truths known antecedently to the proposition in question. If it agrees with any one of these, the hypothesis is correct; if it disagrees with any one of these, the hypothesis is false.

We will give for an illustration of this method, proposition XVII. of the first book of Legendre. "When two right angled triangles have the hypothenuse and a side of the one equal to the hypothenuse and a side of the other, each to each, the remaining parts will be equal each to each, and the triangles themselves will be equal."



In the two right angled triangles BAC and EDF, let the hypothenuse AC be equal to DF, the side BA to the side ED; then will the side BC be equal to the side EF, the angle A to the angle D, and the angle C to the angle F. To prove this proposition we need the following, which have been before proved, viz:

Prop X. (of Legendre). "When two triangles have the three sides of the one equal to the three sides of the other, each to each, the three angles will also be equal, each to each, and the triangles themselves will be equal."

Prop. V. "When two triangles have two sides, and the included angle of the one equal to two sides and the included angle of the other, each to each, the two triangles will be equal.

"Axiom 1. Things which are equal to the same thing are equal to each other."

Axiom 10 (of Legendre). "All right angles are equal to each other."

Prop. XV. "If from a point without a straight line a perpendicular be let fall on the line, and oblique lines be drawn to different points.

"1st. The perpendicular will be shorter than any oblique line.

"2d. Of two oblique lines drawn at pleasure, that which is further from the perpendicular will be the longer."

Now the two sides BC and EF are either equal or unequal. If they are equal, then by Prop. X, the remaining parts of the two triangles are also equal, and the triangles themselves are equal. If the two sides are unequal, one of them must be greater than the other. Suppose BC to be the greater.

On the greater side BC, take a part BG, equal to EF, and draw AG. Then in the two triangles BAG and DEF, the angle B is equal to the angle E by axiom 10, both being right angles. The side AB is equal to the side DE, and by hypothesis, the side BG is equal to the side EF. Then it follows from Prop. V, that the side AG is equal to the side DF. But the side DF is equal to the side AC; hence, by axiom 1, the side AG is equal to AC. But the line AG cannot be equal to the line AC, having been shown to be less than it by Prop. XV; hence, the conclusion contradicts a known truth, and is, therefore, false; consequently the supposition (on which the conclusion rests) is false; therefore, the triangles are equal and all of their parts are equal, each to each.

It is often, though erroneously, supposed that the indirect demonstration, or the "reductio ad absurdum," is less conclusive and satisfactory than the direct demonstration. This impression arises from want of proper analysis of the nature of the reasoning.

For example: in the demonstration just given, it was proved that the two sides BC and EF cannot be unequal, for such a supposition, in a logical argumentation, resulted in a conclusion directly opposed to a known truth, and as equality and inequality are the only general conditions of relation that can subsist between the two quantities, it follows if they are not unequal they must be equal.

In both kinds of demonstration the premises and conclusion agree; that is, they are both true or both false, and the reasoning or argument in both is supposed to be strictly logical. In the direct demonstration the premises are known, being antecedent truths, and hence the conclusion is true. In the indirect demonstration, *one element* is assumed, and the conclusion is compared with truths previously established. If the conclusion is found to agree with any one of these, we infer the assumed element is true; if it contradicts any one of these, we infer that the assumed element is false. The method of reasoning in both cases is precisely the same, being according to the strict rules of logic.

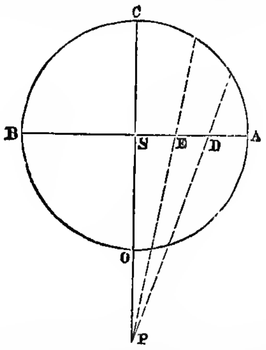
GILL. A measure of capacity, containing one-fourth of a pint, or nearly $8\frac{5}{8}$ cubic inches.

GIV'EN. Something that is known, or whose real value is assumed. Thus we say that a straight line is given in position, when we know its direction with respect to some other line regarded as fixed. A circle is given when we know the position of its plane, its centre, and the radius with which it is described. In analysis a line or surface is said to be given when its equation is given, that is, when we know the form of the equation, and the constants which enter it. The term given is often used to imply that a thing can be found. Thus, we say that a circle is given when three of its points are given, for we then have the means of constructing it by known rules. If we know the ratio between two quantities, they are said to have a *given* ratio; in short, any element of mathematics supposed to be known is said to be given.

GLŌBE. [L. *globus*, a ball]. In Geometry, the same as SPHERE, *which see.*

GLOB'U-LAR. Relating to, or partaking of, the nature of a globe. Thus, we say globular chart, globular projection, globular sailing, &c.

GLOBULAR PROJECTION. In Spherical Projections, that species of projection in which the point of sight is taken in the axis of the primitive circle, and at a distance from the pole of this circle equal to the sine of 45°.



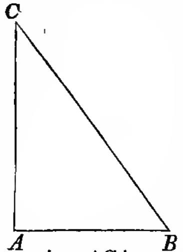
Let AOBC be the great circle of the sphere, cut out by a plane through CP, the axis of the

primitive circle; and let AB be the diameter of the primitive circle lying in this plane; make OP equal to sin 45°. Now if the arc AC or BC be divided into equal parts, and the points of division be projected upon AB by lines drawn to P, then will the spaces AD, DE, ES, &c., be nearly equal, and this is the advantage claimed for this mode of projection over the other methods.

In the stereographic method, the projection of those parts of the sphere near the pole are unduly crowded together, and in the orthographic projection, those near the primitive circle are crowded together, whilst in the globular, this crowding is, in a measure, avoided.

If the primitive plane coincides with the equator, the globular projections of the meridians are straight lines, and the projections of the circles of latitude are circles; in other cases, the projections are ellipses. See *Spherical Projections.*

GNŌ'MON. [Gr. γνωμον, an index]. An instrument employed for measuring the altitude of the sun by means of the lengths of shadows cast. Let AC represent a vertical style, column, or pillar, and BA a horizontal plane. Suppose that the sun, shining upon the style, casts a shadow of the point C at B. Then is BA the length of the shadow of the style, and it may be measured by any scale of equal parts.

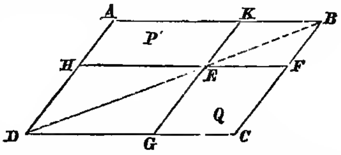


Then, since AC is supposed to be accurately known, we may find, from the right-angled triangle ABC, the value of the angle CBA at the base, which will be the altitude of the sun at the time of observation. If the line BA is in the meridian of the place, the altitude determined is the meridian altitude. This method of determining altitudes admits of considerable accuracy. The gnomon, if attached to a movable plane, susceptible of being leveled by a spirit-level, may be applied to measuring heights by means of shadows, as follows:

Set up the gnomon, and level its horizontal limb. Measure the length of the shadow projected by it, and at the same time measure the length of the shadow cast by the object whose height is desired; then, the length of the shadow of the gnomon is to the length of the shadow of the object, as the length of the gnomon to the height of the object.

Gnomon, in Dialing, is the style or pen, the shadow of which points out the hours. See *Dial.*

Gnomon, in Geometry. The space included between the lines forming two similar parallelograms, the smaller being so situated that it shall have an angle in common with the larger.



Thus, if ABCD and AHEK are similar parallelograms, the space KBCDHEK is a gnomon. If the sides KE and HE be produced to G and F, then is the space FEGDABF a gnomon with respect to the parallelogram BD and FG.

Gnomonic Projection of a sphere, is the projection of the lines of a sphere upon a plane tangent to the surface of the sphere, the point of sight or the eye being taken at the centre of the sphere. In this projection, all great circles of the sphere are projected into straight lines; all small circles, whose planes are parallel to the plane of projection, into concentric circles, having their common centre at the point of contact; and all other small circles into ellipses.

The gnomonic projection is also called the horologiographic projection, on account of its use in dialing.

GNO-MON'ICS. The art of dialing, so called, because it shows how to determine the time of day by the shadow of a gnomon. See *Dialing.*

GŌLD'EN NUMBER. So called, from its having been formerly written in golden letters in the almanac; it is the number denoting the year of the cycle of 19 years, in which the year in question falls. To find the golden number for any given year, add 1 to its number in the christian year, and divide the sum by 19, the remainder is the golden number of the year, unless the remainder is 0, in which case the golden number is 19.

For example, to find the golden number for the year 1854: dividing 1855 by 19, the remainder is 12; hence, 12 is the golden number required.

GOLD′EN RULE. A name given to the Rule of Three, on account of its universal use and great practical value.

GO-NI-OM′E-TRY. [Gr. γωνια, angle, and μετρον, measure]. The art of measuring angles, either upon paper, or on the surface of the earth.

GORGE. In Descriptive Geometry, the throat, or the smallest section of an hyperboloid of one nappe. In general, the section of the gorge is an ellipse, but in the particular case of the hyperboloid of revolution of one nappe, it is a circle, and is called the *Circle of the Gorge.* If we conceive a straight line to revolve about another straight line, not in its own plane, as an axis, the surface generated is an hyperboloid of revolution. If a straight line be drawn perpendicular to the generatrix and the axis, and intersecting both, this will, during the revolution, generate the circle of the gorge. As the distance between the generatrix diminishes, the circle of the gorge diminishes, and when it becomes 0, the circle of the gorge becomes a point, and the surface passes into a cone.

GRĀCE, DAYS OF. See *Exchange.*

GRAD-U-Ā′TION. [L. *gradus*, a step]. The operation of dividing any given scale or arc of a circle into equal parts. This is the most important of the practical operations of the mathematical instrument-maker.

The most simple operation of graduation consists in copying any given graduated limb. Straight scales may be copied as follows:

The original pattern, and the scale on which the copy is to be made, are placed side by side, and strictly parallel; a straight edge with a shoulder at right angles, like a carpenter's square, is made to slide along the original, stopping at each division, when a corresponding stroke is cut by the dividing knife on the copy. With care, this method admits of considerable accuracy.

The divisions of a graduated circle may be copied upon a concentric circle, by laying a straight edge upon each division in succession, so that it shall pass through the centre, and then, with the dividing knife, making the proper mark upon the copy: in this manner, protractors and scales upon rough instruments, might be graduated. The pattern used, is called the dividing-plate. This method of dividing is now nearly superseded by the dividing engine. The first engine of this kind was invented by RAMSDEN, of which the following is a general description:

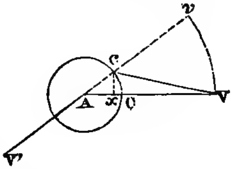
A horizontal circle, four feet in diameter, turns upon a vertical axis; the outer edge is notched; it is put in motion by an endless screw, one revolution of which carries the circle around 10′; the screw is turned by a treadle, and by an ingenious contrivance, the operator can cause the screw to turn through any part of a revolution by one descent of the treadle, whilst, on removing the foot, the screw is prevented from returning. The circle to be divided is fixed upon the engine, and made concentric with the revolving wheel, and a division line is cut, after each descent of the foot. Improvements in this engine have been made by the TROUGHTONS and others, and the engine, thus improved, still continues to be used for graduating instruments of minor importance.

Engines differing from Ramsden's, have been constructed by REICHENBACH, of Germany, and GAMBEY, of Paris. The German method of division is performed by copying. A large circle having been graduated with great care, the copy is placed upon and concentric with it. A microscope is fixed independently over the divided circle, and the divisions are brought in succession under the wires of the micrometer, and a line cut in the copy after each bisection. This process gives better results than the English dividing engine. In all the cases considered, a copy or original circle has to be graduated. We now propose to give a brief sketch of the method of original dividing.

The first attempts of original graduation, according to correct principles, were by GRAHAM, an Englishman, distinguished for his clocks and watches. The tool employed by him was a beam compass, a straight rod of wood or metal on which points of steel are fixed. In order to bisect a line, the points of the beam compass are placed at a distance from each other, nearly equal to half the

length of the line ; supposing the extremities of the line to be marked by dots, one point of the compass is placed at one dot, and a faint arc of a circle struck with the other towards the distant dot, and the operation is repeated with the other dot as a centre. The two faint arcs will either include a small space, or leave a small space between them, which can be most accurately divided with a pointer by the hand, aided by a magnifying glass. GRAHAM applied this principle in graduating the Normal circle, which he erected at Greenwich. To divide the quadrant, he stepped off from the 0 of the limit an arc whose chord was equal to the radius of the circle ; this gave the 60° point ; he next bisected the arc of 60°, and from the point of bisection he again stepped off an arc of 60°, which gave the 90° point. The arcs of 30° were by continual bisections divided, each into 32 parts, and the subdivisions were on the same principle divided into 16 parts. These divisions not being aliquot parts of a degree, had to be reduced by formulas or tables to the standard degree, a matter giving some trouble, but easily enough effected. A great improvement was made upon Graham's system by BIRD. He constructed, by the principle of bisections, a scale of equal parts, having a vernier, and by means of this scale he was enabled, with great exactness, to construct an arc of 85° 20′, which by perpetual bisection gave 5′ spaces on the limb. The details of this operation it is not necessary to give. Finally, by the introduction of magnifying glasses, the method of bisection in connection with the scale of equal parts, attained a great degree of perfection.

A further improvement was made by TROUGHTON. We will suppose a circle to be divided originally. After the edge of the circle is very carefully turned upon its own centre, a small circular roller, 16 revolutions of which carry it exactly around the circle, is prepared and so fitted to the circle by a radial frame joining the two centres, that on turning the frame around, the roller is turned in an opposite direction by the friction between the edges of the circle and roller. The roller is divided into 16 parts, and a microscope placed over the divisions ; as each division comes under the microscope, a small round dot is made upon the circle, which is thus approximately divided into 256 equal parts. By the aid of an *optical* beam compass, the accuracy of these divisions was tested, and the error of position of each dot determined and tabulated. The error being thus determined in the case of each dot, TROUGHTON returned to the roller, and by the aid of a sector which revolved with it and gave him an enlarged scale, he was able to reduce the approximate divisions of 256 parts to 360 equal parts, mechanically, and to cut the actual divisions of degrees upon the circle. Enough has been said to show the principles employed in the graduation of limbs ; nothing more was intended. Should the reader desire further information, he is referred to a valuable article by TROUGHTON, in the Edinburgh Encyclopœdia, entitled *Graduation.* It is to be remarked before leaving the subject, that in any given divided circles, an error due to want of centering, called eccentricity, is of frequent occurrence. To illustrate the extent and effect of this let A represent the axis about which the circle rotates, and Cc the small circle described by the centre of the graduated arc.



Suppose the circle to have revolved through the angular space V Av, then will the angle read off be equal to Vcv, which is evidently in error by the small angle cVA. The sine of the angle cVA is measured by

$$\frac{cx}{cV} = \frac{Ac}{cV} \sin VAv.$$

If we suppose V to be the 0 point of the scale, the error for a given instrument will vary between certain limits, being 0 when Av coincides with, or is in the prolongation of AV, being a maximum when Av is at right angles with AV. It is also apparent, that if the reading of the line Av is also taken at the opposite extremity of the diameter at V′, this will be as much in defect as the first is in excess, and that their mean diminished by 90° will give the true reading. We see from the formula for the error, that it will be 0 when $Ac = 0$, that is, when there is no eccentricity. By the aid of the given formula a table may be constructed for correcting

each reading for error in eccentricity. Such a table would be necessary in a sextant when readings at both extremities of a diameter are impossible.

GRAPH-OM′E-TER. [Gr. γραφω, to describe, and μετρον, measure]. An instrument used for measuring angles whose vertices are at its centre ; the protractor is a graphometer.

GROINS. Lines of intersection of the under surfaces of arched vaultings. Arches of this kind are called groined arches. The groin forms the basis of an extensive system of decorations in architecture. If the lower surfaces of the intersecting arches are cylindrical, having equal circular bases, and their axes lying in the same plane at right angles, the groins are ellipses whose conjugate axes are equal to the diameter of the base of the arch, and whose transverse axes are equal to this diameter multipled by $\sqrt{2}$. This is the simplest case. By using oblique arches, a great variety of lines will be produced.

GROUND′-LINE. In Descriptive Geometry, the line of intersection of the horizontal and vertical planes of projection. The planes of projection being infinite in extent, the ground line is infinite in length.

GROUND′-PLANE. In Perspective, the horizontal plane on which the objects to be put in perspective are situated. The horizontal plane of projection.

GRöUP OF EQUATIONS. In Analysis, several equations considered together or simultaneously. It is a property of any group of simultaneous equations, that in order that they may be determinate, there must be just as many of them as there are unknown quantities entering them. If they are all independent, and their number is greater than the number of unknown quantities, the group is impossible ; if they are fewer in number than the number of unknown quantities. the group is indeterminate.

GUNTER. Inventor of Surveying and Drawing instruments.

Gunter's Chain. A chain, first used by Gunter, for the purposes of land surveying. It is 66 feet, or four rods in length, divided into 100 links, each 7.92 inches in length, every tenth one being marked, for convenience in counting. The advantage of this mode of division is, that a square chain or square link is a decimal fraction of an acre, and consequently, if the area of a piece of ground be found in square chains and links, it may at once be converted into acres by pointing off a suitable number of decimal places. To convert square chains into acres, point off one place of decimals from the right. To convert square links into acres, point off five places of decimals from the right.

Gunter's Line. A sliding scale corresponding to logarithms of numbers, by means of which the operations of multiplication and division may be performed by inspection.

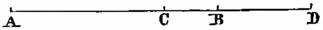
Gunter's Scale. A wooden ruler two feet in length, upon which is marked a great number of different scales, by means of which a great variety of problems in Surveying and Navigation can be solved mechanically by the aid of the dividers alone. On one side of the ruler are scales of equal parts, of sines, of tangents, of rhumbs, &c., on the other side are scales of the logarithms of these various parts.

H. The eighth letter of the English alphabet. As a numeral, it has been used to denote 200 ; with a dash over it, thus, $\overline{\text{H}}$, it denoted 200,000. As an abbreviation, it stands for *hour*.

HăR-MON′IC PROPORTION. The reciprocals of quantities in arithmetical progression are said to be in harmonic proportion, thus, the numbers,

$$\frac{1}{a}, \quad \frac{1}{a+b}, \quad \frac{1}{a+2b}, \quad \frac{1}{a+3b}, \text{ \&c.,}$$

are in harmonic proportion, or, more properly, in harmonic progression.



A given line AB, is said to be harmonically divided when two points, one on the line, and the other on its prolongation, are so placed that

$$AC : CB :: AD : DB.$$

In this construction CD is a harmonic mean between AD and BD or AD ; CD and BD are the reciprocals of distances which are in arithmetical progression.

HEC′TO-GRAMME. [Gr ἑκατον, a hundred, and γραμμα, a gramme]. In the French

system of weights and measures, a weight of 100 grammes.

HEC'TO-LĬ-TRE. [Gr. ἑκατον and λιτρα, a pound]. A French measure of capacity, containing 100 litres.

HEC-TOM'E-TRE. [Gr. ἑκατον and μετρον, measure]. A French measure of length, equal to 100 metres.

HEĪGHT. The third dimension of space, when it is supposed to be reckoned upwards. In general, any distance estimated or measured upwards. In leveling, the height of one point above another is the difference of their distances from the centre of the earth. See *Altitude* and *Leveling*.

HEL'I-COID. [From ἑλιξ, a scroll]. A warped surface, which may be generated by a straight line moving in such a manner that each point of it shall have a uniform motion in the direction of a fixed straight line, and at the same time a uniform angular motion about it. The fixed straight line is the axis or directrix of the surface, and the moving line the generatrix. The conditions of generation require that the same point of the generatrix shall remain upon the directrix, and that the angle between the directrix and generatrix shall be constant. When this angle is a right angle, the helicoid is *right;* when it is oblique, the helicoid is *oblique.* The curve generated by any point of the moving line, is called a *helix.*

The right helicoid forms the under surface or soffit of the spiral staircase; also, the upper and lower surface of the thread of the rectangular threaded screw. The oblique helicoid forms the upper and lower surfaces of the thread of the triangular screw.

The right helicoid is a species of right conoid. Every plane passed through an element of the helicoid is tangent to the surface somewhere along the element. When it passes through the axis, the point of contact is where the element intersects the axis, and if the axis is vertical, the tangent plane makes with the horizontal plane an angle of 90°, and is a maximum. If, now, the plane be revolved about the element, as an axis, the point of contact recedes from the axis, and, finally, when the tangent plane makes with the horizontal plane the same angle that the element does, the point of contact is at an infinite distance from the axis, and the angle is a minimum. To pass a plane tangent to the helicoid at any point, find the tangent line to the helix through this point, and pass a plane through it and the element through the point of contact. The curve of intersection of the oblique helicoid with a plane perpendicular to the axis, is the spiral of Archimedes.

HĒ'LI-O-STAT. [Gr. ἥλιος, the sun, and στατος]. An instrument by means of which a reflected ray of light may be retained in a fixed position, notwithstanding the motion of the sun.

HEL-I-SPHER'IC-AL LINE. A rhumb line, more commonly called the loxodromic curve.

HĒ'LIX. [Gr. ἑλιξ, a winding]. A curve generated by any point of the generatrix of a helicoid in its motion, as described under *Helicoid.* From the nature of the generation, it follows that every point of a helix is equally distant from the axis; hence, when the axis is vertical, the horizontal projection of the helix is a circle; and, further, a tangent to the same helix at each point, makes with the horizontal plane a constant angle. If the horizontal projecting cylinder of any helix be developed upon a plane surface, the helix will develop into a right line, making with the development of the base an angle, equal to the angle of the helix with the horizontal plane. The tangent of the angle which any helix makes with the horizontal plane, is equal to the distance ascended by the helix during any portion of a revolution divided by the horizontal projection of the same portion of the helix. The helix generated by the point in the axis of the helicoid, makes with the horizontal plane an angle equal to 90°, which is a maximum; that generated by the point at an infinite distance, is 0, which is a minimum. The edge of the thread of a screw is a helix, as is also the path described by any point of the surface of the thread when moved in the nut. This curve is principally of importance in the mechanical discussion of the properties of the screw.

HEM'I-SPHERE. [Gr. ἡμισφαιριον, a half sphere]. If a sphere be divided into two equal parts by a plane through its centre, each part is a hemisphere. In ordinary lan-

guage, the term hemisphere is applied simply to the surface; thus we speak of the northern and southern hemispheres, meaning their surfaces.

HEP'TA-GON. [Gr. ἑπτα, seven, and γωνια, an angle]. A polygon of seven angles, or seven sides.

HEP-TAG'ON-AL. Having seven angles, and, consequently, seven sides.

HEPTAGONAL NUMBERS. A kind of polygonal numbers, formed as follows: Let these be the arithmetical progression,

1, 6, 11, 16, 21, 26, &c.,
and 1, 7, 18, 34, 55, 81, &c.,

will be the series of polygonal numbers, called heptagonal. The law of formation is to add each number in the lower line to the next one on the right in the upper line. The general formula for heptagonal numbers is

$$N = \frac{5n^2 - 3n}{2}.$$

in which N denotes the heptagonal number in any place, and n the order of the place.

It is a property of heptagonal numbers that if any one of them be multiplied by 40, and the product be increased by 9, the result will be a perfect square. For,

$$\frac{5n^2 - 3n}{2} \times 40 + 9 = 100n^2 - 60n + 9 = (10n - 3)^2.$$

See *Number*.

HET-E-RO-GĒ'NE-OUS. [Gr. ἑτερος, other, and γενος, kind]. A polynomial is heterogeneous when all of its terms have not the same number of literal factors; thus,

$$a^2 + 2bc^3 - y,$$

is a heterogeneous expression.

HEX'ĀDE. A series of six numbers.

HEX'A-GON. [Gr. ἑξ, six, γωνια, angle]. A polygon of six angles or sides. To inscribe a regular hexagon in a circle, apply the radius six times as a chord, then will there be inscribed a regular hexagon. This construction arises from the fact that the side of a regular hexagon is equal to the radius of the circumscribing circle.

HEX-AG'ON-AL. Having six angles.

HEX-A-HĒ'DRON OR CUBE. [Gr. ἑξ, six, and ἑδρα, a base]. A polyhedron of six faces. See *Cube*.

HEX-OC-TA-HĒ'DRON. A polyhedron of 48 equal triangular faces.

HOGS'HĔAD. A measure of capacity. See *Measures*.

HO-LOM'E-TER. [Gr. ὁλος, all, μετρον]. An instrument for measuring distances of all kinds.

HOM-O-CEN'TRIC. [Gr. ὁμος and κεντρον]. Having the same centre. See *Concentric*.

HO-MO-GĒ'NE-OUS. [Gr. ὁμος, like, and γενος, kind]. In Algebra, a polynomial is homogeneous when each term contains the same number of literal factors; thus,

$$ax^2 + bxy + c^3$$

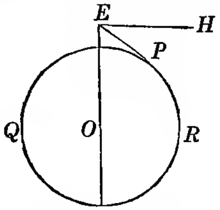
is a homogeneous expression.

HO-MOL'O-GOUS. [Gr. ὁμος, similar, and λογος, proportion]. In Geometry, the parts of similar magnitudes, which are like placed, are called *homologous*. Thus, in similar polygons, the corresponding sides, angles, diagonals, &c., are homologous. In similar solids, the corresponding faces, edges, polyhedral angles, diagonals, &c., in each are *homologous*. In a proportion, the antecedent and consequent of the same couplet, are homologous with respect to each other. This case covers the relations between homologous elements in all cases. Between any two magnitudes whatever, which are similar, the ratio of any two homologous elements of the same name, is always constant. Thus, in two similar polyhedrons, the area of any face of the first, is to the area of its homologous face in the second, as the area of any other face in the first, is to the area of its homologous face in the second. Like results obtain in all other cases.

HO-RĪ'ZON. [Gr. ὁριζων, from, ὁριζω, to bound, and ὁρος, a limit]. The horizon of any point on the surface of the earth regarded as a sphere, is a great circle perpendicular to the radius of the earth, regarded as a sphere, through the point. This is called the *true* horizon, and divides the earth into two equal parts or hemispheres. The *apparent* horizon is the plane passed tangent to the surface of the earth at the point considered. Its poles, like those of the true horizon, are at the zenith and nadir of the place. If the planes of the true and apparent horizon be extended

infinitely till they cut the heavens, the circles cut from the celestial sphere will sensibly coincide.

In Navigation, the horizon is understood to be the circle determined by the intersection of the heavens with a cone whose vertex is at the eye, and whose elements are tangent to lines of the earth's surface.



Let RPQ represent a section of the earth through the point C made by a vertical plane, and let EP be tangent to it; now, if this line EP be revolved about the line EO as an axis, it will generate the surface of a cone, which limits the visible horizon as seen from E. The angle HEP is called the *dip* of the horizon. See *Dip*.

Horizon in Perspective. The intersection of the perspective plane, and a horizontal plane, passed through the point of sight. It is the vanishing line of all horizontal planes, and is the locus of the vanishing points of all horizontal lines. See *Perspective*.

HOR-I-ZON'TAL. Parallel to the horizon. A line is said to be horizontal when it is parallel to the horizon; or, if a short line, parallel to the surface of still water. In leveling, it is the same as a level line. See *Level*.

A horizontal plane is one which is parallel to the horizon.

Horizontal Dial. A dial constructed on a horizontal plane, having its gnomon or style parallel to the axis of the earth. See *Dial*.

Horizontal Distance. In Surveying, any distance estimated in a horizontal direction.

Horizontal Projection. The horizontal projection of a *point*, is the foot of a straight line drawn through the point perpendicular to the horizontal plane of projection.

Horizontal Projection of a *line*, is the intersection of the horizontal plane of projection with a cylinder passing through the line, and having its elements perpendicular to the horizontal plane of projection.

Horizontal Projection of a *surface*, is that portion of the horizontal plane included within a cylinder perpendicular to it, and tangent to, or enveloping the surface. The horizontal projection of a surface is generally determined by projecting some of the principal elements of the surface. See *Descriptive Geometry*.

HO-ROG'RA-PHY. [Gr. ὥρα, hour, and γραφω, to write]. The art of dialing.

HO-ROM'E-TRY. [Gr. ὥρα, hour, and μετρον]. The art of measuring time by hours.

HOUR. [Gr. ὥρα]. A unit of time equal to the twenty-fourth part of a day. Hours are *solar* and *sidereal*.

A *solar hour* is the twenty-fourth part of the time which elapses between two consecutive passages of the sun over the meridian of a place. As this interval varies slightly, from day to day, a mean of all the days in the year is taken, and the hour thus determined is called the mean solar hour.

The *sidereal hour* is the twenty-fourth part of the interval between two consecutive passages of a fixed star on the same meridian. A sidereal hour is a little shorter than the mean solar hour, since 365 solar days correspond to about 366 sidereal days, so that a solar hour exceeds a sidereal hour by nearly 10 seconds.

Hour Circles. Meridians of the sphere making angles of 15° with each other, are called hour circles. There are twelve entire hour circles, which divide the surface of the spheres into 24 equal lines.

Hour Lines. Lines drawn on the face of a dial to indicate the hours of the day. See *Dial*.

HȲ-DROG'RA-PHY. [Gr. ὕδωρ, water, and γραφω, to describe]. That branch of maritime or nautical surveying which has for its object to ascertain the channels, their depth, width, &c., the position of shoals, the depth of water thereon; in general, it embraces all the operations necessary to a complete determination of the contour of the bottom of a harbor or other sheet of water. Hydrographical operations are generally carried on in connection with geodesic surveys of the

country bordering on the coast, and are always intimately connected with them. For the purpose of fixing points of reference on the surface of the water, a sufficient number of buoys are anchored, so that the lines joining them shall form a system of well conditioned triangles, the positions of which are carefully determined from a base line on shore by means of angles measured with a theodolite or other instrument.

From the nature of the case these positions cannot be so readily or so accurately fixed as the stations on shore. The buoys having been established, lines of soundings are taken in every possible direction between the buoys, also between the buoys and stations taken on the shore. By means of these soundings, the depth of water is determined at a great number of points, and we are thus enabled to determine the relative position of these points with respect to a plane of reference, which is generally taken as the plane of mean low water.

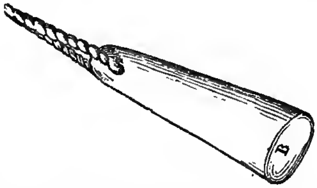
There are several methods of taking soundings and determining the places at which they are made. One of the simplest is as follows: Having provided a suitable boat and crew, the surveyor causes the boat to be rowed uniformly from one buoy to another, or from a buoy to a station on shore, and at intervals of time, generally equal, the lead is cast, and the depth entered in a note-book, together with the time at which the sound is taken; then knowing the length of time occupied in rowing from one station to another, and the length of time between each sounding, the position of the boat at each cast of the lead may be determined by a simple proportion. The length of time between the two stations is to the distance between them as the length of time between any two soundings is to the distance between the two points at which they are taken. Could perfect uniformity in rowing be attained, and currents be avoided, this method would be sufficiently accurate, but neither of these conditions can be fulfilled, and to correct for the necessary errors, the following process is often adopted:

An assistant is stationed on shore with a theodolite, and at a given signal made by the sounder, at every second or third sounding, he measures the angle subtended by the boat and some fixed signal on shore. The position of the boat may then be determined on the plot by plotting the measured angle, and finding where its second side intersects the plot of the line joining the two buoys There is sometimes a little difficulty in keeping the boat on the line between the buoys, on account of currents and unskillfulness in the oarsmen. In order to keep the boat in line, a third flag or signal is generally planted on shore, so as to be in line with the two buoys, between which the soundings are to be taken, and the oarsmen keep one buoy and flag in the same line with the boat, which insures the boat being on the proper course.

By the aid of two assistants with theodolites, on shore, one stationed at each extremity of a line taken as a base, the position of the sounding boat may be accurately determined at each signal.

There is another method of determining the position of a sounding, where the preceding ones are not practicable. Suppose that a sounding is to be taken at some point: the boat is brought to and anchored, the sounding taken, and by means of a sextant the angles subtended by three fixed objects on shore, taken two and two, are carefully measured; then the place of the boat can be found by the problem of three points. See *Problem of Three Points*.

The sounding is made by means of the lead and line. The line is of strong cord and divided from the lead into fathoms, the points of division being marked in such manner that they can easily be counted. The lead is shaped like the frustum of a cone, with the base B hollowed out to hold some grease.



The object of this arrangement is to bring up specimens of the mud, sand, &c., at the bottom, for the purpose of ascertaining the nature of the anchorage. These observations are to be recorded in the note book, and are generally laid down upon the chart.

It has been remarked that the time of

making each sounding is entered on the note book of the surveyor. This, besides aiding in plotting, is also of use in reducing the sounding to a fixed plane of reference.

At some suitable locality a tide gauge is erected, and during the time of sounding an assistant notes, at intervals of 15 minutes, the exact height of the tide; then by comparison the surveyor can at once make the necessary deduction from each sounding to reduce it to what it would have been if taken at low water. This reduction in some localities is often several feet. Having determined the depth of the water at a sufficient number of points, and plotted the positions of these points on the map, the contour lines can be drawn approximately. These are usually taken in planes one fathom distant from each other, the first fathom curve being 6 feet below the plane of mean low water, the second one 6 feet below it, and so on. It is not customary to delineate more than five or six horizontal contour lines, that indicating the nature of the bottom for a sufficient depth to accommodate the largest class of vessels.

Besides these lines, the depth of water and nature of the bottom is set down on the map, so as to indicate the best places for anchorage. The direction and velocities of the currents are also determined and laid down upon the chart at suitable places, and in addition, the magnetic bearings of prominent objects on shore from prominent points in the harbors are laid down. Sometimes there is added a perspective view of the land objects on approaching a harbor, together with suitable sailing directions for the use of mariners. All buoys, light houses, light ships, and other objects interesting to navigators, are to be carefully located and mapped; these, with a suitable topographical representation of the adjoining coast, constitute a complete hydrographical map of the harbor or other sheet of water in question.

For ease of reference, a scale of English, and a second scale of nautical miles ought to be drawn upon the map

In hydrographical surveying in deep water, nothing more is attempted than to ascertain the depth of water and the nature of the bottom, together with the force and direction of the currents at various places, sufficient to guide the navigator.

HȲ-PER'BO-LA. [Gr. ὑπερ, over, and βαλλω, to throw]. One of the conic sections. Its particular cases are the equilateral hyperbola, and two straight lines intersecting each other. The general case of the hyperbola may be cut from a right cone with a circular base by any plane which makes, with the base, an angle greater than that made by an element. In this case all of the elements except two are cut, half of them in the lower and half in the upper nappe. It is composed of two branches, one lying upon one nappe, and the other upon the other nappe of the cone. If a plane is passed through the vertex of any cone with a circular base, so as to cut out two elements, and then if any plane be passed parallel to it, the section is an hyperbola.

If the elements cut out by the plane through the vertex, make an *acute* angle with each other, the hyperbola is *acute;* if an *obtuse* angle, the hyperbola is *obtuse;* if a *right* angle, the hyperbola is *right* or *equilateral.* All the hyperbolas cut out of a given cone by a system of parallel planes are similar curves. The section made by a plane through the vertex is, analytically considered, an hyperbola; hence, two straight lines intersecting each other are considered as a particular case of the hyperbola.

In general, if the lines cut out by the plane through the vertex, be projected upon the plane of any hyperbola whose plane is parallel to them, by straight lines parallel to that joining the vertex of the cone and the centre of the hyperbola, the projections will be asymptotes of the hyperbola.

If we regard the oblique cone with a circular base, any plane parallel to two elements cuts out an hyperbola, as in the case of the right cone. If we suppose the vertex to approach the plane of the base and finally to reach that plane without the base, the cone will reduce to a portion of a plane determined by drawing lines through the vertex tangent to the base. In this case, the cutting plane through the vertex cuts out but one element, or, rather, the two elements have united and become one. Under this supposition the parallel planes cut out straight lines limited towards the centre; hence, a straight line, with a portion removed, or limited towards the centre, is another particular case of the

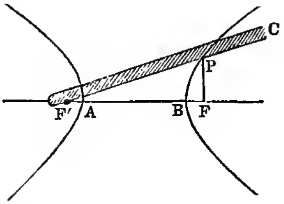
hyperbola. As the plane moves towards the one through the vertex, the limiting points approach each other and finally unite at the vertex; hence a straight line estimated in both directions from a point, is a particular case of an hyperbola.

Although the hyperbola was first suggested to geometers from a consideration of the sections of a cone, it may be defined, and its properties deduced, without any reference whatever to the cone.

The following are some of the definitions that have been given, as well as some of the leading properties of the curve.

1. An hyperbola is a plane curve, such that the difference of the distances from any point of it to two fixed points, is equal to a given distance. The fixed points are called *foci*, and the given distance is equal to that portion of a straight line drawn through the foci, which is included between the two branches of the curve.

This definition gives rise to the following construction by a continuous movement.

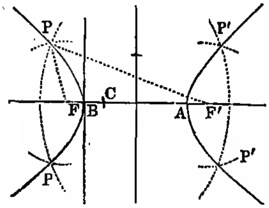


Let F and F be the foci; fasten a ruler F′C of sufficient length so that one end shall remain fast at F′, and be free to revolve about it: attach a string at H, whose length shall be less than the length of the ruler by a given distance AB, less than FF′; fasten the other end of the string at F; press a pencil P against the string, keeping it close to the ruler, and keeping the string tense; then move the ruler about F′; the pencil will describe one branch of the curve; a similar construction, with F as a centre, will give the other branch; or we might make the string longer than the ruler by AB, and then construct the other branch. The reason of this construction is evident: for, in every position of the pencil we have, by the conditions given,

$$F'P - FP = AB,$$

which corresponds to the definition.

The same property enables us to construct the curve by points, as follows:



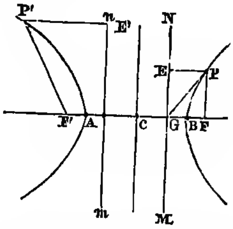
With any distance greater than F′C as a radius, and with F′ as a centre, let an arc of a circle be described, then with a radius equal to that first employed, diminished by AB, and with F as a centre, let a second arc be described, cutting the former one in P; then will P be a point of the curve; for,

$$F'P - FP = AB.$$

By changing centres and using the same radii, two other points P′ can be constructed, and so on for any number of points.

When a sufficient number of points have been formed, let a curve be drawn through them and it will be the hyperbola.

2. A second definition of the hyperbola is this.



Let F be a given point, MN a given straight line, and suppose a point P to move in the same plane, so that the ratio of its distances from the fixed point and the fixed line shall be constant, PF being always greater than PE, then will the point P describe an hyperbola. The other branch is evidently constructed in the same manner as the first. The fixed point is the *focus*, and the straight line, the *directrix*. The distance from the centre C to the directrix is a third proportional to CF and CB, that is,

$$CF : CB :: CB : CG.$$

3. The hyperbola may be defined by any one of its equations. The three most important equations of the hyperbola are the following:

1. When the curve is referred to the centre and a pair of conjugate diameters, its equation is

$$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2,$$

in which x and y are the rectilineal co-ordinates of all the points of the curve, and a' and b', the semi-conjugate diameters to which it is referred. If these are at right angles, $a' = a$, $b' = b$, and the equation becomes

$$a^2 y^2 - b^2 x^2 = -a^2 b^2,$$

which is the equation of the hyperbola, referred to its centre and axes. This is the form in which the rectilineal equation of the curve is most often used.

2. The polar equation of the hyperbola is,

$$r = -\frac{a(1-e^2)}{1-e\cos\phi},$$

in which the pole is at the focus within the right hand branch, e the eccentricity, r the radius vector, a the semi-transverse axis, and ϕ the angle between the principal axis and the radius vector: ϕ is sometimes called the anomaly.

3. The equation of the curve referred to its asymptotes is,

$$xy = M,$$

in which x and y are the co-ordinates of every point of the curve, and

$$M = \frac{a^2 + b^2}{4},$$

a and b being the semi-axes.

Properties of the Hyperbola and useful constructions.

We shall first give some useful definitions, collecting and arranging them, so that they may be together.

1. The *hyperbola* is a plane curve, in which the difference of the distances from any point of it, to two fixed points, is equal to a given distance. The fixed points are *foci*.

2. The straight line through the foci is the indefinite transverse axis; that part of it lying between the two branches of the curve, is the definite *transverse axis*, and it is this which is always meant when the transverse axis is spoken of. Its middle point is the *centre* of the curve.

3. Any straight line that bisects a system of parallel chords, drawn in the curve, is a *diameter*. If the diameter is perpendicular to the chords which it bisects, it is an *axis* of the curve: there are two axes, the *transverse* axis already described, and the *conjugate* axis which passes through the centre, and is perpendicular to the transverse axis.

4. Every straight line through the centre is a diameter: two diameters are *conjugate* when each bisects a system of chords parallel to the other. There are an infinite number of sets of conjugate diameters.

5. The points in which a diameter intersects the curve, are called its vertices. The right hand vertex of the transverse axis is called the *principal vertex of the curve.*

6. The *parameter* of any diameter is a third proportional to the diameter and its conjugate. The parameter of the transverse axis is called the *parameter of the curve.*

7. If a chord be drawn through the focus perpendicular to the transverse axis, and at its extremity a tangent be drawn, it is called the *focal tangent.* If a perpendicular be drawn to the transverse axis at the point in which the focal tangent intersects it, that line is the *directrix* of that branch of the curve.

8. If any point be assumed in the plane of the curve, and chords be drawn through it cutting the curve in two points, then will the tangents drawn to the curve at the points of intersection of each chord, intersect each other upon a straight line, which is called, *the polar line of the point.* The point is correlatively called, the *pole* of the polar line. The directrix is the polar line of the focus of that branch.

An *ordinate to any diameter* is a straight line drawn from any point of the diameter to the curve, and parallel to the conjugate of the diameter. Every chord bisected by a diameter, is a double ordinate to the diameter. The ordinates to the axes are perpendicular to them.

10. A *tangent* to the curve, at any point, is the limit of all secants to the curve drawn through that point. If any secant be drawn through a point of the curve, and then be revolved about that point, as an axis, until the second point of secancy unites with the first, the secant passes to its limit, and becomes a tangent; at the same time, the two secant

points unite, and constitute the *point of contact.* A *subtangent*, on any diameter, is that portion of the diameter included between the point, where the tangent intersects it, and the foot of the ordinate to the diameter drawn through the point of contact.

11. A *normal* to the curve is a straight line perpendicular to a tangent, at the point of contact. A *subnormal*, on any diameter, is that portion of the diameter between the point in which it is intersected by the normal, and the foot of the ordinate to the diameter drawn through the point of contact.

12. *Supplementary Chords* are chords drawn through the extremities of any diameter, and meeting each other at a point of the curve.

13. The distances from the foot of any ordinate to a diameter, to the vertices of that diameter, are called *segments of the diameter*, and sometimes the *abscissas of the diameter.*

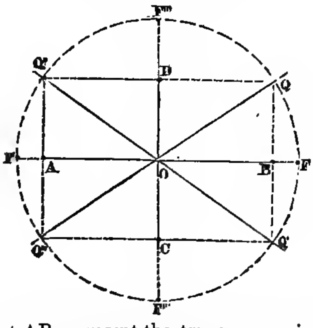
14. The *eccentricity* of the hyperbola is the ratio obtained by dividing the distance from the centre to either focus, by the semi-transverse axis.

15. Two hyperbolas are *conjugate*, when the transverse axis of the one is the conjugate axis of the other, and the reverse.

16. The asymptotes of an hyperbola are two straight lines, to which the curve continually approaches, touches at an infinite distance, but cannot pass. The asymptotes are prolongations of the diagonals of the rectangle, constructed on the axes, or of the diagonals of the parallelogram of any pair of conjugate diameters.

CONSTRUCTIONS.

1. *When the Axes are given.*



Let AB represent the transverse axis, and CD the conjugate axis. Draw DQ parallel to AB, and BQ parallel to CD, intersecting DQ at Q: draw OQ, and with O as a centre, and OQ as a radius, describe a circumference, cutting the transverse axis produced in F and F′; then will F and F′ be the foci.

2. *When the Foci and the Conjugate Axis are given.*

Let F and F′ be the foci, CD the conjugate axis, and O the centre. Through D draw DQ parallel to the line joining F and F′; with O as a centre, and OF as a radius, describe an arc, cutting DQ in Q and Q″; from these points let fall perpendiculars upon FF′: the part AB, intercepted between these perpendiculars, is the transverse axis.

3. *When the Transverse Axis and the Foci are given.*

Let F and F′ be the foci, and AB the transverse axis. On FF′, as a diameter, describe a circle, and through B draw BQ perpendicular to AB, cutting rhe circle in Q and Q′; through these points draw lines parallel to AB, cutting the perpendicular through the middle point of AB in C and D: then is CD the conjugate axis of the curve.

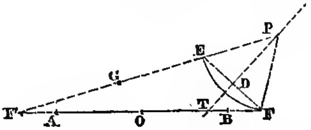
Also, F″ and F‴ are the foci of the conjugate hyperbola, and the lines QQ‴, Q′Q″, are common asymptotes to the two curves.

4. *When the Curve is traced on a plane.*

Draw any two parallel chords in either branch of the curve, and bisect them by a straight line; this will be a diameter of the curve: on that part intercepted between the branches of the curve, as a diameter, describe a semicircle, cutting the curve in a point; draw a pair of supplementary chords through this point and the vertices of the diameter used, and through the centre draw two lines parallel to these chords; the one which cuts the curve will be the transverse axis, and the other one will be the indefinite conjugate axis. At the principal vertex of the curve, erect a perpendicular to the transverse axis, and equal to the semi-transverse axis; join the extremity of this perpendicular with the centre, and with the centre of the curve as a centre, and the line as a radius, describe a circle, cutting the transverse axis in a point; draw the ordinate through this point, and through the point in which it meets the curve, draw a line parallel to the transverse axis, till

it meets the conjugate axis; this point is one extremity of that axis: the remaining elements may be found as already explained.

5. *When the Foci and one point of the Curve are given.*



Let F and F′ be the foci, and P a point of the curve. Draw FF′, and bisect it at O; O is the centre of the curve: with P as a centre, and PF as a radius, describe the arc FE; bisect EF′ in G, and make OA and OB each equal to EG; then is AB the transverse axis. The other axis may be constructed.

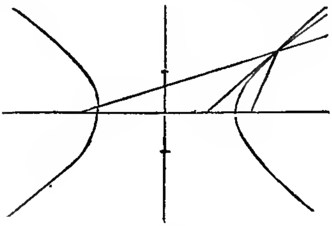
6. *When the Foci and any Tangent to the Curve are given.*

Let F and F′ be the foci, and PT any tangent: draw FE perpendicular to PT, make DE = FD, through E draw F′E, and produce it till it meets PT in P; then is P a point of the curve, and the construction may be made as in the preceding case.

The following properties give rise to useful constructions.

1. The squares of the ordinates to any diameter, are to each other as the rectangles of the corresponding segments of the diameter. This property enables us to construct the curve when a pair of conjugate diameters are given, in a manner entirely analogous to the corresponding construction in the case of the ellipse. See *Ellipse*.

This construction is of little value, as a better one arises from the property of asymptotes.

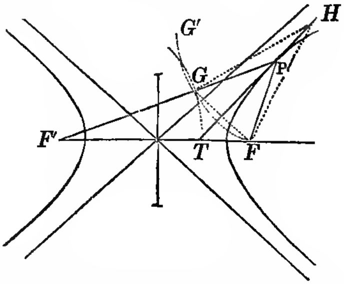


2. If at any point of the curve a tangent be drawn, and two lines to the foci, the tangent will bisect the angle included between the lines drawn to the foci. This property gives rise to the following constructions:

To draw a tangent to the curve at any point P.

Draw PF and PF′, and draw also PT bisecting the angle F′PF; then is PT the tangent required.

To draw a tangent through a point without the curve.



Let H be the point: draw HF, and with HF as a radius, and H as a centre, describe the arc FG; with F′ as a centre, and the transverse axis as a radius, describe the arc GG′, cutting FG in G and G′; draw HG and HF, and bisect the angle between them by HT; then is HT a tangent. Draw F′G, and produce it to the curve at P; P is the point of contact.

3. If any chord be drawn parallel to a diameter, then is its supplementary chord parallel to the tangent at the vertex of the diameter.

This property gives rise to the following constructions:

To construct a tangent to the curve at a point.

Draw a diameter through the point, and through the extremity of any other diameter, draw a chord parallel to the first diameter; draw the supplementary chord, and, parallel to it, draw a line through the given point; it will be the tangent required.

To draw a tangent to the curve parallel to a given line.

Draw a chord parallel to the line, and draw its supplementary chord; draw a diameter parallel to the last chord, and through its vertex draw a line parallel to the given line; it will be the tangent required. Two such tangents may be drawn.

4. *A tangent to the curve is parallel to the chords bisected by the diameter through the point of contact.*

Hence, the following construction for a tangent parallel to a given line. Draw two chords in the curve parallel to the given line, and bisect them by a straight line: through the points in which this intersects the curve, draw straight lines parallel to the given line, and they will be the tangents required. Two such lines can, in general, be drawn. These are the same as the corresponding constructions in the ellipse. See *Ellipse*.

The following properties of the curve are useful in analysis:

1. The angle between two conjugate diameters can never be greater than a right angle. The greatest angle is that between the axes, which is a right angle.

If one of the diameters be revolved about its centre towards the asymptote, the other one will approach the same asymptote, and they will meet upon it. Hence, the asymptote is the locus of coincident conjugate diameters, and we infer that the least angle made by a pair of conjugate diameters is 0.

In the equilateral hyperbola, each diameter is equal to its conjugate, but in the acute and obtuse hyperbolas, there are no equal conjugate diameters.

2. The parallelogram described upon any pair of conjugate diameters, is constant for the same hyperbola.

3. The difference of the squares described upon any pair of conjugate diameters is constant for the same hyperbola.

4. If perpendiculars be drawn from the foci to any tangent, the locus of these points of intersection with the tangent is the circumference of a circle, of which the transverse axis is a diameter.

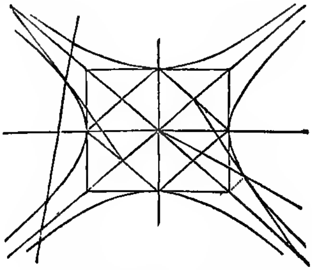
5. The rectangle of any pair of these perpendiculars is equivalent to the square described upon the semi-conjugate axis.

6. Perpendiculars let fall from the foci upon any tangent, are to each other as the focal distances of the point of contact.

7. If two tangents be drawn to the curve, one at the principal vertex, and the other at the vertex of any diameter, each meeting the other diameter, the tangential triangles formed are equal.

8. If the curve be referred to its asymptotes, the parallelogram of the co-ordinates of any point is constant for the same curve.

9. If a tangent be drawn to the curve at any point, and limited by the asymptotes, that portion of it is bisected at the point of contact.



10. If a secant be drawn, cutting the curve in two points, and limited by the asymptetes, the parts intercepted between the curve and asymptotes are equal.

The last mentioned properties give rise to constructions for the curve and tangent that are far simpler than any heretofore considered.

Take any pair of conjugate diameters, axes, or otherwise, and upon them construct a parallelogram; draw its diagonals; these will be the asymptotes; through the vertex of the diameter which cuts the curve, draw any straight line, and lay off from the point in which it cuts one asymptote, a distance from the assumed point to the other asymptote; the extremity of this distance is a point of the curve; in this manner any number of points may be found, and the curve described through them.

To draw a tangent at a given point. Draw through it a line parallel to one asymptote, till it meets the other; from its foot lay off from the centre a distance equal to that from the foot to the centre, and through the point thus found, and the given point, draw a straight line; it will be the tangent required.

It will be observed that the properties of the hyperbola are strikingly analogous to the corresponding properties of the ellipse; in many cases they may be expressed by the same words. See *Ellipse*.

The following analytical expressions are useful.

Let us denote the co-ordinates of any point of the curve by x and y, the co-ordinates of the point of contact by x'' and y'', the semi-transverse axis by a, the semi-conjugate axis by b, any pair of semi-conjugate diameters by a' and b', and the eccentricity by e.

The equation of the curve referred to its centre and axis, is

$$a^2y^2 - b^2x^2 = -a^2b^2 \cdots\cdots (1).$$

The equation of the curve referred to any pair of conjugate diameters, is

$$a'^2y^2 - b'^2x^2 = -a'^2b'^2 \cdots\cdots (2).$$

The equation of the curve referred to any diameter and the tangent at its vertex, is

$$y^2 = \frac{b'^2}{a'^2}(2a'x - x^2) \cdots\cdots (3).$$

The equation of a tangent referred to the centre and axes, is

$$a^2yy'' - b^2xx'' = -a^2b^2 \cdots (4).$$

The equation of a tangent referred to any pair of conjugate diameters, is

$$a'^2yy'' - b'^2xx'' = -a'^2b'^2 \cdots (5).$$

The expression for the sub-tangent, on the axis of X, in the first case, is

$$S = \frac{a^2 - x''^2}{x''} \cdots\cdots (6),$$

and in the second case, it is

$$S = \frac{a'^2 - x''^2}{x''} \cdots\cdots (7).$$

The equation of a normal to the curve at the point (x'', y'') is, when referred to the axes,

$$y - y'' = -\frac{a^2y''}{b^2x''}(x - x'') \cdots (8).$$

And when referred to any pair of conjugate diameters, it is

$$y - y'' = -\frac{a'^2y''}{b'^2x''}(x - x'') \cdots (9).$$

The expression for the sub-normal on the axis of x, in the first case, is

$$S' = -\frac{b^2x''}{a^2} \cdots\cdots (10),$$

and in the second case,

$$S' = -\frac{b'^2}{a'^2}x'' \cdots\cdots (11).$$

The equation of condition for supplementary chords, through the extremities of the transverse axis, and also for conjugate diameters, is

$$\tan a \tan a' = \frac{b^2}{a^2} \cdots\cdots (12),$$

in which a and a' denote the angles which they make with the transverse axis. If the supplementary chords are drawn from the extremities of any diameter, whose length is $2a'$, the equation of condition is

$$cc' = \frac{b'^2}{a'^2} \cdots\cdots (13),$$

in which c and c' are the ratios of the sines of the angles, which the chords or conjugate diameters respectively make with the axes of co-ordinates, or the conjugate diameters.

Any equation of the general form,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \cdot (14),$$

will represent an hyperbola, when

$$b^2 - 4ac > 0.$$

The co-ordinates of its centre are,

$$x' = \frac{2ae - bd}{b^2 - 4ac}, \text{ and } y' = \frac{2cd - be}{b^2 - 4ac}.$$

The polar equation of the hyperbola, when the pole is taken at the right hand focus, is

$$r = -\frac{a(1 - e^2)}{1 + e\cos\phi} \cdots\cdots (15),$$

in which r is the radius vector, and ϕ the angle which it makes with the transverse axis.

Equation (14) represents an equilateral hyperbola, when $b = 0$, and $a = -c$. When $a = 0$ and $c = 0$, it represents an hyperbola referred to lines parallel to its asymptotes.

Hyperbolas of Higher Orders. Every curve whose general equation can be reduced to the form

$$y^m x^n = a,$$

in which m and n are positive whole numbers, is called an hyperbola.

In the case when $m = n = 1$, we have the common hyperbola; all other cases are of the higher orders of hyperbolas. These curves have often, but improperly, been called hyperboloids.

HY-PER-BOL'IC. Appertaining to, or relating to the hyperbola.

Hyperbolic Arc. The arc of an hyperbola. If we denote by d and d' any pair of semi-conjugate diameters, and by y the ex-

treme ordinate of the arc, estimated from the vertex, and make

$$\frac{d^2 + d'^2}{d'^4} = q, \text{ also, } l\left(\frac{y + \sqrt{d'^2 + y^2}}{d'}\right) = A,$$

$$\text{and } \tfrac{1}{2}(y\sqrt{d'^2 + y^2} - d'^2 A) = B,$$

$$\tfrac{1}{4}(y^3\sqrt{d'^2 + y^2} - 3d'^2 B) = C,$$

$$\tfrac{1}{6}(y^5\sqrt{d'^2 + y^2} - 5d'^2 C) = D). \text{ \&c.},$$

we shall have for the length of the arc, denoted by Z,

$$Z = d'\left\{A + \frac{q}{2}B - \frac{q^2}{2\,.\,4}C + \frac{3q^3}{2\,.\,4\,.\,6}D - \frac{3\,.\,5q^4}{2\,.\,4\,.\,6\,.\,8}E + \text{\&c.},\right\} \cdot (1)$$

also the following formula,

$$Z = y\left\{1 + \frac{d^2y^2}{6d'^4} - \frac{d^4 + 4d^2d'^2}{40d'^8} + \frac{(d^6 + 4d^4d'^2 + 8d^2d'^4)\,y^4}{112d'^4}\text{\&c.}\right\} \cdot (2)$$

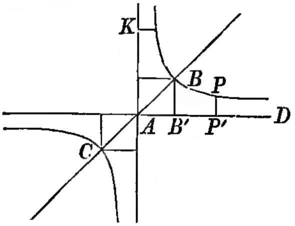
Hyperbolic Area, or Segment. The area of a portion of an hyperbola. To find the area, from the principal vertex to the double ordinate $2y$, we have the following formulas, in which d and d' are, as before, any pair of semi-conjugate diameters.

$$S = 2xy\left\{\frac{1}{3} - \frac{q}{1\,.\,3\,.\,5} - \frac{q^2}{3\,.\,5\,.\,7} - \frac{q^3}{5\,.\,7\,.\,9} - \text{\&c.}\right\} (1)$$

$$S = 2xy\left\{\frac{1}{3} - \frac{1}{5}Aq - \frac{1}{7}Bq - \frac{1}{9}Cq - \text{\&c.}\right\} \cdot\cdot (2)$$

in which A, B, C. &c., and the same as explained in the last article.

Hyperbolic Logarithms. Same as *Naperian logarithms*. See *Logarithms, Naperian.* They are called hyperbolic logarithms, on account of their relation to the area between the hyperbola and its asymptote.



Let B and C be the vertices of an equilateral hyperbola, AK and AD its asymptotes. Draw the line BB′ parallel to AK, and call it 1; draw any ordinate whatever, as PP′ parallel to AK; then will the area BB′P′P be equal to the Naperian logarithm of the abscissa AP′.

There is, however, no reason for calling the Naperian logarithms hyperbolic, rather than any others, for it may easily be shown, that if an acute or obtuse hyperbola be taken instead of an equilateral one, that the corresponding area will be equal to the logarithm of the abscissa of the extreme point, taken in a system whose modulus is equal to the sine of the angle of the asymptotes.

Hyperbolic Paraboloid. A surface whose plane sections are hyperbolas and parabolas. It is a warped surface, and may be generated by a straight line moving in such a manner as to touch two given straight lines, and continue parallel to a given plane; the plane is called the *plane directer.* This is called the *surface of first generation.* If any two elements of the first generation be taken as directrices, and with a plane directer parallel to the directrices of the first generation, and a warped surface be generated, it will be identical with the one already described. This is called the *surface of second generation.* Through any point of this surface two straight lines can always be drawn, which lie wholly in the surface, one an element of the first, the other of the second generation. The plane of these two elements is tangent to the surface at their point of intersection. Hence, to press a plane tangent to the surface at a given point, find the elements of the first and second generations passing through the point; their plane is the tangent plane required.

Every section of the surface made by a plane parallel to a tangent plane, is an hyperbola whose asymptotes are parallel to the elements contained in the tangent plane, and consequently, are similar curves, or conjugate with similar curves. All other sections are parabolas. The elements of the surface divide the directrices proportionally; conversely, if any three straight lines divide two given straight lines proportionally, they are elements of an hyperbolic paraboloid, of which the elements are directrices.

Analytically considered, the hyperbolic paraboloid is a surface of the second order. Its equation may be reduced to the form

$$Mz^2 - Ny^2 + L'x = 0.$$

In this case the sections of the surface parallel to the plane YZ are hyperbolas, and those parallel to the planes XZ and XY are parabolas. The section of the surface by the plane YZ is two straight lines intersecting each other: that is, the plane YZ is tangent to the surface. If two planes be passed through these lines, and the asymptotes of any parallel section, they will include the surface and be asymptotic to it. There are an infinite number of such systems to which the surface may be referred.

HY̆-PER'BO-LOĬD. A surface whose plane sections are either ellipses or hyperbolas. There are two species, those of *one nappe*, and those of *two nappes*.

The hyperboloid of one nappe is a warped surface which may be generated by a straight line moving in such a manner as constantly to touch three straight lines situated in any manner in space. This is called the surface of *first* generation. If we take any three positions of the generatrix as directrices, and move a straight line in such a manner as constantly to touch them, the same surface will be generated. This is called the surface of *second* generation. Through any point of the surface it is always possible to draw two straight lines, which will lie wholly in the surface, viz.: the elements of the *first* and *second* generation. The plane of these lines is tangent to the surface at their point of intersection. Hence, to pass a plane tangent to the surface of an hyperboloid of one nappe at any point, we find the elements of the first and second generation through the point, and pass a plane tangent through them; it will be the tangent plane required. Any plane parallel to the tangent plane intersects the surface in an hyperbola whose asymptotes are parallel to the elements of the surface lying in the tangent plane. All other sections are ellipses.

If the three directrices of either generation are symmetrically disposed with respect to a fourth line, the surface is an *hyperboloid of revolution* of one nappe, the fourth line being the axis of revolution. Every plane through the axis cuts an hyperbola, which, being revolved about the axis, will generate the surface. Every ordinate to the axis, during the revolution, generates a circle; the shortest ordinate generates the smallest circle, called the *circle of the gorge*. The hyperboloid of revolution may also be generated by either one of two straight lines intersecting each other and revolving about an axis parallel to their plane. In this case the perpendicular drawn from their point of intersection to the axis, generates the circle of the gorge. The axis must be so taken that this line shall also be perpendicular to the plane of the lines.

The hyperboloid of two nappes consists of two branches or nappes, each extending to an infinite distance. If a plane be passed tangent to the surface at any point, it will have no other point in common with the surface. Every plane parallel to a tangent plane, which intersects the surface, cuts from it an ellipse. All planes not parallel to a tangent plane intersect the surface in hyperbolas, the rule in this case being exactly the reverse of that in the case of the hyperboloid of one nappe.

Any plane which bisects a system of parallel chords of the surface of an hyperboloid, is called a *diametral* plane. If it is perpendicular to the chords which it bisects, it is a principal plane. The hyperboloids have three principal planes, which intersect in lines called *axes* of the surface. The point common to all the diametral planes is the centre of the surface, and possesses the property of bisecting every straight line drawn through it and terminating in the surface.

If we designate the lengths of the semi-axes of the surface according to their order of magnitude, by a, b, and c, we have, for the equation of the hyperboloid of one nappe, referred to its centre and axes, the equation,

$$a^2b^2z^2 + a^2c^2y^2 - b^2c^2x^2 = a^2b^2c^2 \quad \ldots\ldots (1)$$

and for the equation of the hyperboloid of two nappes,

$$a^2b^2z^2 + a^2c^2y^2 - b^2c^2x^2 = -a^2b^2c^2 \quad \ldots\ldots (2)$$

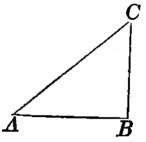
In the first case two of the axes pierce the surface, and the other one does not. The one which does not pierce it coincides with the axis of X.

In the second case the axis coinciding with the axis of X, alone pierces the surface, whilst the other two do not.

If we make $c = b$, the two surfaces become surfaces of revolution, having their axis coinciding with the axis of X

If any number of planes be passed through the centre, cutting out hyperbolas, their asymptotes taken together will form a conic surface asymptotical to the hyperboloid.

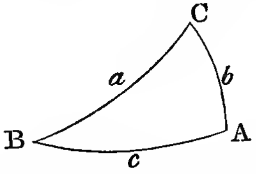
HȲ-POTH′E-NŪSE. [Gr. ὑποτεινω, to subtend]. The side of a right angled triangle opposite the right angle. In a plane triangle the square described upon the hypothenuse is equivalent to the sum of the squares described upon the other two sides.



In the right angled triangle BAC, right angled at B, we have

$AC^2 \asymp BA^2 + CB^2.$

In the right angled spherical triangle, right angled at A, we have the relations expressed in the following



equations, in which the small letters stand for the sides opposite the corresponding angles:

$$\sin a = \cot B \cot C \cdot \cdot \cdot \cdot (1);$$

$$\sin a = \sin c \sin b \cdot \cdot \cdot \cdot \cdot (2).$$

HȲ-POTH′E-SIS. [Gr. ὑποθεσις, a supposition]. A supposition made in the course of a demonstration, or upon the arbitrary constants of a problem during a discussion. The hypothesis made during the course of a demonstration, is introduced as though it were true, and the reasoning continued by the rules for logical argumentation till some result is found which agrees or disagrees with some known truth. If the result agrees with a known truth, the hypothesis is pronounced correct, and is said to be proved; if it disagrees with a known truth the hypothesis is not correct, and the contrary of the hypothesis is proved. In assuming an hypothesis it must be of such a nature that either it or its contrary must necessarily be true. In the discussion of a problem an hypothesis is often made upon the arbitrary constants which enter the equations of the problem, and if real results are obtained, the interpretation shows the consequence of the hypothesis; if imaginary results arise, the hypothesis is pronounced impossible. See *Indirect Demonstration*, and *Discussion*.

I. The ninth letter of the English Alphabet. It forms one of the seven Roman numeral letters, and stands for *one*. When repeated, the number is to be repeated. Thus, II stands for *two*, III for *three*. When written before another numeral letter, one is to be subtracted from the number indicated by that letter; when after it, the number one is to be added. Thus, IV stands for *four*, and VI for *six*.

Ī-CO-SA-HĒ′DRON. [Gr. εικοςι, twenty, and ἑδρα, a base]. A polyhedron bounded by twenty polygons. If the bounding polygons are regular, the polyhedron is a regular icosahedron. See *Regular Polyhedron*.

Ī-DEN′TI-CAL. [L. *idem*, the same]. *Identity* implies sameness under all circumstances. An identical equation is one in which the two members are in reality the same, though often expressed under different forms. Hence, we define an identical equation to be one in which one member is merely the repetition of the other, or in which one member is the result of certain operations indicated in the other.

Sometimes an identical equation takes a form in which one member indicates a certain operation to be performed, whilst the other indicates simply the form of the result which would obtain were the operation performed according to certain indications. Thus,

$$ax + by = ax + by,$$

is an identical equation of the first kind;

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2,$$

is one of the second; whilst

$$\frac{a + bx}{a' + b'x + c'x^2} = P + Qx + Rx^2 + Sx^3 + \&c.$$

is one of the third kind. In it the first member is a fraction, and the second is the indicated form of a series which would result from the division actually performed The quantities P, Q, R, &c., are not determined, but from the nature of identical equations, the conditions which fix their values are

expressed. In every identical equation there is always one or more arbitrary quantities; that is, quantities which may have any value whatever assigned to them without destroying the equality of the members. This follows from the nature and construction of an identical equation. The following properties of identical equations are readily proved:

1st. In every identical equation containing but one arbitrary quantity, in which one member is 0, the co-efficients of the different powers of the arbitrary quantity are separately equal to 0; or when neither member is 0, the co-efficients of the like powers of the arbitrary quantity in the two members are separately equal to each other.

2d. In every identical equation, containing more than one arbitrary quantity, one member of which is 0, the co-efficients of the different powers and combinations of powers of the arbitrary quantities, are separately equal to 0; or when neither member is 0, the co-efficients of the different powers and combinations of powers of the arbitrary quantities in the two members, are separately equal to each other. These two principles are much used in developing analytical expressions into series; they thus afford one of the most potent instruments of analysis. For the practical application of these principles, see *Indeterminate Co-efficients, Taylor's and McLaurin's Theorems.*

IM-AG′IN-A-RY EXPRESSIONS. Indicated even roots of negative quantities, such as

$$\sqrt{-9}, \quad \sqrt[4]{-a^2}, \quad \sqrt[6]{-b^2}, \text{ \&c.}$$

They are called imaginary, because it is impossible to conceive of quantities which they represent, according to the ordinary methods of interpreting Algebraic symbols. We know that any even power of a quantity, whether positive or negative, is always positive, and it is impossible to conceive of such a quantity that being taken an even number of times as a factor, will give a negative result.

Imaginary expressions arise from correct algebraic combinations, and although in an arithmetical point of view their exact value cannot be determined, they are, nevertheless, subject to all the rules of analysis, as much as other expressions. From the greater generality of Algebraic operations, many expressions result which can with difficulty be brought within the range of ordinary interpretation. These expressions are, however, correct expressions of analytical facts, and it only requires a more enlarged view to render their meaning perfectly comprehensible. It will probably be found, on a proper analysis, that the subject of imaginary expressions presents no more difficulties than that of negative quantities, which is now so thoroughly settled as to leave nothing to be desired.

We shall first explain the form to which every imaginary quantity may be reduced, and then give an account of the signification to be attached to imaginary expressions generally. Every imaginary quantity can be reduced to the form

$$a + b\sqrt{-1},$$

in which a and b are real, and $\sqrt{-1}$ imaginary.

To show this, let us assume the well-known formula, for the simplification of radicals,

$$\sqrt{a + \sqrt{b}} = \sqrt{\frac{a+c}{2}} + \sqrt{\frac{a-c}{2}} \quad . (1);$$

in which $c = \sqrt{a^2 - b}$. Making, in formula (1), $b = -b^2$, which gives $c = \sqrt{a^2 + b^2}$, and reducing, it becomes

$$\sqrt{a + b\sqrt{-1}} = \sqrt{\frac{a + \sqrt{a^2+b^2}}{2}} + \sqrt{\frac{a - \sqrt{a^2+b^2}}{2}} \quad \ldots\ldots\ldots (2).$$

In formula (2), the first radical in the second member is positive and real, and may be denoted by c; the quantity under the second radical sign, in the second member, is negative, since $a < \sqrt{a^2 + b^2}$; denoting it by $-d^2$, and reducing, we have

$$\sqrt{a + b\sqrt{-1}} = c + d\sqrt{-1} \quad \ldots (3);$$

which shows that the square root of an expression of the form, $a + b\sqrt{-1}$, is of the same form as the expression itself. Hence, the fourth root of the expression, $a + b\sqrt{-1}$, is of the same form, and so on for the 6th, 8th, or any other even root. This shows that every imaginary expression may ultimately be reduced to the form, $a + b\sqrt{-1}$, which

was to be proved. Hence, in the treatment of imaginary expressions, we need only confine our attention to imaginary expressions of the second degree, and in these we shall only consider the parts involving $\sqrt{-1}$, as a factor. Every such quantity can be placed under the form $b\sqrt{-1}$, by the rule for removing a factor from under the radical sign. For multiplying imaginary expressions together, we make use of the following formulas:

$$(\sqrt{-1})^{4n+1} = \sqrt{-1} \dots\dots (1);$$
$$(\sqrt{-1})^{4n+2} = -1 \dots\dots (2);$$
$$(\sqrt{-1})^{4n+3} = -\sqrt{-1} \dots (3);$$
$$(\sqrt{-1})^{4n} = 1 \dots\dots\dots\dots (4);$$

in which n may be 0, or any positive whole number.

To multiply any number of imaginary expressions together, reduce them to the form $b\sqrt{-1}$; multiply the co-efficients of $\sqrt{-1}$ together for one factor of the product: to find the other factor, look in the above formulas for that one in which the exponent of $\sqrt{-1}$ equals the number of factors to be multiplied together; the second member of it will be the second factor of the required product.

Thus, let it be required to find the 17^{th} power of $\sqrt{-a^2}$. Reducing to the required form, we have $a\sqrt{-1}$; the 17^{th} power of the co-efficient is a^{17}; making $n = 4$, we see that formula 1 is applicable; hence, the second factor is $\sqrt{-1}$, and the required power is $a^{17} \times \sqrt{-1}$.

This rule will cover all operations, which differ from the corresponding operations for real quantities.

With respect to the logical value of the symbol $\sqrt{-1}$, it may be remarked that there are two separate views that may be taken of the expression. In the first place, we may regard it as a symbol of *operation*, in which case it indicates an operation absolutely impossible; for no quantity whatever, taken twice as a factor, can produce -1. In this sense, the quantity indicated by the expression, is truly *imaginary* or *impossible*. The expression may, however, be regarded as a *symbol of interpretation;* that is, it may be an expression resulting from the correct application of the principles of analysis. In this point of view, it admits of complete and satisfactory interpretation. The method of interpretation, which we are about to give, is due to M. Monrey, a distinguished modern analyst. Before proceeding to give an account of his method of interpretation, some preliminary explanations are necessary.

We have seen that every imaginary expression can be reduced to the form $a + b\sqrt{-1}$. The expression, $\sqrt{a^2 + b^2}$, is called the *modulus* of the expression. It is a property of these expressions, that, if two of them be multiplied together, the resulting product will be of the same form as each factor, and its modulus will be equal to the product of the moduli of the two factors.

Thus,

$$(2 + 3\sqrt{-1})(3 - 7\sqrt{-1}) = 27 - 5\sqrt{-1}.$$

The modulus of the first factor is $\sqrt{13}$, that of the second is $\sqrt{58}$, and that of the product, $\sqrt{754} = \sqrt{13} \times \sqrt{58}$.

In general, if any number of factors of the given form be taken, their product will be of the same form, and its modulus will be equal to the product of the moduli of all the factors.

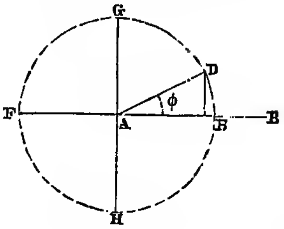
We may now proceed to an examination of M. Monrey's explanation of imaginary results.

If we take the expression $a + b\sqrt{-1}$, and denote its modulus by M, we shall have, for the expression

$$M\left(\frac{a}{M} + \frac{b}{M}\sqrt{-1}\right).$$

By inspection, we see that if $\frac{a}{M}$ is taken for the cosine of an angle ϕ, $\frac{b}{M}$ will represent the sine of the same angle, and by substitution, the expression becomes

$$A(\cos\phi + \sin\phi\sqrt{-1}).$$



Let A be the origin of a system of polar

co-ordinates, AB the initial line, and ϕ the angle made with it by any straight line AD. If now the length of the line AD be taken equal to M, then will the line AD fulfill two conditions, viz.: it is of a given length M, and makes with the initial line an angle ϕ, which conditions make up the *relation* of the line AD to the system.

The angle ϕ is called the *verser*, and the given expression represents the line AD, both in length and position; or, in other words, expresses the relation of the line AD to the system of polar co-ordinates.

If $\phi = 0$, the line takes the position AE, and the given expression reduces to M, and we have also $M = a$. This corresponds to our conventional system of representing a positive quantity by a straight line of definite length, estimated from a fixed point towards the right. If $\phi = 180°$, the expression becomes $-M$, and the line takes the position AF, which also corresponds to the same system. of $\phi = 90°$, the expression becomes $M\sqrt{-1}$, and the line takes the position AG. If $\phi = 270°$, the expression becomes $-M\sqrt{-1}$, and the line takes the position AH. In any given expression, the value of ϕ may be found from the equation

$$\cos\phi = \frac{a}{\sqrt{a^2+b^2}};$$

and this, together with the value of $M = \sqrt{a^2+b^2}$, will serve to determine the relation of the radius vector to the system.

This method of representation conforms perfectly to every case of an expression of the form $a + b\sqrt{-1}$; it now remains to explain the results obtained by operating upon it by the rules of algebra.

Let us consider the result of multiplying

$$a + b\sqrt{-1} \text{ by } c + d\sqrt{-1}.$$

Performing the multiplication, we have for the product,

$$(ac - bd) + (ab + bd)\sqrt{-1}.$$

Now, the angle which the line whose relation is given by this product, makes with the initial line an angle y, whose cosine equals

$$\frac{ac - bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}},$$

that is,

$$\cos\gamma = \frac{ac - bd}{\sqrt{(a^2+b^2)(c^2+d^2)}};$$

If we denote the angles made by the first and second lines with the initial line, by a and β, we shall have,

$$\cos a = \frac{a}{\sqrt{a^2+b^2}},\quad \cos\beta = \frac{c}{\sqrt{a^2+d^2}},$$

$$\sin a = \frac{b}{\sqrt{a^2+b^2}},\quad \sin\beta = \frac{d}{\sqrt{c^2+d^2}};$$

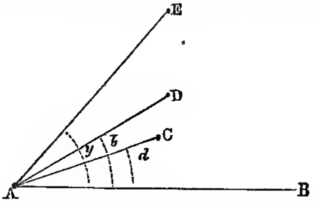
we have also,

$$\cos(a+\beta) = \cos a\cos\beta - \sin a\sin\beta$$
$$= \frac{ac - bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}},$$

and consequently

$$\cos\gamma = \cos(a+\beta).$$

Hence, the product represents the relation of the line AE to the system, whose length AE



is equal to the length AC, taken as many times as there are units of length in AD, and making with the initial line an angle equal to the sum of the angles which the lines AC and AD make with it.

In general, the product of any number of factors of the given form, represents the relation of a line to the system, which is equal in length to the length of any one of the lines taken as many times as there are units in the continued product of the number of units in each of the other lines taken separately, and which makes, with the initial line, an angle equal to the sum of the angles made by each line with the initial line. When this angle is any multiple of 180°, the product becomes real. The entire subject of imaginary quantities may be clearly explained, and as we see, without any impossible circumstances arising.

We see, then, that to interpret the expression $\sqrt{-a^2}$, we have simply to regard it as the representation of a straight line perpendicular to the initial line at the origin, and equal in length to a. Whilst the expression $-\sqrt{-a^2}$ represents a line equal and directly opposed to that represented by $\sqrt{-a^2}$

Thus interpreted, every idea of impossibility disappears from the mind, and the subject becomes as plain as the interpretation of negative results.

IMAGINARY ROOTS. It is a principle of Algebra, that if an equation, having real co-efficients, contains any imaginary roots, it will contain an even number of them, and all of them must, from the preceding article, be particular cases of the general form $a + b\sqrt{-1}$. It has been shown, that for every root of the form $a + b\sqrt{-1}$, there is a root of the form $a - b\sqrt{-1}$; this principle is expressed by saying that imaginary roots enter by pairs.

Sometimes a real root of an equation may be expressed in an imaginary form; we have an example in the roots of a cubic equation as solved by Cardan's method. The number of imaginary roots of an equation may be ascertained by means of Sturm's Theorem; for by it we may find the number of real roots, and then taking this from the number denoting the degree of the equation, the remainder will denote the number of imaginary roots.

IM-PER′FECT NUMBER. [L. *imperfectus*, *in*, and *perfectus*, finished]. A number, the sum of whose divisors is not equal to the number itself.

When this sum is less than the number, the number is said to be *defective*; when greater, it is *abundant*. Thus, 10 is a defective number because $1 + 2 + 5 < 10$, and 12 is an abundant number, because $1 + 2 + 3 + 4 + 6 > 12$. See *Number*.

IMPERFECT POWER. A number whose root cannot be expressed in exact parts of 1, that is, a number such that no whole number or vulgar fraction can be found, which, taken any number of times as a factor, will produce the given number. Thus, 5 is an imperfect power. Some numbers may be imperfect powers of one degree, but perfect powers of another degree; thus, 8 is an imperfect square, but a perfect cube. Hence, in speaking of imperfect powers, it is customary to designate the degree of the power referred to; thus, we say an imperfect square, an imperfect cube, and imperfect n^{th} power, and so on.

IM-PLĬC′IT FUNCTION. [L. *implicitus*, from in, and *plico*, to fold]. An expression in which the form of the function is not directly given, but which requires some operation to be performed, to render it evident. Thus, in the equation,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

y is an implicit function of x. See *Function*.

IM-POS′SI-BLE. [L. *impossibilis*, from in, and *possum*, to be able]. In Analysis, the same as imaginary. We sometimes speak of impossible equations, impossible roots, impossible expressions, &c., meaning the same as imaginary equations, roots, expressions, &c. The term imaginary is preferable. See *Imaginary*.

IM-PROP′ER FRACTION. [L. *improprius*, from in, and *proprius*, proper]. A vulgar fraction whose numerator is greater than its denominator. Thus, $\frac{8}{6}$ is an improper fraction. See *Fractions*.

IN-AC-CESS′I-BLE. In Surveying, a distance or height which cannot be reached, for the purpose of measuring it directly, on account of some obstacle. See *Distances* and *Heights*.

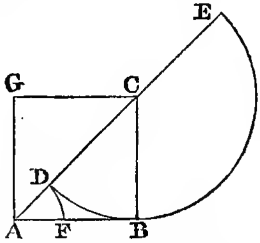
INCH. [L. *uncia*, the twelfth part]. A measure of length equal to the twelfth part of a foot.

IN-CLĬ-NĀ′TION. [L. *inclinatio*, an inclining]. The inclination of one line to another, or of one plane to another, is the same as the angle which they make with each other.

IN-COM-MEN′SU-RA-BLE. Two quantities of the same kind are incommensurable with respect to each other, when they have not a common unit, that is, when there is no quantity of the same kind so small that it is contained in both an exact number of times. Thus, the diagonal and side of a square are incommensurable, for it has been shown, that if we denote the side of the square by 1, the diagonal will be denoted by $\sqrt{2}$; but the square root of 2 is incommensurable with 1, because the square root of an imperfect square cannot be expressed in exact terms of 1; were we to extract the square root of 2 and carry on the operation decimally, there would be an infinite number of decimal places in the result. We may, however, find a number which will approximate to a common unit as closely as we wish; for example, were we to stop the process of extraction of the square root at the 5th place of decimals, and take the result as the true value of $\sqrt{2}$,

then would .00001 be a common unit of 1 and $\sqrt{2}$, approximately. No two whole numbers or vulgar fractions are incommensurable: for, let $\frac{a}{b}$ and $\frac{c}{d}$ be any two vulgar fractions, then since they may be transformed into the equivalent fractions $\frac{ad}{bd}$ and $\frac{bc}{bd}$, it is clear that they have a common unit $\frac{1}{bd}$, the first containing this unit ad times, and the second bc times. Prime quantities which can only be expressed by series, or by decimals having an infinite number of places of figures, are incommensurable with those which can be otherwise expressed.

The following is the geometrical method of proving that the side of a square and its diagonal are incommensurable, and it also gives a very clear idea of the meaning of the term incommensurable. If the two lines have no common divisor they are of necessity, incommensurable.



Let ABCG be a square, and AC one of its diagonals. We first apply CB to CA. For this purpose, let the semi-circumference DBE be constructed from C as a centre, with the radius CB, and produce AC to E. It is evident that CD is contained once in CA, with the remainder AD. The first result is, therefore, a quotient 1, with the remainder AD. This remainder must now be compared with CB, or its equal AB.

Since the angle ABC is a right angle, AB is tangent to the arc DBE, and AE a secant terminating in the concave arc, whence

$$AD \;.\; AB :: AB \cdot AE.$$

Hence, in the second operation, where AD is compared with AB the equal ratio, AB to AE may be taken instead; but AB or its equal CD is contained twice in AE, with the remainder AD. The result of the second operation, therefore, is a quotient 2, with the remainder AD; and this must be again compared with AB. Then, the third operation consists in comparing AD with AB, which gives, as before, the quotient 2, with a reremainder AD, and so on indefinitely. Since the process will never terminate, there is no remainder which is contained an exact number of times in the preceding divisor, and consequently the lines AC and BC have no common measure; they are therefore incommensurable.

IN-COM-PLĒTE′ EQUATION. An equation, some of whose terms are wanting; or an equation in which the co-efficient of some one or more of the powers of the unknown quantity are equal to 0. See *Equation.*

IN-CON′′GRŪ-OUS NUMBERS. See *Congruous Numbers*.

IN-CRĒASE′. [L. from *in* and *cresco*, to grow]. To augment, to make greater by addition.

INCREASING FUNCTION. A function that increases as the variable increases, and of course decreases as the variable decreases. See *Function.*

IN′CRE-MENT. [L. *incresco*, to increase]. A quantity, generally variable, added to the independent variable in a variable expression. The function also undergoes a corresponding change, which is called an increment or decrement, according as the function is increasing or decreasing. When the increment or decrement is infinitely small, it is called a *differential.*

IN-DEF′IN-ĬTE. [L. *indefinitus*, indefinite]. Unbounded or unlimited. That portion of a straight line included between any two of its points is definite, and is called a definite straight line; but if the direction of the line only is given, it is supposed to extend in both directions from any point of it without limit; such a line is, properly speaking, an indefinite line. If we speak of that portion of a straight line which lies entirely on one side of any point of it, it is said to extend indefinitely in that direction. A plane extends indefinitely in all directions, unless limited by a boundary: it may be limited in one or more directions by a line or lines, and indefinite in all other directions.

Space is indefinite in all directions, unless limited by a surface; when so limited, it is indefinite in all other directions. Properly speaking, space is indefinite in the most enlarged sense of the word, but for convenience of speaking, we are led to admit the distinctions above drawn.

The term indefinite is often and erroneously used as synonymous with infinite. Thus, it is common to speak of a magnitude as indefinitely great or small, of a polygon with an indefinite number of sides, &c., in all of which cases, it is better to use the term infinite, as that is the only correct term to express the idea intended to be conveyed.

Whenever lines or surfaces are given by their equations, if they are not from their nature necessarily limited, the equation stands for them in their indefinite sense; thus, the equation

$$y = ax + b,$$

is the equation of a straight line indefinite in length.

There is another sense in which the word indefinite is used in analysis; for example, in the equation above given, so long as the constants a and b are not given, but remain arbitrary, the position of the line is said to be indefinite. In this case. the term *arbitrary* is better.

IN-DE-PEND′ENT. One quantity is said to be independent of another with which it is connected, when it does not depend upon it for its value. In this case, the term is nearly synonymous with *arbitrary*, but not quite, as we shall presently show. In an equation containing more than one variable, as does the equation of any magnitude, all the variables, except one, are independent; that is, any value may be assigned to them at pleasure, and the corresponding value of the other will be found for the solution of the equation. Thus, in the equation of the straight line,

$$y = ax + b,$$

we may take x as the independent variable, in which case, whatever be the value assigned to it, the corresponding value of y may be found. The assumed and deduced values determine a point upon the line. The variables x and y represent, at the same instant, the co-ordinates of every point of the line, x independently, y dependently; that is, subject to the form of the equation, and to the values of a and b. The quantities a and b serve to determine the position of the line, with respect to the co-ordinate axes, and may be assumed at pleasure. These are called *arbitrary*. That is, the arbitrary quantities admit of *any set of values*, whilst the variables have necessarily at the same instant every possible value that will satisfy the equation. This constitutes the difference between *independent* and *arbitrary* quantities, which is exactly the same as that between the words *any* and *every*. Equations are independent when they have no connection with each other; that is, when the quantities entering the different equations are not at all dependent upon each other. See *Simultaneous Equation*.

IN-DE-TERM′IN-ATE. A *quantity* is *indeterminate* when it admits of an infinite number of values. In the equation of a straight line,

$$y = ax + b,$$

x represents the abscissa of any point of the line, and is indeterminate when considered only with reference to its value; when considered with reference to its connection with y, it is *independent* of it, provided we agree to assume it as the independent variable. See *Independent*.

Indeterminate Equation. An equation is indeterminate when the unknown quantities which enter it admit of an infinite number of values; the equation of the right line is an example of an indeterminate equation; in general, most of the equations used in analysis are indeterminate.

Whenever an equation contains more than one arbitrary or unknown quantity, that, considered by itself is indeterminate, for any number of sets of values may be attributed to all the unknown quantities, except one, and the value of that one deduced. The assumed and deduced values satisfy the equation, and therefore the unknown quantities admit of an infinite number of systems of values, each of which satisfies it; hence it is indeterminate by definition. In like manner, a group of equations containing more unknown quantities than there are equations, is indeterminate.

Indeterminate Problem. A problem is

indeterminate when it admits of an infinite number of solutions. This will always be the case when there are fewer imposed conditions than there are unknown or required parts; for, in that case, the equations which express the imposed conditions will be fewer than the number of unknown quantities which enter them; consequently, they will be indeterminate, and of course the problem itself will also be indeterminate.

INDETERMINATE ANALYSIS. A branch of analysis which has for its object the solution of indeterminate problems. A problem is indeterminate when it admits of an infinite number of solutions. In all cases, when the conditions of a problem do not furnish as many independent equations as there are unknown quantities, the equations of the problem, and consequently the problem itself, is indeterminate. In most cases, the conditions require the solutions to be expressed in whole numbers, and these conditions often greatly diminish or restrict the number of solutions. Indeterminate analysis may be of the *first*, *second*, or *higher* degrees, according as the equations arising are of the first, second, or higher degrees.

As an example, let it be proposed to divide 159 into two such parts, that one shall be divisible by 8, and the other by 13.

If we denote the quotients by x and y respectively, we shall have, for the equation of the problem,

$$8x + 13y = 159 \ldots (1).$$

Let it be required, in addition, that the results shall be whole numbers.

We have, from equation (1),

$$x = \frac{159 - 13y}{8} = 19 - y + \frac{7 - 5y}{8}.$$

In order that x and y may be whole numbers, it is necessary that

$$\frac{7 - 5y}{8}$$

should be a whole number also. Denote this number by n, and we shall have

$$\frac{7 - 5y}{8} = n \quad \text{or} \quad 8n + 5y = 7 \ldots (2),$$

an equation of the same form as (1). Finding the value of y, we have

$$y = \frac{7 - 8n}{5} \quad \text{or} \quad y = 1 - n + \frac{2 - 3n}{5}.$$

Now, as before, since y and n are to be whole numbers, so must

$$\frac{2 - 3n}{5}$$

be a whole number: denoting this by n', we have

$$\frac{2 - 3n}{5} = n';$$

or, clearing of fractions,

$$3n + 5n' = 2 \ldots (3).$$

Continuing this process of transformation, we shall obtain the equations

$$2n' + 3n'' = 2 \ldots (4),$$
$$n'' = 2n''' \ldots (5).$$

For any value assigned to n''', which is entire, all the quantities n'', n', n, y and x, will be entire, and the last two will form answers to the problem. Collecting the equations, and substituting, we have

$$\left.\begin{aligned} x &= 19 - y + n \\ y &= 1 - n + n' \\ n &= \quad - n' + n'' \\ n'' &= 2n''' \end{aligned}\right\} \ldots (6).$$

Combining the equations in group (6), and eliminating, we have, finally,

$$x = 15 + 13n''' \quad \text{and} \quad y = 3 - 8n''' \ldots (7)$$

Making n''' successively equal to

$$0, 1, 2, 3, \text{\&c.}, -1, -2, \text{\&c.},$$

we shall find all the values of x and y, which will satisfy the conditions of the problem. If the conditions of the problem had required that the solutions should all be positive whole numbers, such values must be given to n''' as will make both $15 + 13n'''$ and $3 - 8n'''$ positive. The only values for n''', which will satisfy these conditions, are

$$n''' = 0 \quad \text{and} \quad n''' = -1;$$

these, being substituted in equation (7), give for the only solutions

$$x = 15, \ y = 3, \quad \text{and} \quad x = 2, \ y = 11.$$

All other indeterminate problems of the first degree, involving but two unknown quantities, may be solved in a similar manner, and by extending the principles, rules may be formed for solving all indeterminate problems of the first degree, involving any number of unknown quantities.

For the method of solving indeterminate problems of the second degree, the reader is

referred to the works of Legendre, Gauss, Barlow, Euler, Lagrange, &c.

The following indeterminate formulas are taken from Barlow's Theory of Numbers:

1. $ax - by = \pm c,$

Value of x . . . $x = mb \pm cq,$
" " y . . $y = ma \pm cp.$

In which m is indeterminate, and p and q result from the solution of the equation

$$ap - bq = \pm 1.$$

2. $ax + by = c.$

Value of x . . . $x = cq - mb,$
" " y . . . $y = ma - cp.$

In which m is indeterminate, and p and q may be found from the equation $ap - bq = \pm 1$.

3. $ax + by + cz = d.$

Value of $x = (d - cz)q - mb,$
" " $y = ma - (d - cz)p.$

In which m is indeterminate, z any whole number less than $\frac{d}{c}$, and p and q the same as in (1) and (2).

4. $x^2 - ay^2 = z^2.$

Value of x . . $x = p^2 + aq^2,$
" " y . . . $y = 2pq,$
" " z . . . $z = p^2 - aq^2$

in which p and q may be assumed at pleasure.

5. $x^2 + ay^2 = z^2.$

Value of x . . . $x = p^2 - aq^2,$
" " y . . . $y = 2pq,$
" " z . $z = p^2 + aq^2;$

in which p and q are entirely arbitrary.

6. $ax^2 + bxy + y^2 = z^2.$

Value of x . . $x = 2pq + bq^2,$
" " y . . . $y = p^2 - aq^2,$
" " z . . . $z = p^2 + bpq + aq^2;$

in which p and q are entirely arbitrary.

7. $ax^2 + bx = z^2.$

Value of x . . $x = \frac{bq^2}{p^2 - aq^2},$

" " z . . . $z = \frac{bnq}{p^2 - aq^2};$

in which p and q are arbitrary.

8. $m^2x^2 + bx + c = z^2.$

Value of x . . . $x = \frac{p^2 - cq^2}{bq^2 - 2mpq},$

" " z . . . $z = \frac{mp^2 + mcq^2 - bpq}{bq^2 - 2mpq};$

in which p and q are indeterminate.

9. $ax^2 + bx + m^2 = z^2.$

Value of x . . . $x = \frac{bq^2 - 2mpq}{p^2 - aq^2},$

" " z . . . $z = \frac{mp^2 + amq^2 - bpq}{p^2 - aq^2};$

in which p and q are, as before, entirely arbitrary.

Many other forms might be given, but the reader desirous of examining them is referred to Barlow's theory of numbers.

Indeterminate Co-efficients. It has been stated that an identical equation is true for all values of the arbitrary quantity or quantities which enter it. If, in such an equation, all the terms be transposed to one member of the equation, the co-efficients of the different powers of the arbitrary quantity are called *indeterminate co-efficients*, not because they are themselves indeterminate, but because they are co-efficients of indeterminate quantities. These co-efficients are in reality each equal to 0. This is called the principle of indeterminate co-efficients, and may be enunciated as follows:

1. In every identical equation, containing but one indeterminate quantity, the second member of which is 0, the co-efficients of the different powers of that quantity are separately equal to 0.

Or, in every identical equation, containing but one indeterminate quantity, the co-efficients of the different powers of that quantity in the two members are separately equal to each other.

2. In every identical equation containing more than one indeterminate quantity, the second member of which is 0, the co-efficients of the different powers and combinations of powers of these quantities, are separately equal to 0.

Or, in every identical equation containing more than one indeterminate quantity, the co-efficients of the different powers and combinations of powers in the two members, are separately equal to each other.

These principles are of extensive application in analysis. We shall give but a single example, namely, that of developing an expression into a series.

Let it be required to develop the expression $\frac{1}{1+x}$ into a series, according to the ascending powers of x.

Assume a development of the required form

$$\frac{1}{1+x} = P + Qx + Rx^2 + \&c.$$

in which P, Q, R, &c., are to be determined so that the equation shall be true for *all values* of x; that is, so that the equation shall be identical. Clearing the equation of fractions,

$$1 = P + \left.\begin{matrix} Q \\ +P \end{matrix}\right| x + \left.\begin{matrix} R \\ +Q \end{matrix}\right| x^2 + \&c.$$

Equating the co-efficients of the like powers of x in the two members, we have

$$1 = P,\ Q + P = 0,\ R + Q = 0,\ \&c.;$$

whence,

$$P = 1,\ Q = -1,\ R = +1,\ \&c.$$

Hence, by substitution,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \&c.$$

Any other expression may be treated in a similar manner; hence the following rule:

Place the expression to be developed equal to a series of the proposed form, the co-efficients of which are to be determined so as to make the equation identical. Clear the equation of fractions and equate the co-efficients of the like powers of the unknown quantity in the two members separately, and from these equations find the values of the co-efficients; substitute these in the assumed development. and the result will be the required series. We have only explained the manner of developing an expression containing but one unknown quantity; but the rule may be extended to the development of a quantity containing any number.

The various applications of the principle of indeterminate co-efficients are too numerous to admit of numeration in an article like this; in fact they extend throughout every branch of the higher mathematics and philosophy, and upon them depends a large share of the most important demonstrations of both.

IN′DEX. [L. *in*, from, and *dico*, to say]. The index of a radical is a number written over the radical sign to denote the degree of the root to be extracted. Thus, in the expressions

$$\sqrt[3]{a},\quad \sqrt[4]{b},\quad \&c.,$$

3, 4, &c., are called indices. An index is generally a whole number greater than 2. When the square root is indicated, the index is generally omitted, being understood.

There is a second method of representing radicals by means of fractional exponents; thus,

$$\sqrt[3]{4},\quad \sqrt[7]{5},\quad \sqrt[n]{c},\quad \&c.,$$

may be written

$$(4)^{\frac{1}{3}},\quad (5)^{\frac{1}{7}},\quad (c)^{\frac{1}{n}},\quad \&c.$$

In these cases the fractional exponent is often called the fractional *index*.

IN-DI-RECT′ DEMONSTRATION. See *Demonstration*.

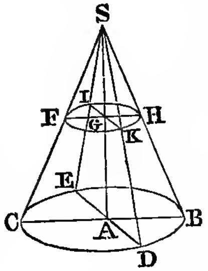
IN-DI-VIS′I-BLE. That which cannot be exactly divided. One quantity is said to be indivisible by another when no commensurable expression can be found, which, being multiplied by the latter, will give the former. See *Division*.

IN-DI-VIS′I-BLES. In ancient geometry the same as infinites—small or infinitely small quantities.

According to the views of the first inventor of the system of indivisibles, lines are made up of an infinite number of points, surfaces of an infinite number of lines, and volumes of an infinite number of surfaces. This corresponds with the idea of fluents and fluxions, as originally conceived. We subjoin an example of the method of reasoning. Let it be required to deduce an expression for the volume of a cone.

Let SBC be a cone, S its vertex, SA a perpendicular let fall upon the base. Let SA be divided into an infinite number of equal parts, and through each point of division let a plane be passed parallel to the base According to the idea of indivisibles each of these sections is an element of the volume, and the sum of these elements is equal to the volume of the cone Now, from the princi-

ples of elementary geometry these elements are to each other as the squares of their distances from the vertex. If we denote the area of the base by A, and that of the parallel element FH by a, we shall have



$A : a :: SA^2 : SG^2$;

whence, $$a = A \cdot \frac{SG^2}{SA^2} = \frac{Ah'^2}{h^2},$$

h denoting the altitude of the cone, and h' the distance of the element FH from the vertex. If V be taken to represent the volume of the cone, it will be equal to $\frac{A}{h^2}$, multiplied by the sum of the different values of h' for every possible section. But the distances h' increase from the vertex as the series of natural numbers 0, 1, 2, 3, 4, &c., up to h inclusive. But the sum of the series from 0 up to the limit h, the whole number of terms being infinite, is equal to $\frac{1}{3}h^3$; hence

$$V = \tfrac{1}{3} A \times h,$$

a well known formula for the volume of a cone.

IN-DUC′TION. [L. *inductio*, from *in* and *duco*, to lead]. The method of induction in its true sense is not known in pure mathematics, except perhaps in the processes of establishing the axioms. It is often, however, employed in inferring principles which are afterwards submitted to a process of rigid demonstration. There is a mathematical process of demonstration which possesses somewhat the character of induction, inasmuch as a general truth is gathered from the examination of particular cases, but it differs from it inasmuch as each successive case is made to depend upon the preceding one. This process has been called the process of *successive induction*. The following demonstration is sufficient to illustrate the process.

Let it be required to prove that the difference between the like powers of two quantities is exactly divisible by the difference of the quantities.

Let $x^n - a^n$ denote the difference of the like powers of two quantities, n being any whole number; then will $x - a$ denote the difference between the quantities. We have the relation

$$x^n - a^n = x(x^{n-1} - a^{n-1}) + a^{n-1}(x - a),$$

as may readily be shown by performing the operations indicated in the second member, and reducing. In the second member the last term is evidently divisible by $(x - a)$, consequently the second member itself, and of course the first member, will be divisible by $x - a$ if $x^{n-1} - a^{n-1}$ is divisible by $x - a$. This shows that *the difference of the like powers of two quantities is exactly divisible by the difference of the quantities, if the difference of the powers of the quantities of a degree less* 1 *is thus divisible.* Now we know that $x^2 - a^2$ is divisible by $x - a$, hence from the above principle $x^3 - a^3$ is thus divisible. Again, since $x^3 - a^3$ is divisible, $x^4 - a^4$ is thus divisible, and so on. Since this process may be carried on to any extent, we infer generally that $x^n - a^n$ is divisible by $x - a$, n being any whole number whatever.

There are cases where we reason by successive steps, as above, but infer in every case from two preceding conclusions. For example, to demonstrate the universality of the formula

$$x^n + \frac{1}{x^n} = 2\cos(n\theta);$$

this is a consequence of the relation

$$x + \frac{1}{x} = 2\cos\theta.$$

No one case of this can be proved without showing that the preceding two cases hold true. Thus, if it is true when $n = 1$ and $n = 2$, it is necessarily true when $n = 3$. If true when $n = 2$ and $n = 3$, it is so when $n = 4$, and so on; hence the generality of the formula is proved.

IN-E-QUAL′I-TY. [L. *in*, and, *æqualitas*, equal]. In Algebra, the expression of two unequal quantities, connected by the sign of inequality. Thus, $2 < 3$, $4 > 1$, are inequalities. Every inequality consists of two parts: that on the left of the sign of inequality, is called the first member; that on the right is called the second member.

Two inequalities are said to exist in the same sense when the first members are both

greater, or both less, than the second members. They exist in a contrary sense when the first member is greater than the second in one inequality, and the second member greater than the first in the other.

The inequalities,

$$4 > 2, \quad 5 > 3, \quad \text{or} \quad 7 < 9, \quad 11 < 13,$$

are said to exist in the same sense. The inequalities, $7 < 9$ and $8 > 4$, exist in a contrary sense.

The following transformations may be made in inequalities:

1. If we add the same quantity to both members of an inequality, or subtract the same quantity from both members, the resulting inequality will exist in the same sense.

2. If two inequalities exist in the same sense, and we add them member to member, the resulting inequality will exist in the same sense. But this is not always the case when we subtract them member from member.

3. If both members of an inequality be multiplied by the same positive quantity, the resulting inequality will exist in the same sense. If both members be multiplied by the same negative quantity, the resulting inequality will exist in a contrary sense.

4. If both members of an inequality are positive, and if both be squared or raised to any power, the resulting inequality will exist in the same sense.

If both members of an inequality are negative, and if both members be squared or raised to any even power, the resulting inequality will exist in a contrary sense.

These principles enable us to find from a given inequality another, in which one member will contain the unknown quantity only. Such operation is called solving the inequality.

IN-E-QUA′TION. The same as inequality. See *Inequality.*

IN′FER-ENCE. [From *in*, and *fero*, to bear]. A conclusion—a truth drawn from another which is admitted, or which has been proved.

IN-FĒ′RI-OR. The inferior *limit of the roots of an equation*, is a number less than the least root of the equation. It is evident from this definition, that there may be an infinite number of such limits. The greatest one, or the greatest one in whole numbers, is the one generally referred to. See *Limit.*

IN-FIN′I-TY. A term employed in mathematics, to express *a quantity greater than any assignable quantity of the same kind.* Mathematically considered, *infinity* is always a limit of a variable quantity, resulting from a particular supposition made upon the varying element which enters it.

In order to illustrate, let us consider the fraction $\frac{a}{x}$, in which a retains the same value throughout, whilst x is entirely arbitrary. If, now, the value of x become smaller and smaller, that of the fraction will become greater and greater. If x becomes exceedingly small with respect to a, the value of the fraction becomes exceedingly great, and, finally, when x becomes smaller than any assignable quantity, the fraction becomes *greater than any assignable quantity;* it is this value that we call *infinity*, and designate by the symbol ∞.

In consequence of the technical meaning of the term, infinity, having been confounded with its absolute or popular meaning, a great deal of metaphysical discussion has arisen as to the propriety of employing it in mathematics.

Without entering upon any of these discussions, which after all are merely verbal, we shall endeavor to explain as clearly as possible the proper signification of the term. This may best be done by citing some particular instances of its appropriate application and use.

In Arithmetic, *infinity* is the limit or last term of the series of natural numbers. This series is an arithmetical progression, each term of which is derived from the preceding one by the addition of the unit 1. It is plain that each term of the series is greater than the preceding one, and if a term be taken sufficiently remote, it may be regarded as greater than any assignable number, or as infinite. In like manner, if we regard the decreasing series of natural numbers,

$$0, \; -1, \; -2, \; -3, \; \&c.,$$

we may regard its final limit as minus infinity; hence, the two limits of all numbers, both positive and negative, are

$$+\infty \quad \text{and} \quad -\infty.$$

In Algebra, the idea of infinity may be obtained by considering the following problem:

Two couriers travel on the same line, and in the same direction, the foremost courier at the rate of m miles per hour, and the rearmost one at the rate of n miles per hour. At a certain time they are distant from each other a miles; in how many hours from that time are they together?

If we designate the required number of hours by t, we shall find, by solving the problem, the relation,

$$t = \frac{a}{n - m}.$$

In assigning particular values to m and n, and interpreting the results, there arises the case in which $m = n$. This supposition gives

$$t = \frac{a}{0} = \infty.$$

To understand the meaning of infinity in this case, we have only to consider the nature of the problem in a common sense point of view. If m is not quite equal to n, but a little smaller, then will t be very great, and the value of t will increase as the difference $n - m$ is diminished; and, finally, it is plain that t is greater than any assignable number, or, according to the definition of the term, is *infinite*. In fact, it is plain, that if the couriers are separated by a distance n miles, and travel both at the same rate, as the supposition indicates, they can never be together, and this is the interpretation put upon the result $t = \infty$, an interpretation entirely consistent with the nature of the case. Of this nature are all of the cases in which infinity appears in algebraic results.

In *Geometry*, if we inscribe a regular polygon in a circle, and then bisect each arc subtended by a side of the polygon, and join the points of bisection with the vertices of the adjacent angles, a new polygon, regular and inscribed, will be formed, having double the number of sides. This polygon will coincide more nearly with the circle than the preceding. If we again form a third regular polygon, in like manner, having double the number of sides that the second has, it will coincide still more nearly with the circle, and so on. If we conceive this process of bisection and formation of polygons, each having double the number of sides of the preceding one, to be continued, the varying polygon will continue to approach the circle in area, but it is evident that no polygon having a finite number of sides, can ever be exactly equal to the circle, though a polygon can always be found which will differ from the circle by less than any assignable quantity. The circle is the limit towards which the varying polygon approaches as the number of sides increases; hence, we say with propriety that *the circle is a regular polygon, having an infinite number of sides.* In like manner, every curve may be regarded as a polygon, obeying a certain law, and having *an infinite number of sides.* The sphere, the cone, and the cylinder, are polyhedrons, obeying certain laws, and having an infinite number of faces.

In *Trigonometry*, the tangent of an arc is the portion of the tangent drawn at one extremity of the arc, and limited by the prolongation of the radius through the other extremity. If the arc be increased from 0° towards 90°, the length of the tangent will increase, and as the arc approaches 90°, the prolonged radius becomes more nearly parallel to the tangent; and, finally, at 90° it becomes absolutely parallel to it, and the length of the tangent becomes greater than any assignable line. Hence, we say that the tangent of 90° is *infinite;* in like manner, the tangent of 270° is $-\infty$, the secant of 90° is $+\infty$, that of 270° is $-\infty$, and so on.

In Analysis, the equation of the common hyperbola, referred to the diagonals of the rectangle on the axes, is

$$xy = m,$$

in which x and y are the co-ordinates of every point, and m is constant. In this equation, as x diminishes, y increases, and when x becomes less than any assignable quantity, y becomes greater than any assignable quantity, or *infinite*. In like manner, for all similar cases in analytical geometry. The interpretation of the case just considered is that for an abscissa 0, there is no ordinate whose length can be expressed in finite terms.

In all the cases considered, which have been purposely selected from the different branches of mathematics, we have seen that infinity denotes *a limit of a varying magnitude or quantity*, and that it admits of an interpretation entirely in accordance with the

established principles of reason, and of mathematical deduction. In this point of view, the consideration and interpretation of infinite results, presents no greater difficulties than arise from the consideration and interpretation of any other results.

We come next to show that two infinite quantities are not necessarily, nor, indeed, are they generally, equal to each other. To illustrate this principle, let us consider the case of a *plane angle*, which is defined to be, *that portion of a plane lying between two straight lines, meeting at a common point.* The lines meet at a point, and are limited in that direction, but extend indefinitely in the direction of the angle. From these considerations, it is evident that the area of that portion of the plane which constitutes the angle, is *infinite.* It is the limit of the sector of a circle, having its centre at the vertex, when the radius becomes infinite.

Now, as the angle increases, the area being infinite, is continually increased by an infinite quantity, till when the angle becomes twice its original magnitude, the infinity obtained is twice that obtained in the first case. If two equal angles be compared, we have the case of two equal infinite quantities, equal because one may be so placed upon the other as to coincide with it throughout its whole extent. If two unequal angles be compared, we have the case of two unequal but infinite quantities, which must have the same relation to each other as the angles themselves. We see, also, from this discussion, that in certain cases, infinities may be compared, measured, or computed, in the same manner as finite quantities.

It follows also, that one infinite quantity may be infinitely small with respect to another. To make this more clear, let us take the identical and continued equation

$$\frac{1}{x} = \frac{x}{x^2} = \frac{x^2}{x^3} = \frac{x^3}{x^4}, \text{ \&c., \&c.}$$

In which x is supposed to be less than any assignable quantity; then from the definition, $\frac{1}{x}$ is infinite, that is, the quantity 1 is infinitely great in comparison with the quantity x; hence, $\frac{x}{x^2}$ is infinite, or x^2 is infinitely small in comparison with x; also, x^3 is infinitely small in comparison with x^2 and so on; x is infinitely small compared with 1, and is called an infinitely small quantity of the first order; x^2 is an infinitely small quantity of the second order; x^3, x^4, &c., x^n are infinitely small quantities of the third, fourth, &c. n^{th} orders. The order of an infinitely small quantity is determined by the number of infinitely small factors of the first order which it contains. The last principle finds an application in the processes of the Differential and Integral Calculus.

IN-FIN-I-TES'I-MAL. An infinitely small quantity. Infinitesimals are of different orders. No quantity is great or small except in comparison with some other quantity. An infinitely small quantity of the first order is one that is infinitely small with respect to a finite quantity, that is, so small that it may be contained in it an infinite number of times. An infinitely small quantity of the second order is one that is infinitely small with respect to an infinitely small quantity of the first order. In general, an infinitely small quantity of the n^{th} order is one which is infinitely small with respect to an infinitely small quantity of the $(n-1)^{th}$ order. When several quantities, either finite or infinitesimal, are connected together by the signs plus or minus, all except those of the lowest order may be neglected without affecting the value of the expression. Thus,

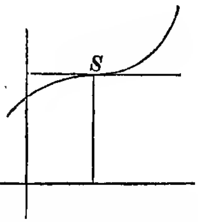
$$a + dx + dx^3 = a,$$

also,

$$dx + dx^2 + dx^3 = dx,$$

dx being infinitely small with respect to a; dx^2 infinitely small with respect to dx, &c.

IN-FLEX'ION. A point at which a curve ceases to be concave and becomes convex, or the reverse, with respect to a straight line not



passing through the point. The point S is a point of inflexion. If we take a system of

co-ordinates, such that the axis of X shall not pass through a point of inflexion, we shall have one of the following cases.

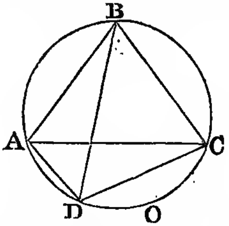
1. If, just before reaching the point of inflexion, the curve is convex with respect to the axis of X, and the ordinate of any point and the second differential co-efficient of the ordinate, taken at the same point, have the same sign; then just after passing the point of inflexion the curve will be concave, with respect to the axis of X, and the ordinate and second differential co-efficient will have contrary signs. Now, since the sign of the ordinate has not changed, that of the second differential co-efficient must have changed.

2. If, just before reaching the point of inflexion, the ordinate and second differential co-efficient have contrary signs, then just after passing it, they have the same signs; hence, in this case, the second differential co-efficient of the ordinate must have changed its sign in passing the point of inflexion. Now, a quantity can only change sign by reducing to 0 or ∞. Hence, we have the following rule for finding all of the points of inflexion of any given line:

Differentiate the equation of the curve twice; combine the resulting and given equations and find the value of the second differential co-efficient of the ordinate of the curve in terms of x; place this equal to 0 and ∞, and deduce the roots of the resulting equations; these will include all of the values of x that can possibly belong to points of inflexion. Substitute each value of x, increased and diminished by an infinitely small quantity, for x in the expression for the second differential co-efficient, and see if they give contrary signs; if so, the value of x belongs to a point of inflexion, and this point may be found by substituting this value in the equation of the curve, and deducing therefrom the corresponding value of y.

The radius of curvature may be 0 or ∞ at a point of inflexion, but it can never be *finite*.

IN-SCRĪBED′ LINE. [L. from *in*, and *scribo*, to write]. A straight line is said to be inscribed in a circle when its two extremities lie in the circumference of the circle. Thus, AB is inscribed in the circle ABCD. An angle is inscribed in a circle when its vertex lies in the circumference, and when its sides form chords of the circle. The angles ABC, ABO, &c., are inscribed angles.



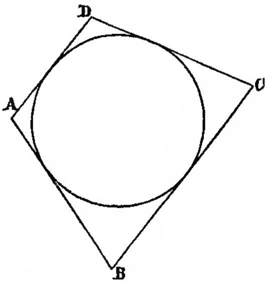
A polygon is inscribed in a circle when all of the vertices of its angles lie in the circumference. Thus, the polygons ABC, ACD, ABCOD, are inscribed in the circle ABCD. In like manner, we say that a line, angle, or polygon, is inscribed in an ellipse or other plane curve. A polyhedron is inscribed in a sphere or other curved surface, when its vertices are all contained in the surface.

IN-SCRIP′TI-BLE. A polygon is said to be inscriptible when it can be inscribed in a circle, or when the circumference of a circle can be passed through all its vertices. All regular polygons are inscriptible. A quadrilateral is inscriptible when the sum of any two opposite angles is equal to 180°.

A polyhedron is inscriptible when the surface of a sphere can be passed through all of its vertices.

A circle is inscribed in a triangle or other polygon, when it is tangent to every side of the polygon. A sphere is inscribed in a polyhedron when it is tangent to every face of the polyhedron.

A circle can always be inscribed in any triangle. A circle can always be inscribed in a quadrilateral, when the sum of two opposite sides is equal to the sum of the other two



opposite sides. Thus, in the quadrilateral

AC, if the sum of the sides DC and AB is equal to the sum of the sides DA and BC, then can a circle be inscribed in it. The only parallelograms in which a circle can be inscribed, are the *square*, and the *rhombus*, or *lozenge*. A circle can always be inscribed in a regular polygon of any number of sides.

A sphere can be inscribed in any regular polyhedron. A sphere can also be inscribed in any triangular pyramid.

IN-SŬR'ANCE. An agreement by which an individual or a company agrees to exempt the owners of certain property, as ships, goods, houses, &c., from loss or hazard.

The agreement is generally in writing, and the instrument is called a policy. The amount paid by the owner of the property insured, as a compensation for the risk assumed, is called the premium. The premium is generally computed at a certain rate per cent., which varies according to the nature of the risk taken. The amount of premium may be found by the same rules as used in computing simple interest.

In all cases, the first thing towards determining the *rate*, is to ascertain the probability that the loss insured against will take place. From the very nature of the case, this element, as an isolated case, cannot be determined with any degree of accuracy. For example, the loss of a ship at sea is contingent upon hundreds of events, which cannot be embraced in any mathematical formula, such as storms, fires, hidden reefs, &c. The only clue that can be had upon this subject, is the record of past experience; but this cannot be of fixed value, on account of the continual change that is going on in the method of constructing and navigating vessels. The aid which science is daily affording, in devising better models for vessels, in seeking for and mapping down hidden dangers, and particularly in systematizing the science of currents and ocean storms, serves to render the records of the past of less avail than they would otherwise be. In the case of insurance against fire, the exact appreciation of the risk is quite as difficult as in marine insurance. Here, too, the recorded experience of the past is made a basis of calculation.

The thing aimed at, in all kinds of insurance, is to reduce to an average value the profits arising from speculations of the same kind, however numerous they may be. The result to the insured is the same, as though each one contributed to a common fund a certain sum, from which fund all losses were to be paid. From the necessary competition between rival companies, excessive premiums are prevented, and the rates are reduced nearly to their minimum.

The principle of mutual insurance consists in each of the insured paying into a common treasury a certain amount of money, and executing an obligation to pay a certain other amount, should the losses require such payment.

A mercantile firm employing a great number of ships, or a large property-holder having a great variety of buildings in different localities, would be little benefited by insuring; since the amount of premiums that he would have to pay, would soon be sufficient to cover all probable losses. It is upon this principle, that the United States Government never insures any of the supplies that are being continually transported from one part of the country to the other.

IN'TE-GER. A whole number as distinguished from a fraction; that is, it is a number which contains the unit 1 an exact number of times; 2, 13, 42, 25, 16, &c., are integers.

IN'TE-GRAL. In Arithmetic, it denotes a whole number. In Calculus, an expression which, being differentiated, will produce a given differential. See *Calculus*.

INTEGRAL CALCULUS. See *Calculus*.

IN-TE-GRĀ'TION. The operation of finding the integral of a given differential. See *Calculus*.

IN-TER-CEPT'. [L. *intercipio*, to stop]. To include between. When a curve cuts a straight line in two points, the part of the straight line lying between the two points, is said to be intercepted between the two points. And, in general, that part of a line lying between any two points, is said to be intercepted between them.

IN'TER-EST. An allowance made for the use of borrowed money. The money, on which interest is to be paid, is called the *principal*. The money paid is called the *interest*. The principal and interest, taken together, are called the *amount*. The ratio of the prin-

cipal to the interest, per annum, is the *rate*, or *rate per cent.*

Interest is either *simple* or *compound.*

Simple Interest is the interest upon the principal, during the time of the loan.

Compound Interest is the interest, not only upon the principal, but upon the interest also, as it falls due.

Simple Interest.

Denote the principal by p, the rate by r, the interest by i, the number of years by t, and the amount by s; then will the following formulas be sufficient to solve every problem that can arise in simple interest:

$$i = ptr \ldots\ldots (1). \quad s = p(1+tr) \ldots\ldots (2).$$

$$p = \frac{s}{1+tr} \ldots (3). \quad t = \frac{s-p}{pr} \ldots\ldots (4).$$

$$r = \frac{s-p}{pt} \ldots\ldots (5).$$

t may be fractional, as, when the interest is for 60 days,

$$t = \frac{60}{365} = \frac{12}{73}.$$

If the rate is 4 per cent, then is $r = .04$; if 5 per cent, $r = .05$, and so on.

Compound Interest.

Assuming the same notation as in simple interest, and supposing the interest to be compounded annually.

At the end of one year, we shall have, from formula (2),

$$s = p(1+r).$$

This sum now becomes a new principal, and, from the same formula, at the end of two years, we shall have

$$s = p(1+r)(1+r) = p(1+r)^2.$$

This again becomes a new principal, and, as before, at the end of the third year, we have

$$s = p(1+r)^2(1+r) = p(1+r)^3 \ldots (1);$$

and so on indefinitely: hence, the amount at the end of t years is given by the formula

$$s = p(1+r)^t;$$

or, by taking the logarithms of both members.

$$\log s = \log p + t\log(1+r) \ldots\ldots (1),$$

a formula well adapted to computing interest in a given case, or for computing tables for practical use.

From formula (1) we deduce the following:

$$p = \frac{s}{(1+r)^t} = s(1+r)^{-t}; \quad \text{or}$$

$$\log p = \log s - t\log(1+r) \ldots\ldots (2).$$

$$t = \frac{\log s - \log p}{\log(1+r)} \ldots\ldots\ldots\ldots (3).$$

$$r = \left(\frac{s}{p}\right)^{\frac{1}{t}} - 1 \ldots\ldots\ldots\ldots (4).$$

As an example, let it be required to find the number of years that it will take a sum of money to double itself at the rate of 5 per cent per annum. In formula (3) we have

$$s = 2p, \ 1 + r = 1.05;$$

hence,

$$t = \frac{\log 2s - \log s}{\log 1.05} = \frac{\log 2}{\log 1.05} = \frac{0.301030}{0.211899}$$
$$= 14.2067 \text{ nearly}.$$

Hence it would double in $14\frac{1}{5}$ years.

Here we have supposed that interest is added to principal at the end of each year. Were it added oftener, r would represent the rate per cent for the period. For instance, if it were added half yearly, and the rate per cent per annum were 6, then would $r = .03$ in the above formulas; and in like manner for any shorter interval. It is an advantage to the lender to have the interest added to the principal as often as possible. If it is added semi-annually the annual interest will be

$$\left(1 + \frac{r}{2}\right)^2 = 1 + r + \frac{r^2}{4}$$

instead of $1 + r$, as it would be, were it added annually; hence in this case the advantage for one year would be $\frac{r^2}{2}$. Were it added quaterly, the annual interest would be

$$\left(1 + \frac{r}{4}\right)^4 = 1 + r + \frac{3r^2}{8} + \frac{r^3}{16} + \frac{r^4}{256},$$

or an advantage of

$$\frac{3r^2}{8} + \frac{r^3}{16} + \frac{r^4}{256};$$

and generally, were the interest added every $\frac{1}{m}$th part of a year, the advantage would be expressed by

$$\frac{m-1}{2m}r^2 + \frac{(m-1)(m-2)}{2 \cdot 3m^2}r^3 + \&c.$$

If r is very small, all powers of it, greater

than the second, may be neglected, and the advantage would be $\frac{m-1}{2m}r^2$. If $r = \infty$, in which case the interest is added continually, the total interest for the year will be

$$1 + r + \frac{r^2}{1 \cdot 2} + \frac{r^3}{1 \cdot 2 \cdot 3} + \frac{r^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{\&c.}$$

The sum of this series is equal to the number whose logarithm is .4342945. Let us take the case in which $r = .05$, then will the amount of one dollar for one year, under the last hypothesis, be $1.05127, which exceeds the nominal rate by $0.00127.

IN-TER-FĀ'CIAL. Included between two plane faces. An interfacial angle of a polyhedron is a diedral angle included between two faces of the polyhedron. All interfacial angles of a regular polyhedron are equal to each other.

IN-TĒ'RI-OR. [L. *intra*, within]. Lying within. An interior angle of a polygon is an angle included between two adjacent sides and lying within the polygon. The term is used in contradistinction to exterior, the exterior angle being included between any side and an adjacent one produced. See *Angle.*

IN-TER-MĒ'DI-ATE TERMS. [L. *inter*, between, and *medius*, middle]. In a progression the first and last terms are called extremes, the remaining ones are called intermediate terms or simply means.

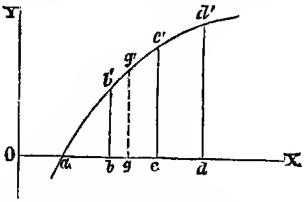
IN-TERN'AL ANGLES. Same as interior angles. See *Interior.*

IN-TER-PO-LĀ'TION. [L. *interpolo*, to interpolate]. The operation of finding terms between any two consecutive ones of a series which shall conform to the law of the series. In most cases the law of the series is not given, but only numerical values of certain terms of the series, taken at fixed and regular intervals. In this case we may approximate to the interpolated term by the formula

$$T = a + \frac{n}{1}d_1 + \frac{n \cdot (n-1)}{1 \cdot 2}d_2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}d_3 + \text{\&c.} \quad \cdot \cdot \quad (1)$$

Formula (1) expresses any term of a series whose terms are computed for values of the variable in arithmetical progression; a denotes the term of the series preceding the interpolated term, d_1, d_2, d_3, d_4, &c., are the first terms of the successive orders of differences, counting from the term a, and n denotes the order of the interpolated term. To illustrate the process of interpolation, let us take the equation $y = f(x)$. Now, by assigning values to x, and deducing corresponding values of y, we shall have sets of values of x and y which may be regarded as the co-ordinates of a plane curve that may be constructed.

Suppose OX and OY to be the axes of co-ordinates, and $ab'c'd'$, &c., the curve; let



bb', cc', dd', &c., be ordinates taken at equal intervals, that is, so that

$$Oa = ab = bc = cd, \text{ \&c.}$$

Now, if the curve were accurately constructed, any ordinate gg' between bb' and cc', might be found by drawing gg' parallel to OY and measuring the length of it by means of a scale of equal parts, but if the curve were only approximately given, the value of gg' could only be approximately determined.

Now, if we have tabulated a series of values of y for values of x in arithmetical progression, we can by interpolation obtain, to any degree of exactness, any intermediate ordinate. In order to apply formula (1), to find the value of gg', we should make in it

$$a = bb', \quad n = \frac{bg}{bc},$$

and taking the tabulated values of bb', cc', dd', &c., find the successive order of differences to any required degree of accuracy, and make d_1, d_2, d_3, &c., equal to the first terms of the successive orders of differences. Substituting these expressions. in formula (1), the value of T will be the ordinate required, or the interpolated term.

To illustrate, let it be required to find from the tabulated values of the logarithms of the numbers 12, 13, 14, and 15, the value of the logarithm of $12\frac{1}{2}$.

Numbers	12	13	14	15
Log.	1.079181	1.113943	1.146128	1.176091
1st or. diffs.	0.034762	0.032185	0.029963	
2d do.		−0.002577	−0.002222	
3d do.			+ 0.000355.	

Counting from log 12, we have

$$a = 1.079181, \quad n = \tfrac{1}{2}, \quad d_1 = 0.034762$$
$$_2 = -0.002577 \qquad d_3 = 0.000355.$$

Substituting these values in formula (1), and neglecting all the terms after the fourth, as inappreciable, we have

$$T = \log. 12\tfrac{1}{2} = a + \tfrac{1}{2}d_1 + \tfrac{1}{8}d_2 + \tfrac{1}{16}d_3 + \&c.$$
$$= 1.079181 + 0.017381 + 0.000322$$
$$+ 0.000022 = 1.096906.$$

Had it been required to find the logarithm of 12.39, we should have made $n = 0.39$, and the process would have been the same as above. In like manner we may interpolate terms between the tabulated terms of any mathematical table.

The method of interpolation is of extensive use, not only in pure analysis and geometry, but also in various other subjects of mathematical inquiry and computation, particularly in Astronomy. In this latter branch of investigation it is the means of saving, in many cases, immensely laborious computations. Thus, for example, in finding the places of some of the planets whose motions are not very rapid, it will be sufficiently accurate to compute their places for every fourth or fifth day, and then by interpolation, to find their places for intermediate days. Again, in finding the moon's place for any particular hour, supposing its place for every 3, 6, or 12 hours to be given, the method of interpolation may be applied with great success, the results differing inappreciably from those of actual computation.

By this means, also, the place of a comet at any particular time may be determined from observations made previous and subsequent to that precise period. In a word, Astronomy has derived more assistance from this principle than from almost any other mathematical device.

IN-TER-PRE-TA'TION. [L. *interpretatio*, explanation]. The process of explaining results arrived at by the application of mathematical rules. When, for example, an algebraic definition is laid down, there is frequently some restriction implied in making the definition, so that the result to which it leads presents more cases than can be explained by it, or even than was contemplated by it. Thus the abbreviation of aa, aaa, into a^2, a^3, and the rules which spring from it, lead to results of the form

$$a^{-2}, \quad a^0, \quad a^{\frac{1}{2}}, \quad a^{-\frac{3}{4}}, \ \&c.$$

These results, until interpreted, are without any intelligent algebraic meaning.

When such results arise, the province of interpretation begins; their meaning and force are investigated and explained, and the definitions heretofore too narrow, are extended so as to cover these and other results.

The rule to be adopted in interpreting new expressions obtained by applying known processes, is to attribute to them such a meaning as to make the whole of the process true by which they were obtained. For example: the formula

$$a^m \times a^n = a^{m+n}$$

is perfectly intelligible so long as m and n are whole numbers. Suppose it were required to interpret the symbol a^0, that is, to give to it such a meaning, that the above formula shall be true in that case. Making $m = 0$, the formula becomes

$$a^0 \times a^n = a^{0+n} = a^n;$$

hence, $a^0 = 1$. Again, suppose it were required to interpret the symbol $a^{\frac{1}{2}}$. Make $m = \frac{1}{2}$ and $n = \frac{1}{2}$, and the formula becomes

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a,$$

hence, $a^{\frac{1}{2}} = \sqrt{a}$, for $\sqrt{a} \times \sqrt{a} = a$, by definition.

Besides the application of the principles of interpretation to the explanation of new symbols, another very important application consists in making suppositions upon certain arbitrary quantities which enter formulas, and then comparing the results with known facts, thus deducing new truths. As an example of this method of interpretation let us take the equation of the ellipse

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

and suppose $x > a$, finding the value of y in terms of x, we have

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2},$$

from which we see that for all values of x

greater than a, y is *imaginary*. Now an imaginary result indicates an impossibility in the assumption. Hence, we interpret the result as indicating that no point of the ellipse can lie at a greater distance from its conjugate axis than the extremity of the transverse axis.

In integrating the differential of a transcendental function by an algebraic rule, a result ∞ is reached, which is manifestly absurd, since no function can be ∞. We interpret this as indicating that the rule fails in the case considered.

IN-TER-SECT′. [L. from *inter*, between, and *seco*, to cut]. To cut each other. Two lines are said to intersect when they cross each other, having a point in common. Two surfaces intersect when they cut each other, having a line, or lines, in common.

To find the point in which two lines, given in a plane by their equations, intersect, combine the equations of the lines and find the values of the variables, these will be the co-ordinates of the point of intersection. The number of sets of real values found will indicate the number of points of intersection.

To find the points of intersection of two lines in space, combine the equations of their projections upon the plane of XZ; find the values of z and set them aside; combine the equations of the projections of the lines upon the plane YZ, and find the values of z, and set them aside; for each pair of real and equal values of z found, there will be a point of intersection, the co-ordinates of which may be found by substituting this value of z, for z in the equations of the line, and finding the corresponding values of x and y. In either case, where there are no sets of real values found for the variables, the lines do not intersect.

To find the intersection of two surfaces whose equations are given: combine the equations and eliminate one of the variables; the resulting equation is that of the projection of the line of intersection on the plane of the other two; combine the equations again, eliminating a second variable; the resulting equation is that of the projection of the intersection on the plain of the other two. These equations, taken together, fix the position of the line of intersection.

In Descriptive Geometry, the line of intersection of two surfaces is found by points, as follows: Pass auxiliary surfaces intersecting both the given surfaces in lines, the points in which these lines intersect, are points of the line of intersection of the two surfaces. Having found a sufficient number of these points, draw a line through them, and it will be the line of intersection required.

IN-VĀ′RI-A-BLE. Unchanging, constant.

Invariable Function. A function which enters an equation, and which may vary under certain circumstances, but which does not vary under the conditions imposed by the equation, is called the *invariable of the equation.*

In a common differential equation which holds true for all values of x and y, the only invariables must be absolute constants; but in an equation of differences in which the value of x only passes from one whole number to another, any function which does not change value whilst x passes from one whole number to another, may be an *invariable.*

For example, let it be required to find the integral of the equation of finite differences,

$$\Delta y = x + 1,$$

in which the value of x only changes from one whole number to another. From the rule for integrating finite differences, we have,

$$y = \tfrac{1}{2}(x^2 + x) + c,$$

in which c may have any value consistent with the conditions of the question. Since $f(\cos 2\pi x)$ does not change its value, whilst x passes from one whole number to another, we may place it for c, giving,

$$y = \tfrac{1}{2}(x^2 + x) + f(\cos 2\pi x),$$

the required solution. In this case, $f(\cos 2\pi x)$ is the *invariable of the equation.*

IN-VERSE′. [L. *inverto*, to turn about]. Two operations are inverse, when the one is exactly contrary to the other, or when being performed in succession upon a given quantity, the result will be that quantity. Addition and subtraction are inverse operations, for, if we add to a the quantity b, and from the sum subtract the quantity b, the result will be a. Multiplication and division, raising to powers and extracting roots, differentiation, and integration, are all inverse operations.

If two variable quantities are connected together by an equation, either one is a function of the other. If it be agreed to call the first a direct function of the second, then is the second an inverse function of the first. The *forms* of direct and inverse functions, as dependent upon the connecting equation, may be determined by solving the equation with respect to each function separately.

Let $x^2 - 2x$ be the form of a direct function, required that of its inverse. Assuming the equation,

$$y = x^2 - 2x,$$

and solving it with respect to x, we find,

$$x = 1 + \sqrt{y+1}, \text{ and } x = 1 - \sqrt{y+1}.$$

The second members indicate that both

$$1 + \sqrt{x+1} \text{ and } 1 - \sqrt{x+1},$$

are inverse functions of $x^2 - 2x$. In this case there are two inverse functions, in other cases there may be more than two. If we denote the form of any direct function of x by the symbol ϕ, and that of its inverse by ϕ^{-1}, there may be two cases; 1st. When both of the equations

$$\phi [\phi^{-1}(x)] = x, \text{ and } \phi^{-1}[\phi(x)] = x,$$

are satisfied; and 2d. When both are not satisfied. When both are satisfied, the inverse is said to be *convertible*, when both are not satisfied, it is said to be *inconvertible*. Every function has one *convertible inverse* and only one, the remaining ones being inconvertible. In the preceding example, $1 + \sqrt{x+1}$ is the convertible inverse of $x^2 - 2x$, for

$$1 + \sqrt{(x^2 - 2x) + 1} = 1 + (x - 1) = x,$$

also,

$$(1 + \sqrt{x+1})^2 - 2(1 + \sqrt{x+1}) = x:$$

but $1 - \sqrt{x+1}$ is an inconvertible inverse, for,

$$1 - \sqrt{(x^2 - 2x) + 1} = 2 - x,$$

whilst as before,

$$(1 - \sqrt{x+1})^2 - 2(1 - \sqrt{x+1}) = x.$$

There is, however, a function of a function of x, of the given form, to which $1 - \sqrt{x+1}$ is a convertible inverse, which is

$(2-x)^2 - 2(2-x)$; for, $(2-x)^2 - 2(2-x) = 2^2 - 2x$, also,

$$1 - \sqrt{(2-x)^2 - 2(2-x) + 1} = 1 - (2 - x - 1) = x.$$

It may be shown that every function of x, which has more than one inverse, is not only a function of x, but the same function of other functions of x, differing simply in form, and the whole number of forms is exactly equal to the number of inverse functions; furthermore, each inverse is the convertible inverse of one of these forms, and of only one.

Having the convertible inverse of a given function, to find the remaining inverses. Solve the equation

$$\phi [f(x)] = \phi(x),$$

and let the forms found be

$$f(x), f'(x), f''(x), \text{\&c.}$$

Then, $\phi^{-1}(x)$ being the convertible inverse of $\phi(x)$, the remaining inverses are

$$f'[\phi^{-1}(x)], f''[\phi^{-1}(x)], f'''[\phi^{-1}(x)], \text{\&c.}$$

Thus, in the preceding example, $\phi^{-1}(x)$ being the convertible inverse, the other is $2 - \phi^{-1}(x)$.

There is a remarkable class of functions each of which is its own inverse; thus, $1 - x$, $\frac{1}{x}$, $\sqrt{1 - x^2}$. Now, if $\phi(x) = \phi^{-1}(x)$, we have, $\phi[\phi(x)] = x$, a periodic function of the second order, to which class those mentioned evidently belong.

IN-VER′SION. [L. *inverto*, to turn about]. The operation of changing the order of the terms, so that the antecedent shall take the place of the consequent and the reverse, in both couplets. Thus, from the proportion,

$$a : b :: c : d,$$

we have, by inversion,

$$b : a :: d : c.$$

INVERSION OF SERIES. See *Series*.

IN-VERT′. [L. *inverto*, from *in*, and *verto*, to turn]. To place in a contrary order. To invert the terms of a fraction is to put the numerator in place of the denominator, and the reverse.

IN′VO-LUTE. [L. *involutio*, that which is inwrapped, or infolded]. If a thread be tightly wrapped about a given curve and then unwrapped, being kept stretched, each point of it will generate a curve, called an *involute* of the given curve. The given curve with respect to any of its involutes is called an evolute. From the preceding definition, we see that any given curve has an infinite number of involutes, and in order to fix the position of any one of them, it is necessary to

know not only the evolute, but also one point of the involute.

To construct the involute of a given curve, when one point of it is given. Wrap a thread tightly about the evolute until it passes through the given point and is tangent to the evolute: fix a pencil at the point and unwrap the thread, keeping it stretched; the point of the pencil will describe the required involute.

To find the equation of the involute, under the same conditions, let the equation of the evolute be

$$f(a, \beta) = 0, \ldots \ldots \ldots (1),$$

assume the second and third equations of condition for osculatory circles, which are

$$y - \beta = -\frac{dx^2 + dy^2}{d^2y} \quad \ldots \; (2),$$

and
$$x - a = -\frac{dy}{dx}(y - \beta) \quad \ldots \; (3).$$

Combining equations (1), (2), (3), and eliminating a and β, there will result a differential equation of the second order, which is the differential equation of the whole class of involutes. To find the equation of the particular involute, let the equation be integrated twice, and the constants of integration be determined in accordance with the conditions of the problem; the resulting equation is that of the particular involute in question. This problem is not of so much importance as its converse, that is, the method of finding the equation of the evolute corresponding to a given involute. The terms involute and evolute are correlative, neither having any signification without reference to the other.

IN-VO-LŬ'TION. [L. *involutio*, that which is unfolded]. In Arithmetic and Algebra, the operation of finding any power of a given quantity. It is the reverse of evolution, which is the operation of finding a root of a given quantity. The operation of involution may be directly performed by continued multiplication, but it is often performed by means of formulas, particularly by the binomial formula.

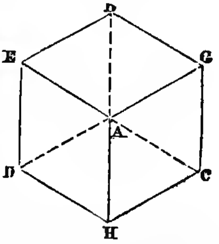
IR-RĂ''TION-AL. [L. *in*, and *rationalis*, from ratio]. Any quantity which cannot be exactly expressed by an integral number, or by a vulgar fraction; thus, $\sqrt{2}$ is an irrational quantity, because we cannot write for

it either an integral number, or a vulgar fraction; we may, however, approximate to it as closely as may be desired. In general, every indicated root of an imperfect power of the degree indicated, is *irrational*. Such quantities are often called surds. See *Incommensurable*.

IR-RE-DŪ'CI-BLE. In Algebra, the irreducible case of a cubic equation in which Cardan's rule fails to give the roots. This case arises when all the roots are real. For the method of treating the irreducible case, see *Cubic Equation*.

Ī-SO-MET'RIC-AL PROJECTION [Gr. ἴσος, equal, and μετρον, measure]. A species of orthographic projection, in which but a single plane of projection is used. It is called *isometrical* from the fact, that the projections of equal lines, parallel respectively to three rectangular axes, are equal to each other. This kind of projection is principally used in delineating buildings or machinery, in which the principal lines are parallel to three rectangular axes, and the principal planes are parallel to three rectangular planes passing through the three axes.

If we conceive a cube to be placed so, that its edges shall be parallel, respectively, to the principal lines of the figure to be projected, and then draw a diagonal of this cube, this diagonal is called the axis of the projection, and all the projecting lines of the points are parallel to it. The plane of projection is taken at right angles to the axis of the projection. The three edges of the cube, meeting at the vertex through which the diagonal is drawn, are projected into equal straight lines, making angles of 120° with each other; the remaining edges are also projected into equal and parallel lines.



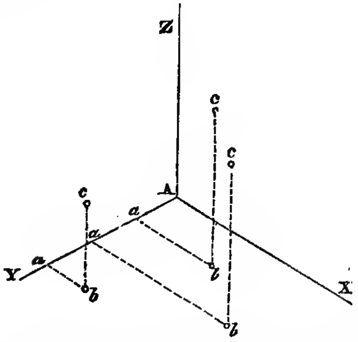
Let A be the vertex of the angle through which the axis of projection passes: draw AB, AC, and AD, making angles of 120° with each other, and lay off the distances AB, AC, AD, respectively equal to each other. Draw CH parallel to AD; DH parallel to

AC; draw DE, HA, CG, parallel and equal to AB, and join BE, BG, GA, and EA: the figure formed is the isometrical projection of the cube. The cube which we have assumed, is called the *directing cube.* The vertex A, through which the axis of projection and the plane of projection both pass, is called the *centre of projection*, and the projections of the indefinite edges of the directing cube, which pass through A, are called *directing lines of the projection.*

The planes passing through the edges of the cube, which meet at the centre of projection, are called *co-ordinate planes.* One of these planes is usually taken parallel to the horizon, and is called the *horizontal plane;* a second is supposed to be in front of the point of sight, or the point from which the projection is to be viewed, and is called the *frontal plane;* the other one, perpendicular to these, is supposed to be to the left of the point of sight, and is called the *lateral plane.*

If we construct a scale of equal parts upon a line parallel to one of the edges of the cube, and project the scale upon the projection of that edge of the cube, the projection, thus obtained, is called the *scale of the directing line.* The scale of each directing line is the same, and may be assumed at pleasure.

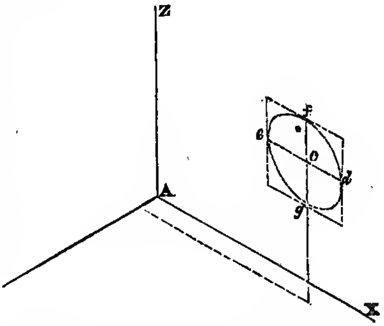
Points of objects to be represented by this mode of projection, are given by means of their distances from the co-ordinate planes, and their projections may be construeted as follows



Draw the directing lines AX, AY, AZ, making angles of 120° with each other. Lay off from A, on the directing line AY, from the scale of the directing line, a distance A*a*, equal to the distance of the given point from the frontal plane; draw *ab* parallel to AX, and make it equal to the distance of the point from the *lateral plane;* draw *bc* parallel to AZ, and make it equal to the distance of the point from the horizontal plane; then will *c* be the projection of the point. In like manner, any number of points may be constructed, and by joining these by suitable lines, the projection of the body may be constructed.

A circle, whose plane is parallel to either co-ordinate plane, is projected into an ellipse, having a pair of equal conjugate diameters parallel to the directing lines of that plane.

Suppose, for example, that the plane of the circle to be projected is parallel to the frontal plane, and that the centre is 4 feet above the horizontal plane, 8 feet to the right of the lateral plane, 1 foot in front of the frontal plane, and that the radius of the circle is 2 feet:



Construct the projection of the centre *c*, as before described. Draw *cd* parallel to AX, and make $cd = ce = 2$ feet, from the scale of the directing line; draw *fcg* parallel to AZ, and make $fc = cg = 2$ feet, taken from the same scale. Then, on *ed* and *fg*, as conjugate diameters, construct an ellipse, and it will be the projection required.

In like manner, the projection of a circle parallel to either of the co-ordinate planes may be constructed.

The projections of all circles parallel to the co-ordinate planes are similar ellipses. Practically, it is usual to draw those projections by means of elliptical disks cut out of card board of different sizes, to suit the different radii of the circles to be projected.

This species of projection is principally employed in representing frames and machinery; and if the co-ordinate planes are properly selected, the principal lines of the objects to be represented will be projected parallel to the directing lines, and the principal circles of the objects will be parallel to the co-ordinate planes.

Further details of this method of projection need not be given. By a judicious combination of the principles already laid down, every possible figure may be constructed, either accurately or approximately.

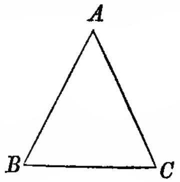
Ī-SO-PER-I-MET'RIC-AL. Relating to figures having equal perimeters.

Ī-SO-PE-RIM'E-TRY. [Gr. ισος, equal, περι, around, and μετρον, measure]. That branch of Higher Geometry which treats of the properties and relations of isoperimetrical figures, viz.: of surfaces having equal perimeters, volumes bounded by equal surfaces, &c.

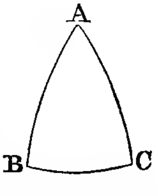
The simplest of the isoperimetrical problems is to find, amongst all the curvilineal areas bounded by the equal perimeters, which may be shown by elementary geometry to be the circle.

In all isoperimetrical problems, there are two conditions to be fulfilled: according to the *first* of which, a certain *property* is to remain constant, or to belong to all individuals of the species; and according to the *second*, another property is to be the greatest or least possible. In the problems of this class, which were first considered, the *first* property was the length of the perimeter of a curve, and it was from this circumstance that the term isoperimetrical was derived. More recently, the principles used in the investigation of these problems have been greatly extended, and have given rise to a new branch of mathematical analysis, now known as the Calculus of Variations. See *Calculus of Variations.*

Ī-SOS'CE-LĒS. [Gr. ισοσκελης; from ισος, equal, and σκελος, a leg]. A triangle is isosceles, when two of its sides are equal.



Thus, in the triangle ABC, if AB=BC, the triangle is isosceles. It is a property of an isosceles triangle, that the angles opposite the equal sides are equal, and this is true, whether the triangle is plane or spherical. ABC is a spherical isosceles triangle.



If, in a plane or spherical isosceles triangle, a line be drawn from the vertex formed by the meeting of the equal sides, to the middle point of the base, it will be perpendicular to the base; and conversely, if the line is perpendicular to the base, it will bisect it, and also will bisect the angle at the vertex. If the third side is equal to the other two, the triangle becomes equilateral: hence, an equilateral triangle is a particular case of the isosceles triangle.

K. The eleventh letter of the English alphabet. As a numeral, K has been used to denote 250; with a dash over it, thus, $\overline{\text{K}}$, it stood for 250,000.

KĪND. Genus, generic class. In technical language, *kind* answers to *genus*. The term is, however, loosely used for sort. We say that a line or surface is given in kind when the form of its equation is given, the constants which enter it being arbitrary.

KNŌWN QUANTITIES. Those whose values are given or determined. They are generally denoted by the leading letters of the alphabet, or by the final letters, with one or more dashes; as,

$$a, b, c, \quad x', y'', z''', \text{ \&c.}$$

L. The twelfth letter of the English alphabet. As a numeral character, it stands for 50; with a dash over it, thus, $\overline{\text{L}}$, it stands for 50,000. It is used as a symbol for *pounds* in the system of sterling currency. Generally, the written symbol is crossed by a horizontal mark; thus, £.

LAND'-SURVEYING. See *Surveying*.

LAND MEASURE. A collection of numbers, constructed according to a varying scale, by which we designate the quantity of land contained in a small portion of the earth's surface. The principal unit of this measure is 1 acre, which is divided into 4 equal parts, each of which is called a rood, and each rood is again divided into 40 equal parts, called

perches. The acre contains 10 square chains; that is, it is equivalent to a parallelogram whose base is 10 chains $= 660$ feet, and breadth $= 1$ chain $= 66$ feet.

LAN″GUAGE OF MATHEMATICS. [From *lengua*, a tongue]. The symbols and combinations of symbols, employed in mathematical reasoning and in mathematical operations. Language is an instrument of thought, and one of the principal helps in all mental operations. Any imperfection in the instrument, or in the mode of using it, will materially affect any result attained through its aid. Every branch of science has, to a certain extent, its own appropriate language; this is particularly the case with mathematics.

The language of mathematics is mixed. Although made up mainly of symbols which are defined with reference to the uses which are made of them, and which, therefore, have a precise signification, it also is composed in part of words transferred from ordinary language The symbols, though arbitrary signs, are nevertheless entirely general as signs and instruments of thought, and when their meaning is once fixed by definition, or by interpretation, they always retain that meaning under the same circumstances throughout the entire analysis. The meaning of the words borrowed from the common vocabulary, are often modified, and sometimes totally changed when transferred to the language of science. They are then used in a particular sense, and are said to have a *technical* signification.

The great power and universality of the mathematical language depends upon its conciseness, its generality, and the definite nature of the terms employed. By it, all quantities, whether abstract or concrete, are presented to the mind by arbitrary symbols. These representatives of quantity are reasoned upon and operated upon by means of another set of symbols called *signs;* and the signs and letters, with the words borrowed from the ordinary language, make up, as we have stated above, the language of pure mathematics. By means of this language, we are able to state the most general proposition, and present to the mind, in their proper order, every elementary principle employed in its demonstration. By its generality, it reaches over the whole field of the pure and mixed sciences, and presents in condensed forms, all the conditions and relations necessary to the development of particular facts and general truths. Each branch of mathematics has its own particular language, and it is from this fact that the different branches present such widely divergent methods, though the reasoning process is the same in all. See *Notation, Symbols*, &c.

LAT′ER-AL. [L. *lateralis*, from, *latus*, a side]. Appertaining to the side. The lateral faces of a pyramid are those which meet at the vertex: the lateral faces of a prism are those which have a side lying in the perimeter of each base.

The term *lateral* equation was formerly used instead of the more common appellation, equation of the first degree.

LAT′I-TUDE. [L. *latitudo*, breadth]. The latitude of a place on the surface of the earth, is its angular distance from the equator, measured on the meridian of the place. Latitude is north or south, according as the place is north or south of the equator. Circles whose planes are parallel to that of the equator, are called circles of latitude, or parallels of latitude, because the latitude of every point of each circle is the same. The latitude of a place is always equal to the inclination of the axis of the earth to the horizon of the place, and conversely. See *Geocentric* and *Geographic.*

LATITUDE IN SURVEYING. The distance between two east and west lines drawn through the two extremities of a course. If the course is run towards the north, the latitude is called *northing*, if towards the south, it is called *southing.*

The latitude of any course may be computed from the following formula,

$$L = D \times \cos a,$$

in which L denotes the latitude, D the length of the course, and a the bearing in degrees.

In Navigation, the term *difference of latitude of two points*, is the arc of any meridian intercepted between the parallels of latitude through the points, expressed in degrees. When the two latitudes are of the same name, the algebraic difference is the same as the arithmetical difference of the latitudes; when they are of different names, the alge-

braic difference is the arithmetical sum, the southern latitude being regarded as negative.

MIDDLE LATITUDE. In Navigation, the mean of two latitudes found by taking half of their algebraic sum.

LĂ'TUS REC'TUM, of a conic section. The same as the parameter. It is a straight line drawn through either focus perpendicular to the transverse axis, and limited by its intersection with the curve. See *Parameter*.

LAW. An order of sequence. In Mathematics, the term law is oftentimes used as nearly synonymous with rule; there is this distinction, however, the term law is more general than the term rule. The law of a series is the order of succession of the terms, and explains the relation between each and the preceding ones. A rule, assuming the facts expressed by the law, lays down the necessary directions for finding each term of the series.

The laws of series are expressed in the general formula,

$$T = a + nd_1 + \frac{n(n-1)}{1 \,.\, 2}d_2 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}d_3 + \&c.$$

In which T denotes any term, the n^{th}, estimating from a given one, a denotes the given term, and d_1, d_2, d_3, &c., the first terms of the successive orders of differences.

The mathematical law of a phenomena is nothing more than the expression, by means of mathematical language, of the invariable order of sequence, or of relation between the quantities considered.

LĒAD'ING LETTER of an expression or series. The letter with reference to which its terms are arranged.

LEE'WĀY. The distance made by a ship at right angles to the course steered, in consequence of imperfect sailing, currents, &c. See *Navigation*.

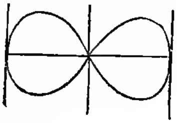
The leeway, expressed angularly, is the angle made by the keel of the ship, and the course actually described through the water.

LEG of a triangle. The same as side. We generally understand, by the term leg, one of the sides about the right angle of a right angled triangle.

LEG of an hyperbola. The same as branch. Hyperbolic legs are branches of a curve which partake of the nature of an hyperbola in having an asymptote.

LEM'MA. [Gr. λημμα, from, λαμβανω, to receive]. An auxiliary proposition, demonstrated on account of its immediate application to some other proposition. The conclusion of the lemma becomes requisite to the demonstration of the main proposition, and rather than encumber that proposition, a separate demonstration is introduced. The idea of a lemma is, that it is introduced out of its natural place, and this serves to distinguish it from ordinary propositions which, entering in their proper places, are of more or less use in demonstrating subsequent ones. The 11th, 12th, and 13th propositions of Davies' Legendre, Book VIII, are Lemmas.

LEM-NIS'CATA. [L. *lemniscus*]. The lemniscata is the locus of the points in which the tangent to the hyperbola intersects the perpendicular let fall upon it from the centre. If the equation of the hyperbola is



$$y^2 - x^2 = a^2,$$

that of the lemniscata, referred to the same axes, is

$$(x^2 + y^2)^2 = a^2(y^2 - x^2),$$

and its polar equation,

$$r^2 = a^2 \cos 2v.$$

If the equation of the hyperbola is

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

that of the lemniscata i

$$(x^2 + y^2)^2 = a^2x^2 - b^2y^2.$$

The general equation of curves of this kind, is

$$y^2 = mx^2(a^2 - x^2).$$

LENGTH. One of the three attributes of extension. Length generally implies extension in a horizontal direction, and generally is the greatest horizontal dimension of a body.

LESS. The comparative of little. One quantity is less than another when the latter exceeds the former in measure.

LEV'EL. A surface is said to be level when it is concentric with the surface of the sea, or the surface which the ocean would have were the surface of the globe entirely covered

with water. For small areas, that is, for an extent of a few miles, we may regard a level surface as that of a sphere which is osculatory to the ellipsoidal surface of the earth. The level surface which we have considered, is one of *true* level; a surface of *apparent* level, at any point, is a plane drawn tangent to the surface of true level at the point. For ordinary surveying, it is sufficiently accurate to consider the surface of the earth as spherical; in this case, a surface of *apparent* level is a horizontal plane at the point.

A line of true level is any line of a surface of true level. A line of apparent level is any line of a surface of apparent level. The instruments employed for leveling, indicate lines of apparent level, and have to be reduced to lines of true level by certain corrections called *corrections for curvature.*

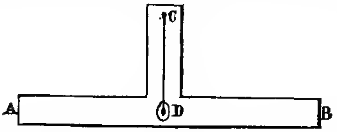
LEVEL. An instrument employed in leveling. There are several kinds of levels, more or less used; some of the most important of which we shall proceed to describe. Levels are constructed on three different principles.

1st. The line of apparent level is determined by means of a plumb line.

2d. It is determined by means of the surface of a fluid in equilibrium; and

3d. It is determined by means of an optical property of reflected rays of light.

1. *Levels in which the plumb line forms an essential part.* These are those generally used by bricklayers, carpenters, &c. They are constructed on many different plans, but the general arrangement is as follows:



A frame or board is prepared, having one edge AB perfectly straight, and a line CD drawn on the board or frame exactly at right angles to the straight edge; and at some point C of this line a string is attached, carrying a plummet D; when the frame is so placed that the plumb line, hanging freely, coincides with the straight line drawn on the frame, the line AB will be horizontal, and by its direction will point out a line of apparent level. This instrument is of little use in field leveling.

2. *Levels which depend upon the surface of a fluid in equilibrium.* These are of various kinds. The most important is the Y level.

This instrument consists, essentially, of a telescope mounted in supports, which, from their shape, are called Y's or wyes; to the telescope is attached a delicate spirit level; the Y's are attached to a bar or limb, which is connected with a supporting tripod by means of a ball and socket joint, so arranged that the instrument can be leveled by the aid of leveling screws. The telescope bears an internal diaphram, with cross hairs and antagonistic screws, by means of which their intersection may be brought into the axis of the telescope. The attached level has two sets of adjusting screws, by means of which its axis may be rendered parallel to that of the telescope. The Y's have also an arrangement of adjusting screws, by means of which the axes of the telescope and level may be made perpendicular to the axis of the limb.

The axis of the instrument is the line that remains fast when the instrument is turned round horizontally. The axis of the telescope is the line that remains fast when the telescope is revolved in the Y's.

Before using the Y level it must be adjusted, that is, all its parts must be brought to their proper relative positions. There are three adjustments.

FIRST ADJUSTMENT. *To fix the intersection of the cross hairs in the axis of the telescope.* Turn the telescope on its axis till one of the hairs is horizontal, and direct it to some fixed and well defined object; then turn it in the Y's, through 180°, till the same hair is again horizontal, and see if it remains upon the object; if it does, that hair is adjusted; if not, move it through half of the displacement by means of two of the antagonistic adjusting screws. Then repeat the operation and continue approximating till the hair remains in both positions upon the fixed objects. Next turn the telescope about its axis 90°, making the other hair horizontal, and then adjust it in the same manner as the first. If both have been properly adjusted, the intersection of the cross hairs will remain upon the same point during an entire revolution of the telescope in the Y's, in which case the first adjustment is complete.

SECOND ADJUSTMENT. *To make the axis*

of the level parallel to the axis of the telescope. This adjustment consists of two parts. *First,* bring the level over two of the leveling screws, and by means of them bring the bubble to the middle of the tube; then take the telescope from the Y's and reverse it, that is, change it end for end, and see if the bubble still remains in the middle of the tube; if it does, this part of the adjustment is complete; if not, raise the depressed, or depress the elevated end of the level half way, by means of the proper adjusting screw, and repeat the operation till by continual approximation the bubble is brought so as to remain in the middle of the tube in either position of the telescope.

Second. Turn the telescope in the Y's, and watch the bubble; if it remains in the middle of the tube, the second part of the adjustment is complete; if not, make the correction by means of the lateral adjusting screw. When the bubble remains in the middle of the tube, the telescope being revolved in the Y's, the second adjustment is complete.

Third Adjustment. *To make the axis of the telescope perpendicular to the axis of the instrument.* Bring the bar or limb over two of the leveling screws, and by turning them, bring the bubble to the middle of the tube; revolve the telescope about the axis of the instrument 180°, and observe whether the bubble remains in the middle; if so, the adjustment is complete; if not, make half the correction with antagonistic screws, which connect the Y's with the limb. Repeat the examination, continuing the approximation till the bubble remains exactly in the middle in both positions of the instrument; the adjustment of the instrument is then complete.

The adjustments should be examined from time to time, as they are liable to become deranged from a variety of causes.

Troughton's improved Level.

This instrument is constructed on the same general principle as the Y level already described. The telescope rests on a horizontal bar, which turns about an axis at right angles to its length in the same manner as the Y level. On the top of the telescope, and partly imbedded within its tube is a spirit level, over which is supported a compass box standing on four pillars. The bubble is long enough so that each end may be seen beyond the compass-box. The telescope is achromatic and inverting, and being placed nearer the horizontal bar, it is much firmer than the telescope in the Y level.

The adjustments are essentially the same as in the Y level; there are but two of them. The line of collimation, or axis of the telescope, and the axis of the level, must be made parallel to each other, and both must be made perpendicular to the axis of the instrument. The adjustment of the level is effected by means of capstan screws, which attach the telescope to the horizontal bar. The spirit level being firmly attached to the telescope by the maker, the line of collimation must be adjusted to it, which can be done by two screws near the eye end of the telescope.

To adjust the line of collimation, set up the instrument on a level piece of ground, level the telescope by the parallel plate screws, and direct it to a staff held by an assistant at from ten to twenty chains distance; let the vane of the staff be run up till its central line coincides with the horizontal cross hair of the telescope, and measure the height above the ground; now measure the height of the telescope above the ground, and from these heights find the difference of level between the two points. Let the instrument and staff change places, and the difference of level be determined, as before. If these differences of level are the same, the adjustment is complete; if not, take half the difference between the results and elevate or depress the cross wires that quantity, according as the last result gives a greater or less difference than the first. Again, direct the telescope to the staff, and make the coincidence of the horizontal wire and the central line of the bar by turning the collimation screws.

A sheet of water furnishes an easy mode of adjusting the line of collimation. A mark being fixed at some convenient distance at exactly the same height above the water that the instrument is, (allowing for curvature), the cross wires are made to intersect at that point.

The telescope is generally provided with two vertical wires, and one horizontal one. Some instruments have also a finely divided micrometer scale for reading off any portion of the rod that may be intercepted between the horizontal wire and the upper or lower

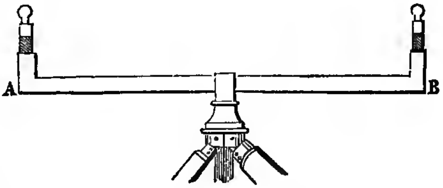
division. Such a scale may be used for determining the distances of the leveling staves from the instrument. This requires a table prepared by a series of successive observations with the instrument.

Gravatt's level.

The advantages claimed for this level are, that it possesses the power of a larger instrument without being of inconvenient length.

It has a telescope having a diaphram and cross hairs, as in the Y level. The internal tube which carries the eye piece, is nearly equal in length to the external tube. The internal tube is drawn in or thrust out by a rack and pinion turned by a milled head screw. The spirit level is placed above the telescope, being attached to two bands which embrace it; two capstan screws serve to adjust the axis of the level so that it shall be parallel with the axis of the telescope; a small level, placed at right angles to the axis of the telescope is used in setting up the instrument. A mirror plate, on a hinge joint, is used to reflect the image of one end of the air bubble to the eye, so that the observer can see, whilst reading the rod, that the instrument retains its position. The parallel plates and screws are similar to those in the Y level. The adjustments of this instrument are similar to those of Troughton's improved level. On account of its dumpy appearance it is often called the *Dumpy Level.*

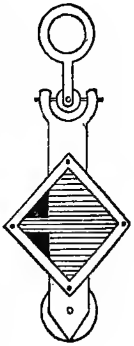
Water Level.



The Water Level is an instrument that possesses the advantage of never requiring adjustment, and also of being very cheap, in fact, any ordinary workman can construct one. Having no telescope, it is impossible to take long sights, but for such work as is required to be done by an ordinary surveyor, it gives very good results. Two brass or tin cups an inch in diameter, and four in height, are soldered to a hollow tube, three feet long and half an inch in diameter. The cups are for the purpose of receiving the ends, E and F, of two vials, the bottoms of which have been cut off and fixed in the tubes with putty; the projecting axis works in a hollow cylinder which forms the top of a stand. The tube, when the instrument is required for use, is filled with water (colored with lake or indigo), till it nearly reaches the necks of the bottles. After placing the stand tolerably level by the eye, withdraw both corks, and the surface of the water in the bottles will indicate a horizontal line in whatever direction the tube is turned. This level is well adapted to tracing contour lines.

3. *Levels which depend upon the reflection of light.*

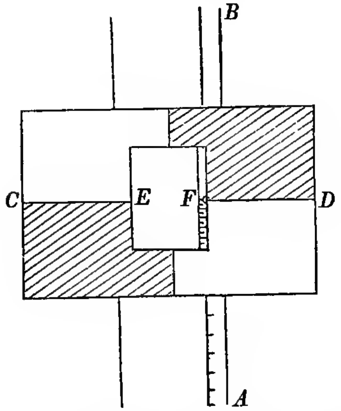
The reflecting level consists of a small piece of looking glass set in a metal frame, and suspended from a point so adjusted that the plane of the glass shall always be vertical. It is evident, that when we see the reflection of our own eye in the mirror, that the line from the pupil to its reflected image must be perpendicular to the plane of the glass, and therefore to the direction of gravity, or horizontal. By shifting the instrument, we have the means of tracing any number of horizontal lines. The instrument may be used for tracing contour lines. Reflecting levels are not very accurate in practice, though beautiful in theory. The figure represents a level of this class.



LEV'EL-ING STAVES. Rods used to determine the point in which a given horizontal line intersects a vertical one, to show its height above the surface of the ground. There are several kinds.

1. One of the best consists of a staff of hard wood capped with metal, from 12 to 15 feet in length, and graduated to feet, tenths, and hundredths. A sliding vane is made to move up and down by a cord and pulleys, and on the vane is a vernier by means of

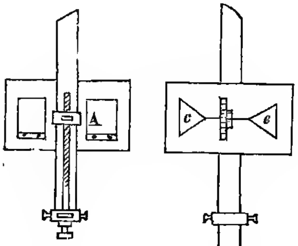
which the reading of the staff may be effected to thousandths of a foot.



AB represents a portion of the staff, DC the movable vane, with an opening EF, through which the graduation on the staff is seen. F is the vernier of the vane, the 0 being determined by the transverse line DC. To render this line more distinct, the vane is divided into four quarters, and the alternate ones are painted black; by their contrast with the white quarters the line DC is distinctly shown.

2 A second variety of leveling staff is formed of two pieces of hard wood, each about six feet in length, one of which slides in a groove of the other, and bears a vane similar to that already described, except the opening. The rod is graduated to feet, tenths, and hundreths, and a vernier may be attached for reading smaller divisions. The line of sight of the telescope is directed to the vane, and when it intersects the rod at less than six feet from the ground the staff is reversed, the vane run up the staff, and the readings made by means of the reversed figures at the right. When the line of sight intersects the rod more than six feet from the ground, the staff is used directly, and the reading is then made at the top of the lower half, and the figures indicating the height of the vane are found on the sliding tongue of the rod.

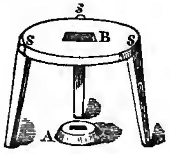
3. Another variety of leveling rod is used, where great accuracy is required, in which the vanes are of metal. There are two vanes on each rod facing in opposite directions, and when possible, the vane used in one observation is not disturbed till the next one is obtained. The vanes move on the staff by means of a sliding clasp, which is connected by an adjusting screw, with a spring clasp B, somewhat lower down, which can, when necessary, be firmly attached to the staff by means of clamping screws. The adjusting screw D, is finely cut so as to admit of deli-



Back. Front.

cate motions of the vane. When the vane is so high on the staff that the arm cannot reach to manipulate it, a rod is used, having a universal joint, and fitting the head of the screw.

A triangular space *c*, is cut from each side of the vane, so that the line joining the vertices shall be perpendicular to the rod and pass through the 0 of the vernier. At the back of these openings small mirrors are fixed, turning upon hinges, so as to reflect light towards the telescope at different angles of incidence. In observing, the horizontal line is seen sharp and well defined upon the faces of the mirrors, and is made to bisect the opposite angles of the openings, and thus to coincide with the 0 of the vernier. The rod is graduated so that with the vernier, readings may be taken accurately to within a thousandth of a foot, and approximately by the eye to another place of decimals. The rod is supported upon a tripod stand, having on its top a strong brass plate to which a horizontal motion can be given by means of screws *sss*. The staff is passed through an opening B in this, and rests upon a massive iron shoe A, in which there is an opening to receive it,



the vertical position is then given to the rod by means of the screws *sss*.

4. There is a fourth kind of leveling rod which is coming into common use, and is particularly used with the Gravatt level. This rod has no vane, but the graduation is made so distinct and clear that the observer can take the reading of the rod, and thus avoid errors of reading by an assistant. The telescope used is achromatic and inverting, which requires that the figures should be inverted upon the rod. The staff rests upon an iron shoe, within which it turns upon a point, without being lifted from the ground.

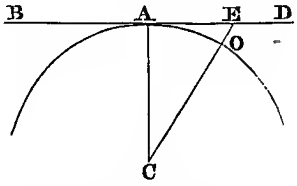
LEV′EL-ING. The operation of finding the difference of level between two points on the surface of the earth, that is, the distance between two level surfaces passed through the two points. One point is said to be above another point, in leveling, when it is farther from some level surface taken as a surface of reference, and the difference of the distances of two points from a fixed level surface of reference is the difference of level of the two points. The operation of leveling is carried on by means of a level and leveling rods.

The operation may be undertaken,

1st. For the purpose of determining the difference of level between two points.

2d For the purpose of obtaining a section or profile along a given route, as in the reconnaissance for establishing a line of railroad, canal, or other work of internal improvement.

3d. In determining the contour lines in a topographical survey. Before considering the operations to be performed, we shall deduce a formula for correcting the readings of the rod.



Let AO be a section of a level surface regarded as spherical, by a plane through the earth, C the centre of the earth, AD a line of apparent level, lying in the plane of the section considered. and CE a vertical line at O. The instrument indicates at A the line of apparent level AD, and the distance OE is a correction that must be subtracted from the reading of the rod at O, in order to reduce the reading to what it would be if the level had pointed out a line of true level. Now, if we denote the correction at any point, as O by x, the diameter of the earth by d, and the distance AO, which may be taken equal to AE, by h, we shall have from Elementary Geometry,

$$h^2 = x(d + x),$$

or, since x is so small with respect to d, that it may be neglected in comparison with it, we shall have,

$$h^2 = dx, \quad \text{or,} \quad x = \frac{h^2}{d} \cdots\cdots (1).$$

Now, since d remains constant, or sensibly so, we see that the correction varies as the square of the horizontal distance from the level to the rod.

If in formula (1) we make h equal to 1 chain, 2 chains, 3 chains, &c., and find the corresponding values of x, and arrange them in a table, the table formed will enable us to make the corrections in any reading, when we know the distance from the level to the rod. When great accuracy is required, a small correction has to be made for refraction.

TABLE.—*Showing the Correction for Curvature in thousandths of a foot for distances from* 1 *to* 100 *chains.*

CHAINS.	FEET.	CHAINS.	FEET.
1	0.000	22	0.050
2	0.000	23	0.055
3	0.001	24	0.060
4	0.002	25	0.065
5	0.003	26	0.070
6	0.004	27	0.076
7	0.005	28	0.082
8	0.007	29	0.088
9	0.008	30	0.094
10	0.010	31	0.100
11	0.013	32	0.107
12	0.015	33	0.113
13	0.018	34	0.120
14	0.020	35	0.128
15	0.023	36	0.135
16	0.027	37	0.143
17	0.030	38	0.150
18	0.034	39	0.158
19	0.038	40	0.167
20	0.042	41	0.175
21	0.046	42	0.184

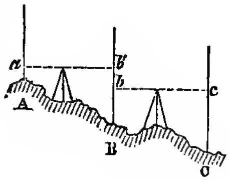
CHAINS.	FEET.	CHAINS.	FEET.
43	0.193	72	0.540
44	0.202	73	0.555
45	0.211	74	0.570
46	0.220	75	0.586
47	0.230	76	0.602
48	0.240	77	0.618
49	0.250	78	0.634
50	0.260	79	0.650
51	0.271	80	0.666
52	0.281	81	0.683
53	0.293	82	0.700
54	0.304	83	0.718
55	0.315	84	0.736
56	0.327	85	0.753
57	0.338	86	0.770
58	0 350	87	0.788
59	0.362	88	0.807
60	0.375	89	0.825
61	0.388	90	0.844
62	0.400	91	0.863
63	0.414	92	0.882
64	0.427	93	0.901
65	0.440	94	0.920
66	0.454	95	0.940
67	0.468	96	0.960
68	0.482	97	0.980
69	0.496	98	1.001
70	0.510	99	1.020
71	0.525	100	1.040

Observing that for 80 chains, $=1$ mile, the correction is .666, or two-thirds of a foot, and that the correction varies as the square of the distance, we have the following easy rule for finding the correction in feet:

The correction for curvature in feet, is equal to two-thirds of the number of miles from the level to the staff.

1. *Difference of level between two points.*

When it is proposed to find the difference of level between two points or stations, all readings taken in the direction of the first point are called *back-sights*, all taken in the direction of the second point, are called *fore-sights.* We shall suppose that the rod used is of the last kind described; that is, having no vane.



The line to be leveled is divided into convenient lengths, so that the rods will not be more than 130 or 140 yards apart. The level is placed at some convenient station, nearly equi-distant from the rods, so as to avoid correction for curvature and refraction.

Beginning at the first station, the instrument is set up between it and the second station and leveled; the rod being at A, the reading is taken and recorded under the head of *back-sights*, as in the following table:

Station.	Back-sight.	Foresight.	Diff. level.	Tot. diff.
1	2.046	7.931	– 5.885	– 5.885
2	1.998	6.021	– 4.023	– 9.908
..	....	....	....	....
..	....	....	....	....

Then the rod being set up at B, the reading is taken and recorded under the head of *fore-sights.* Suppose the readings to be 2.046 and 7.931. The level is then moved so as to be nearly equi-distant from B and C, and the readings taken and recorded as before. Suppose them to be 1.998 and 6.021. The instrument is again moved, and the operation continued till the last station is reached. Then the sum of the back-sights, minus the sum of the foresights, is equal to the difference of level between the two points. If the remainder is positive, the second point is higher than the first; if negative, the reverse is the case. The columns of differences of level, and of total differences, show respectively the difference of level between each two positions of the rod, and the difference of level between each position of the rod and the first point. The method of forming the first is to subtract each foresight from the corresponding back-sight and enter it in the column of difference of level. To form the second column, each difference in the second column is added to the algebraic sum of all the preceding ones in the same column. These columns serve as a check upon the accuracy of the work.

We have supposed the level to be taken at equal distances from the rods in each case; if it is not so taken, another column must be ruled for the distances from the level to the forward rods, and a second one for the distances to the hindmost rod, and with these distances the corrections for curvature must be taken from the table already given, and subtracted from the readings, which must

then be treated as we have explained. This correction may be avoided, if, when we take the level nearer one rod than the other, we at the next station reverse it so that the level shall be nearer the second than the first. The remarks on correction for curvature, are in general applicable to the other methods of leveling yet to be described.

2. *To level for section or profile.*

The general method of proceeding is the same as in the preceding case. The annexed table shows the additional measurements that have to be taken. When a plan, as well as profile, is wanted, bearings from point to point may be taken with a compass.

FORM OF NOTE-BOOK.

Station.	Distances in Feet.	B. Sight.	Foresight.	Diff. between B.S. and F.S.	Total diff. of Level.	Remarks.
1	650	2 35	14.55	−12.20	−12.20	Commenced at bench mark A.
2	700	3.56	9.58	− 6.02	−18.22	
3	750	10.34	6.21	+ 4.13	−14.09	
4	650	14.55	0.25	+14.30	+ 0 21	
5	600	9.98	1.67	+ 8.31	+ 8.52	
6	650	3.62	14.54	−10.92	− 2.40	
	B.M.	1.23	13.45	−12.22	−14.62	Bench mark on rock.
7	500	2.23	12.05	− 9.82	−24.44	Terminating at B. M. on
8	750	6.20	19.55	−13.35	−37.79	oak-tree.

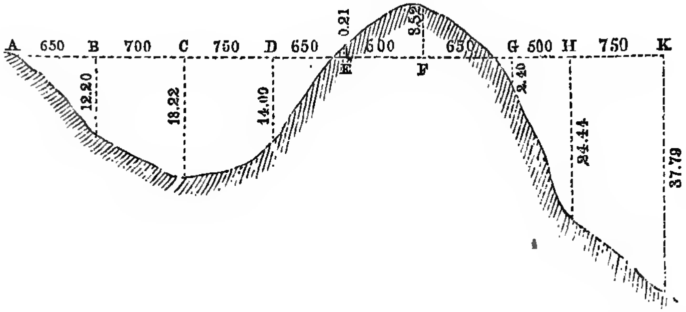
It will be seen that the point of termination is 37.79 feet below the starting point.

Plotting the Section.

The vertical distances being small in comparison with the horizontal ones, two different scales become necessary in plotting a profile.

In order that the vertical distances may be fully represented in the plot, the scale used for them must be much longer than is used for horizontal distances. This becomes absolutely necessary when long lines of profile with gentle slopes are to be plotted, as is always the case in the trial section of a railroad survey. We shall illustrate the manner of plotting. by drawing the section indicated by the field-notes just given.

Draw a horizontal line AK, called a *datum line*, and assume some point A to represent the point of beginning: lay off on the datum line distances, equal to the measured distances, 650, 700, 750, &c., feet, to K, using in this case say a scale of 1500 feet to the inch. At the points B, C, D, E, &c., thus determined, erect perpendiculars, making them equal to the corresponding differences of level taken from the field-book on a scale, say of 25 feet to the inch. Through the upper ends of these perpendiculars draw the irregular line APLM, and it will represent the surface of the ground along the line in which the section is made. We have supposed the section to be developed by means of rolling its projecting cylinder upon a tangent plane. In plotting the plan, we make use of the compass notes as determined in the survey. It is usual for the compass party to precede the leveling party, and to leave at suitable intervals marked stakes, which are used as stations by the following party of levelers. The compass party may also determine the general topographical features of the country.

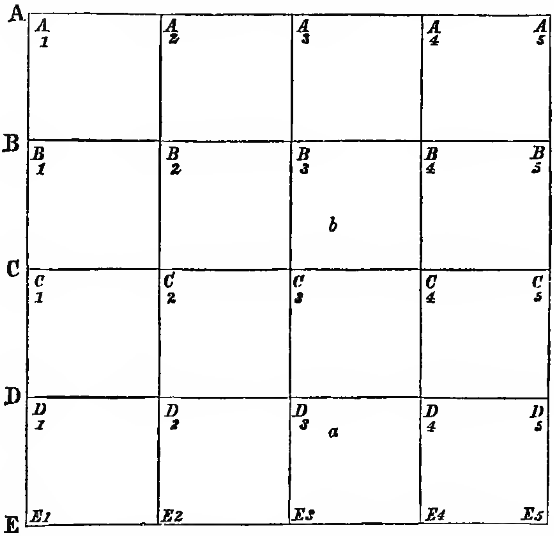


3. *To level for a Topographical Survey.*

A Topographical survey is undertaken for the purpose of determining the form and accidents of the ground, and for making such a plan as will show the minute details of the surface, its hills, its valleys, streams, &c. In

order to represent the irregularities of the surface on a plane, we conceive a succession of planes to be passed at equal distances from each other; these secant planes cut, from the surface of the ground, curves which are called *contour lines*. Now, if these contour lines be projected upon a horizontal plane, the projection will show the form of the surface; where the slope is gentle the projections will be distant from each other, and where it is rapid they will be close to each other. To a practiced eye such a delineation conveys a perfect image of the surface. The object of leveling is to determine the contour lines. The following explanation will show the general method of proceeding:

By means of a theodolite or transit, range a line of stakes, A, B, C, &c., along one side or through the middle of the ground to be surveyed, at equal and convenient distances apart, say 50 or 100 feet. Mark with a piece of red chalk on each stake, the letters A, B, C, &c., in their order. At A range a line of stakes perpendicular to AE, planting the



stakes at the same interval (of 50 or 100 feet), and mark them with the letters A_1, A_2, A_3, &c., in their order. At B range a line of stakes at right angles to AE, at the same distance apart, and mark them B_1, B_2, B_3, &c. Do the same at C, D, &c., until all the stakes are planted, dividing the area to be surveyed into squares of 50 or 100 feet on a side. The letters and figures should be plainly marked on the side of the stakes, so that there may be no difficulty in making the record.

The next step consists in determining the height of each stake above some plane taken as the plane of reference. If the sea is near, the surface of mean low water mark will afford the most natural plane of reference. If not, let the plane be taken through the lowest point of the surface, or below that point. In the example that we have taken, the plane is assumed through the lowest point of the field, supposed to be E_3. Set up the level at some convenient point, as *a*, take the reading of the leveling rod at E_3, and enter the reading in the column of back-sights; then take the reading of the rod at as many stakes as can be reached from the position of the level *a*, entering them as foresights, endeavoring to make the last reading as small as possible. At the last stake C_4 drive a small peg to serve as a bench mark. Move the level to a second point *b* and take a back-sight to C_4, and as many foresights as possible, and so on till the rod has been placed at each stake The following form will show how the height of each stake above the plane of reference is found:

FIELD NOTES.

BACK-SIGHTS.		FORESIGHTS.		DIFFERENCE.	TOTAL DIFF. OF LEVEL ABOVE E_3.		REMARKS.
Object.	Reading.	Object.	Reading.		Object.	Reading.	
					E_3	0.000	
E_3	11.432	D_3	1.211	+ 10.221	D_3	10.221	
		C	0.897	+ 0.314	C_4	10.535	Check 10.535
C_4	11.112	E_2	5.281	+ 5.831	E_2	16.366	
		E_4	6.154	− 0.873	E_4	15.493	
		D_4	6.001	+ 0 153	D_4	15.644	
		D_2	1.182	+ 4.819	D_2	20.465	
		C_3	2.917	− 1.735	C_3	18.730	
		B_5	6.080	− 3.163	B_5	15.567	
		C_5	0.921	+ 5.159	C_5	20.726	Check $\frac{10.999}{21.534}$
		B_4	0.113	+ 0.808	B_4	21.534	
B_4	11.882	E_1	8.019	+ 3.863	E_1	25.397	
		B_3	3.990	+ 4.029	B_3	29.426	
		D_1	4.118	− 0.128	D_1	29.298	
		C_2	1.880	+ 2.238	C_2	31.536	
		A_4	5.000	− 3.120	A_4	28.416	
		A_5	9.928	− 4.928	A_5	23.488	
		D_5	1.675	+ 8 253	D_5	31.741	
		E_5	1.111	+ 0 564	E_5	32.305	
		A_3	0.108	+ 1.003	A_3	33.308	
		C_1	0.004	+ 0.104	C_1	33.412	Check $\frac{11.878}{33.412}$
C_1	11.149	B_2	4.181	+ 6.968	B_2	40.380	
		B_1	2.008	+ 2.173	B_1	42.553	
		A_2	0.817	+ 1.191	A_2	43.744	Check $\frac{10.332}{43.744}$
A_2	10.102	A_1	4.332	+ 5.770	A_1	49 514	Check $\frac{5.770}{49.514}$

If we subtract the first foresight (D_3) from the back-sight (E_3), the difference, entered in the column headed *difference*, is evidently the height of D_3, above the plane of reference, through E_3, and we accordingly enter it under the column headed *total difference of level*, as well as in the column of differences.

If we subtract the foresight C_4 from the foresight D_3, the difference entered in the column of difference is evidently the height of C_4 above D_3; and if we add this difference to the previous total, we shall find the height of C_4 above E_3. Subtracting the foresight (E_2) from the back-sight (C_4), we get the difference of level between E_2 and C_4, which, added to the previous total, gives the height of E_2 above the stake E_3. In subtracting the foresight E_4 from the foresight E_2, we find a negative result, which shows that E_4 is below E_2. We then enter this difference with its negative sign, and to get the total subtract it from the previous total, and so on.

As a check on the work, subtract the foresight (C_4) from the back-sight (E_3), and the difference will give the height of C_4 above the surface of reference through E_3.

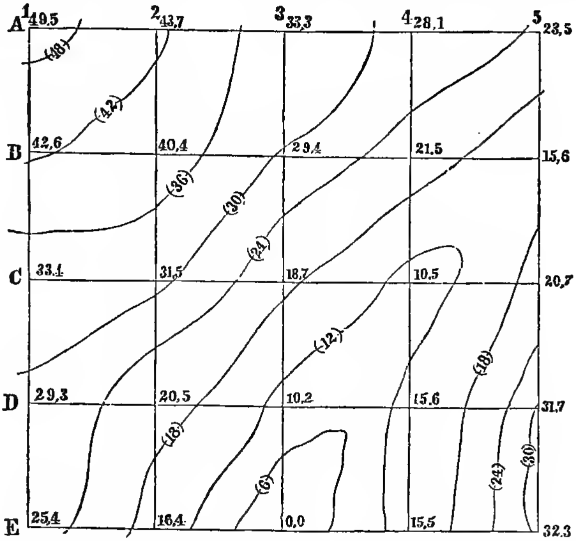
Again, subtract the foresight (B_4) from the backsight (C_4), and add the remainder to the height of C_4, and we shall find the height of B_4, which should agree with the height of B_4 as found under the heading *total difference of level;* and so on for each time that the level is moved.

Plotting the Work.

Draw on a piece of paper a straight line AE. From a scale of equal parts, set off distances AB, BC, &c., each to represent 50 or 100 feet, as the case may be. Suppose in this case 50 feet. Erect perpendiculars at each of the points A, B, C, &c., and set off on these perpendiculars from A to 2, from 2 to 3, &c., distances to represent 50 feet, and through the points 2, 3, 4, &c., draw lines parallel to AE. These, by their intersections with the lines drawn through A, B, C, &c., will determine the plot of the stakes A_1, A_2, &c.

Write in red ink on the plot the height of each stake above E_3, taken from the column of total differences in the field book.

Let us suppose the level surfaces are taken at distances of six feet apart. We may find the points in which the contour lines in pro-



jection cut the lines drawn at right angles to each other by a simple proportion, or perhaps still better by taking the plot into the field, and sketching in the lines by the eye. See *Topography*, *Topographical Surveying*, *&c.*

Leveling with the Theodolite.

To determine the difference of level between two points on a line, by means of the theodolite, set up the theodolite and examine the adjustments with great care, and level it. Measure the exact height of the axis of the telescope above the ground at the station where the theodolite is placed, and set a vane on a leveling rod at the same height, and send the rod forward to a second station. Direct the instrument upon the vane, and read the vertical limb of the instrument. Measure carefully the horizontal distance between the first and second stations. Change places with the instrument and rod, and repeat the observation of the angle of inclination ; this, with the preceding observation will give us the angle at the base of a right angled triangle, and the measured distance will be the base. Compute the perpendicular, and this will be the difference of level between the first and second stations. Now let the operation be repeated, using the second, third, fourth, &c., stations, until the final point of the line is reached. Then the algebraic sum of the partial differences of level will be the total difference of level between the two points. This method is not very accurate in practice, though there is no objection to it in theory.

Leveling with the Barometer.

When lines of levels are to be run several hundred miles in length, and only approximate results are required, resort may be had to the method by the barometer, or the boiling points of water. We shall first explain the method of finding the difference of level of two points *near each other*, by barometrical measurement.

Set up the barometer, and take the reading of the height of the mercury in the tube, the reading of a thermometer attached to the barometer, and of a thermometer detached from it, and hanging freely exposed to the air ; note also the time of making the observation. Suppose the first set of observations to be made at the lower station. Then pro-

ceed to the second or upper station, and make the same observations as soon as possible, after the first set. At an interval of time equal to that between the time of taking the first and second set of observations, let a third set be taken at the lower station. Reduce the height of the barometer's column, as last observed, to what it would have been, had it been of the same temperature as when the first set of observations were taken, by the formula

$$h'' = h\,[1 + (T - T') \times 0.0001];$$

in which h'' denotes the required height, h the observed height, T the height of the attached thermometer at the first set of observations, and T' the height of the attached thermometer at the third set of observations. Take a mean of this, and the height of the column at the first observation, as the height of the barometer at the first station; take also a mean of the temperatures of the air at the first station, and with the temperature of the mercury, as first observed, let these be considered as a set of observations made contemporaneously with the set at the second station.

The difference of level between the two stations may then be found by means of the formula

$$x = 60345.51\,[1 + .001111\,(t + t' - 64)] \times$$

$$log \left\{ \frac{h}{h'} \times \frac{1}{1 + .0001\,(T - T')} \right\} \times$$

$$(1 + 0.002695 \cos 2\phi).$$

In which ϕ denotes the latitude of the place, h the height of the barometer at the lower station, h' that at the upper station, T the temperature of the mercury at the lower station, T' that at the upper station, t the temperature of the air at the lower station, t' that at the upper station. Make

$$A = log\,(60345.51\,[1 + .001111\,(t + t' - 64)]).$$

$$B = log\,[1 + .0001\,(T - T')].$$

$$C = log\,[1 + .002695 \cos 2\phi].$$

$$D = log\,h - (log\,h' + B).$$

Then shall we have the equation

$$log\,x = A + C + log\,D\,;$$

a formula from which the value of x may be very easily computed by the aid of the following tables:

TABLE I.—THERMOMETER IN THE OPEN AIR.

$t+t'$	A	$t+t'$	A	$t+t'$	A
1°	4.74914	61°	4.77919	121°	4.80732
2	.74966	62	.77968	122	.80777
3	.75017	63	.78016	123	.80822
4	·75069	64	.78065	124	.80867
5	.75120	65	.78113	125	.80912
6	.75172	66	.78161	126	.80957
7	.75223	67	.78209	127	.81002
8	.75274	68	.78257	128	.81047
9	.75326	69	.78305	129	.81092
10	.75377	70	.78352	130	.81137
11	.75428	71	.48400	131	.81182
12	.75479	72	.78449	132	.81227
13	.75531	73	.78497	133	.81272
14	.75582	74	.78544	134	.81317
15	.75633	75	.78592	135	.81362
16	.75684	76	.78640	136	.81407
17	.75735	77	.78688	137	.81452
18	.75786	78	.78735	138	.81496
19	.75837	79	.78783	139	.81541
20	.75888	80	.78830	140	.81585
21	.75938	81	.78878	141	.81630
22	.75989	82	.78925	142	.81675
23	.76039	83	.78972	143	.81719
24	.76090	84	.79019	144	.81763
25	.76140	85	.79066	145	.81807
26	.76190	86	.79113	146	.81851
27	.76241	87	.79160	147	.81895
28	.76291	88	.79207	148	.81939
29	.76342	89	.79254	149	.81983
30	.76392	90	.79301	150	.82027
31	.76442	91	.79348	151	.82071
32	.76492	92	.79395	152	.82115
33	.76542	93	.79442	153	.82159
34	.76592	94	.79488	154	.82203
35	.76642	95	.79535	155	.82247
36	.76692	96	.79582	156	.82291
37	.76742	97	.79629	157	.82335
38	.76792	98	.79675	158	.82379
39	.76842	99	.79722	159	.82423
40	.76891	100	.79768	160	.82466
41	.76941	101	.79814	161	.82510
42	.76990	102	.79860	162	.82553
43	.77039	103	.79907	163	.82596
44	.77089	104	.79953	164	.82640
45	.77138	105	.79999	165	.82683
46	.77187	106	.80045	166	.82727
47	.77236	107	.80091	167	.82770
48	.77285	108	.80137	168	.82813
49	.77334	109	.80183	169	.82857
50	.77383	110	.80229	170	.82900
51	.77432	111	.80275	171	.82943
52	.77481	112	.80321	172	.82986
53	.77530	113	.80367	173	.83030
54	.77579	114	.80412	174	.83073
55	.77628	115	.80458	175	.83116
56	.77677	116	.80504	176	.83159
57	.77726	117	.80550	177	.83201
58	.77774	118	.80595	178	.83244
59	.77823	119	.80641	179	.83287
60	.77871	120	.80687	180	.83329

TABLE II.—ATTACHED THERMOMETER.

T−T′	B.	T−T′	B.	T−T′	B.
°		°		°	
0	0.00000	20	.00087	40	.00174
1	.00004	21	.00091	41	.00178
2	.00009	22	.00096	42	.00182
3	.00013	23	.00100	43	.00187
4	.00017	24	.00104	44	.00191
5	.00022	25	.00109	45	.00195
6	.00026	26	.00113	46	.00200
7	.00030	27	.00117	47	.00204
8	.00035	28	00122	48	.00208
9	.00039	29	.00126	49	.00213
10	.00043	30	.00130	50	00217
11	.00048	31	.00135	51	.00221
12	.00052	32	.00139	52	.00226
13	.00056	33	.00143	53	.00230
14	.00061	34	.00148	54	.00234
15	.00065	35	.00152	55	.00239
16	.00069	36	.00156	56	.00243
17	.00074	37	.00161	57	.00247
18	.00078	38	.00165	58	.00252
19	.00083	39	.00169	59	.00256

TABLE III.—*Latitude of Place.*

ϕ	C	ϕ	C	ϕ	C
0°	.00117	35°	.00040	70°	.99910
5	.00115	40	.00020	75	.99900
10	.00110	45	.00000	80	.99890
15	.00100	50	9.99980	85	.99885
20	.00090	55	.99960	90	.99883
25	.00075	60	.99942		
30	.00058	65	.99925		

Example.

	Upper Station.	Lower Station.
Height of barometer,	$h' = 23.66$	$h = 30.05$
Attached thermometer,	$T' = 70.4$	$T = 77.6$
Detached "	$t' = 70.4$	$t = 77.6$

$$B = 0.00031 \qquad \log D = 9.01502$$

$$\log h' = \underline{1.37401} \qquad C = 0.00087$$

$$1.37432 \qquad A = \underline{4.81939}$$

$$\log h \quad \underline{1.47784} \qquad 3.83528$$

$$D = 0.10352 \qquad x = 6843.7 \text{ feet.}$$

In long lines of barometric levels, it is impossible to make contemporaneous observations ; in that case the barometer used ought to be carefully compared with a standard kept at some fixed station, and the mean of a great number of observations taken at the fixed station, should be taken as the cotemporaneous observations made for each set of observations along the line.

Leveling by means of the boiling point of water.

Under the same circumstances of atmospheric pressure, temperature, and hygrometric state of the atmosphere, the boiling point of pure water is constant. As the pressure is diminished the boiling point is lowered, and generally, as the barometric readings are diminished the boiling point is lowered These facts furnish an indication of the method to be pursued, in determining the altitude of a point above some fixed level surface, taken as a surface of reference.

Previous to using a thermometer, it should be tested at the level of the sea, to ascertain its height when immersed in boiling water ; this ought to be 212°, but on account of the little care taken in the construction, it often happens that the boiling point differs from 212° as much as a degree or two. This error should be ascertained, and applied as a correction to each result obtained. The observer should have as many as two or three thermometers, and should use each one, at every point of observation. The thermometer should be fitted up with a scale, having a hinge joint, so that the bulb may be freely immersed. The most convenient apparatus for practical use, consists of a tin pot 9 inches deep and 2 inches in diameter ; a sliding tube of tin fits into the top, the head of which admits of the insertion of the thermometer through a collar of cork ; slits on the side of the tube permit the escape of steam, and keep up the communication with the external air. From 4 to 5 inches in depth of pure water is put into the pot, and the thermometer inserted in its collar of cork ; the tin slide is then moved up or down, till the bulb of the thermometer is within two inches of the bottom. Heat is applied, and after ebullition has been continued for ten or fifteen minutes, the reading is taken several times, and the temperature of the air noted. Similar operations are then performed, using a second and third thermometer. The mean of the results may be taken as a single observation.

The corresponding heights of the barometer column may then be found from the following table, and the difference of level computed, as though the height of the barometer had been read.

Table of barometric heights corresponding to different temperatures of boiling water.

Boiling Point.	Tenths of a Degree Fahrenheit.				
	0	2	4	6	8
°	Inches.	Inches.	Inches.	Inches.	Inches.
185	17.048	17.123	17.199	17.274	17.350
186	.425	.502	.578	.655	.731
187	.808	.886	.964	18.042	18.120
188	18.198	18.277	18.357	.436	.516
189	.595	.676	.756	.837	.918
190	.999	19.081	19.163	19.245	19.328
191	19.410	.493	.577	.661	.744
192	.828	.913	.998	20.083	20.169
193	20.254	20.341	20.427	.514	.601
194	.688	.776	.864	.952	21.041
195	21.129	21.219	21.309	21.398	.488
196	.578	.669	.761	.853	.944
197	22.036	22.129	22.222	22.315	22.409
198	.502	.597	.692	.786	.881
199	.976	23.072	23.169	23-265	23.362
200	23.458	.556	.654	.752	.850
201	.948	24.047	24.147	24.247	24.346
202	24.446	.547	.648	.750	.851
203	.952	25.055	25.158	25.261	25.364
204	25.467	.572	.677	.781	.886
205	.991	26.097	26.204	26.311	26.417
206	26 524	.632	.741	.849	·957
207	27.066	27.176	27.286	27.397	27.507
208	.617	.729	.841	.954	28.066
209	28.178	28.292	28.406	28.521	.635
210	.749	.865	.981	29.098	29.214
211	29.330	29.448	29.566	.685	.803
212	29.921	30.041	30.161	30.281	30.402

The following tables afford pretty good approximate results. The barometric heights from which Table I. is computed, are a little greater than those given in the preceding table, being generally those due to a boiling point about .3 of a degree Fahrenheit lower than in the above table.

Table I.

Boiling Point.	Altitude above the sea.	Boiling Point.	Altitude above the sea.
185°	14548 ft.	200°	6250 ft.
186	13077	201	5716
187	13408	202	5185
188	12843	203	4657
189	12280	204	4131
190	11719	205	3607
191	11161	206	3085
192	10606	207	2566
193	10053	208	2049
194	9502	209	1534
195	8953	210	1021
196	8407	211	509
197	7864	212	0
198	7324	213	−507
199	6786	214	−1013

Table II.

Temp're of air.	Multiplier.	Temp're of air.	Multiplier.
°		°	
32	1.000	62	1.062
33	1.002	63	1.064
34	1.004	64	1.066
35	1.006	65	1.069
36	1.008	66	1.071
37	1.010	67	1.073
38	1.012	68	1.075
39	1.015	69	1.077
40	1.017	70	1.079
41	1.019	71	1.081
42	1.021	72	1.083
43	1.023	73	1.085
44	1.025	74	1.087
45	1.027	75	1.089
46	1.029	76	1.091
47	1.031	77	1.094
48	1.033	78	1.096
49	1.035	79	1.098
50	1.037	80	1.100
51	1.039	81	1.102
52	1.042	82	1.104
53	1.044	83	1.106
54	1.046	84	1.108
55	1.048	85	1.110
56	1.050	86	1.112
57	1.052	87	1.114
58	1.054	88	1.116
59	1.056	89	1.118
60	1.058	90	1.121
61	1.060	91	1.123

To use the above tables, enter Table I. with the observed boiling point, and take out the corresponding height above the level of the sea; taking proportional parts for fractions of a degree: then, take from Table II. the multiplier corresponding to the observed temperature of the air, and form the product of these two numbers, the result will be the approximate altitude required.

1. Boiling point on a hill 204°.2, temperature of the air 76°: required the altitude of the hill above the level of the sea.

From Table I. for 204°, height,	4131 ft.
Prop. part for 0°.2, deduct,	104
	4027
Multiplier from Table II. for 76°,	1.091
Approximate height required,	4393 ft.

LIFE'-ANNUITY. See *Annuity.*

Life Assurance, or, Insurance. See *Assurance, and Insurance.*

LIFE. Of a thousand lives, equally good, any one may expect to endure till 500 are

extinct. This period has been denominated the *probable* life. The mean duration of life is found from the tables of mortality, which give out of a certain number born, the number living at each successive birthday. If the absolute average law of human life were given, and if $f(x)\,dx$ represented the chance of an individual aged n, living precisely x moments of time, then would $\int f(x)\,dx$ between the limits 0 and the *longest term of life*, represent the average duration of life of persons aged n. The tables, however, are so imperfect that it is useless to attempt the accurate application of the formula; nothing more is necessary than a very rough approximation to its application.

The tables for mean duration of life are constructed as follows. Let a denote the number living at the age in question, of whom b, c, d, &c., are left alive at the end of the successive years. Then, $a - b$ die in the first year, and as their deaths occur, scattered through the year, they enjoy amongst them $\frac{1}{2}(a - b)$ years of life, whilst the b survivors enjoy the whole year; consequently, the a persons enjoy in the first year of the calculation

$$b + \tfrac{1}{2}(a - b) = \tfrac{1}{2}(a + b) \text{ years};$$

in like manner, they enjoy

$$\tfrac{1}{2}(b + c),\ \tfrac{1}{2}(c + d), \text{ \&c. years};$$

in the second, third, &c. years of the calculation. If these results be summed, and the result divided by a, we shall have

$$\frac{\frac{1}{2}(a+2b+2c+2d+\text{\&c.})}{a} = \tfrac{1}{2} + \frac{b+c+d+\text{\&c.}}{a}$$

for the mean duration. Hence, to find the mean duration, add together the numbers left at every age, greater than that given, divide by the number alive at the given age, and to the quotient add $\frac{1}{2}$, this will be the number of years in the mean duration.

LIKE QUANTITIES. Same as similar quantities. See *Similar Quantities*.

LIMB, of an instrument, the graduated part. Any scale of equal parts, whether straight or circular, is called the limb with respect to a second scale of equal parts applied to it, for the purpose of reading to a greater degree of accuracy, than one of the equal parts which is called a vernier.

In the theodolite there are two limbs. The plane of the first is perpendicular to the axis of the instrument, and consists of a complete circle whose centre is in the axis, and whose circumference is divided into a number of equal parts, dependent upon the diameter of the circle and the accuracy of the instrument; this is called the *horizontal* limb. The *vertical* limb is generally a semi-circle whose plane is perpendicular to that of the horizontal limb, and is graduated into equal parts, whose number depends upon the diameter and the accuracy of the instrument. In the leveling rod, the graduation on the rod is often called the limb, whilst that on the vane is called the vernier.

LIM'IT. [L. *limes*, a limit; Fr. *limites*]. A quantity towards which a varying quantity may approach to within less than any assignable quantity, but which it cannot pass. Thus, the quantity $a^2 + 2ax^2$ varies with x, or it is a function of x. and approximates towards a^2 in value, as x is diminished, and may, by giving a suitable value to x, be made to differ from a^2 by less than any assignable quantity. Hence, a^2 is, properly speaking, a limit of the expression, which in this case may be found by making $x = 0$.

If a regular polygon be inscribed in a circle, and the number of sides be increased, the area of the inscribed polygon approaches that of the circle, and may be made to differ from it by less than any assignable quantity; finally, when the number of sides becomes infinite, we may regard the two areas as equal, but no supposition can cause the area of the polygon to exceed that of the circle. Hence, in this case, the area may be regarded as the limit of the area of inscribed regular polygons.

In like manner, the circumference of the circle is the limit of the perimeter of all inscribed regular polygons.

The surface and volume of the cone and cylinder are also limits of inscribed pyramids and prisms, having regular bases.

In all such cases, any property which is true for all states of the varying magnitude, is true also at the limit.

This principle is of use in deducing many useful properties of lines, surfaces, and solids. In Analysis, the principle of limits is of extensive application, and is now made the basis

of demonstration of the principles of the differential calculus.

The differential co-efficient of a function of one variable, being the limit of the ratio of an increment given to the variable, to the corresponding increment of the function, the whole of the theoretical part of the differential calculus reduces to the different processes of finding the ratios of these simultaneous increments, and from these, the ratio of their limits.

In the theory of tangents, the principle of limits is also of much value. We may define a tangent, at any point of a curve, to be the limit of all secants through the point. That is, if any secant be drawn through a point of the curve cutting it in some other point, and then be revolved about the first point till the second approaches it, and finally coincides with it, at that instant the secant becomes a tangent, or *passes to its limit.* If the revolution be continued, the line again becomes secant, cutting the curve on the other side of the assumed point.

A tangent plane to the surface at any point, is a limit of all secant planes through that point. It may be conceived, as follows: Let a plane be passed through a straight line, tangent to a section of the surface at the point; it will, in general, be a secant plane. Now, if this plane be revolved about this line as an axis, till the section cut out reduces to its limit, the plane becomes a tangent plane. No single principle has been more fruitful in results, both geometrical and analytical, than that of limits.

Limit of the Roots of a Numerical Equation. A number greater or less than any one of the roots of the equation. In this sense, there must be an infinite number of limits, and the term limit departs from its true meaning and becomes purely technical.

A Superior Limit of Positive Roots of a numerical equation, is any number greater than the greatest positive roots of the equation. The numerical value of the greatest co-efficient, increased by 1, is always a superior limit, and of course all greater numbers are also superior limits. This limit, in general, is unnecessarily great; hence, we usually seek what is called, the ordinary superior limit. This is equal to 1, increased by that root of the numerical value of the greatest negative co-efficient, whose index is the number of terms preceding the first negative term. If the co-efficient of any term is 0, that term must still be counted. The *least superior limit of positive roots*, in whole numbers, may be found by the following rule:

Write down the first member of the given equation, the second member being 0, and also its successive derived polynomials. Find by trial, such a number as will make all these polynomials, including the first member of the given equation, positive; then, if this number diminished by 1 will make the first member of the given equation negative, it is the least superior limit of the positive roots. This rule is easily applicable in most cases.

The *inferior limit of positive roots* is a number, less than any positive root of the equation. To find it, transform the equation into another by substituting $\frac{1}{y}$ for x, and by any of the preceding methods, determine the superior limit of positive roots of the resulting equation: the reciprocal of this limit is the inferior limit required.

The *superior limit of the negative roots* (numerically considered), is a negative number numerically greater than any of the negative roots of the equation. To find it, transform the given equation into another, by substituting $-y$ for x; then find by any of the preceding methods, the superior limit of positive roots of the transformed equation, and this taken with a contrary sign will be the limit required.

The *inferior limit of negative roots* (numerically considered), is a negative number numerically less than the least negative root. To find it, transform the equation into another by substituting $-\frac{1}{y}$ for x. Find by any of the preceding methods, the superior limit of positive roots of the transformed equation; the reciprocal of this limit, taken with a contrary sign, is the limit required.

The superior limit of positive and negative roots may be found very readily, by the aid of Sturm's theorem. Having determined the polynomials as directed by Sturm's rule, find by trial, two numbers which will give the same number of variations of sign, as $+\infty$ and $-\infty$; these will be superior limits of positive and negative roots, numerically considered. The inferior limits may be found by

transforming the given equation, by substituting $\frac{1}{y}$ for x; then finding the superior limits as just explained and taking their reciprocals. This method is at once the most complete and satisfactory, and the one that will, in most cases, be found most expeditious.

LIM'IT-ED. Bounded; we say that a line is limited, when only a definite portion of it is taken; this portion is said to be limited by its extreme points. A limited surface is a surface bounded by a fixed and definite line, and a limited portion of space is a portion bounded by a fixed and definite surface. In these cases the limits are of one degree inferior to the magnitude limited; that is, points limit lines, lines limit surfaces, and surfaces limit volumes.

LIM'IT-ING. Bounding; we speak of the limiting points of lines, the limiting lines of surfaces, and the limiting surfaces of volumes. Analytically, we speak of limiting values of expressions, which are nothing else than the limits of those expressions.

LINE. [L. *linea*, a line. Gr. λινον, flax]. A magnitude which has length, but neither breadth nor thickness. It possesses one, and only one, attribute of extension. In Elementary Geometry, lines are classed as *straight* and *curved*. A *straight line* is one which does not change its direction between any two of its points. A *curved line* is one which changes its direction at every one of its points. Such a line is often called a curve. A *broken* line is one made up of limited straight lines lying in different directions.

In Descriptive Geometry, lines are regarded as being generated by points, moving in accordance with some mathematical law. The moving point is called the *generatrix* of the line, and the position which the generatrix takes immediately after leaving any given point, is said to be consecutive with the given point. In accordance with this view of the subject, every line is regarded as made up of rectilineal elements, each less than any assignable line, or infinitely small.

Lines, in Descriptive Geometry, are classed as in Elementary Geometry, into *straight* and *curved* lines. Curved lines are subdivided into two classes, curves of *single curvature* and curves of *double curvature*, according to the method of generation.

A Curve of Single Curvature is one which may be generated by a point moving in such a manner that all its positions shall lie in the same plane, as the circle, the conic sections, &c.

A Curve of Double Curvature is one which may be generated by a point moving in such a manner that no more than three of its consecutive positions shall lie in the same plane, as a spiral, &c.

In either case, if any element of the curve be prolonged in the direction of the motion of the point, it is tangent to the curve at the first of the two consecutive points through which it passes.

Lines, in Analysis, are classed as *algebraic* and *transcendental*.

An Algebraic Line, is one whose rectilineal equation may be expressed by the ordinary operations of algebra, that is, by addition, subtraction, multiplication, division, raising to powers denoted by constant exponents and extraction of roots indicated by constant indices.

Transcendental Lines are those whose rectilineal equations cannot be expressed by the ordinary operations of algebra.

Algebraic lines are classed into orders, according to the degree of their equations. Lines whose equations are of the first degree, constitute the *first* order, those whose equations are of the second degree, the *second* order, and so on.

All algebraic lines of the first order are straight lines; all those of the second order are conic sections. Transcendental lines are sometimes classed according to the nature of the transcendental quantity employed in expressing their rectilineal equations, as the *logarithmic curves*, *curve of sines*, *curve of tangents*, &c. As yet no systematic classification of transcendental lines has been established.

Lines of Shade and Shadow. *The line of shade* on a body, is the line of contact of an enveloping cylinder of rays with the surface of the body.

The line of shadow, on a body, is the intersection of the surface of the body by a cylinder of rays which envelopes the body casting

the shadow. The line of shadow may be regarded as the shadow cast by the line of shade.

The line of shade in a body separates the shade from the illuminated part, and the line of shadow separates the shadow from the illuminated part. The line of shade is always a line of contact, the line of shadow is always a line of intersection. We have supposed the light to proceed from a point so far distant that the rays may be regarded as parallel; if such is not the case, the term cone may be substituted wherever the term cylinder is used, and the preceding explanation will hold true in that case. The cone of rays is a cone enveloping both the luminous body and the body casting the shadow.

LINE OF MEASURES. The line of measures of a circle, in *spherical projections*, is the line of intersection of the primitive plane with a plane passed through the axis of the primitive circle and that of the given circle. When the circle to be projected is parallel to the primitive plane, any straight line through the centre of the primitive circle is a line of measures; in all other cases there is but one line of measures, and that is found by drawing a straight line through the centre of the primitive circle, perpendicular to the trace of the plane of the circle to be projected upon the primitive plane. The line of measures possesses this property, viz.: that one of the axes of the projection of any circle always lies in it. See *Spherical Projections*.

LIN'E-AR. [L. *linearis*, pertaining to a line.] Appertaining to a line. Thus, we speak of linear dimensions, &c.

A linear expression is one, whose terms are all of the first degree. A linear equation is an equation of the first degree. A linear problem is one that can be solved by the use of right lines only.

LINK. A unit of measure employed in land surveying, equal to the hundredth part of Gunter's chain. The chain being 66 feet in length, the link must be 7.92 inches. A square link is equal to $\frac{1}{100000}$ of an acre. See *Chain*.

LIT'ER-AL. [L. *litera*, a letter]. Expressed by means of letters. A literal equation is one in which some of the known quantities are expressed by letters, as

$$ax + by = c.$$

It is so named to distinguish it from a numerical equation in which all the known quantities are expressed by numbers. A literal expression is one in which some of the quantities entering it are expressed by letters. A literal factor is a factor denoted by a letter, or some power of a letter, as a, a^2, &c.

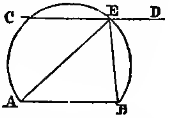
LI'TRE. A French measure of capacity, in the decimal system. It is a cubic decimeter, or a cube each edge of which is 3.9371 English inches. It contains 61.028 cubic inches; four and a half litres make about an English imperial gallon. The litre is therefore a little less than a quart, or more precisely, it is .22009687 gallon.

LŌ'CUS. [L. *locus*, a place]. The locus of a point is the line generated by the point when moving according to some determinate law.

The locus of a line is the surface generated by a line moving according to some fixed law. Thus, if a point moves in the same plane in such a manner that the sum of its distances from two fixed points of the plane is constant, the locus of the point is an ellipse. If a straight line move in such a manner that it shall always pass through a fixed point, and continually touch a fixed curve, the locus of the line is a conic surface.

To find the equation of a locus we have only to express the law of motion by means of one or more indeterminate equations.

The geometrical analysis of the Greeks depended much upon the investigation of loci; the following example will serve as an illustration. Let it be required to find a triangle whose base is equal to a given line, whose area is equal to a given area, and whose vertical angle is equal to a given angle.



Let AB be the given base; it is easy to construct the line CD, which is the locus of all the vertices of triangles having the given area and the given base. Next upon the line AB, as a chord, construct an arc of a circle which shall contain the given angle. If the parallel and arc intersect, the point of intersection E will be the vertex of the required angle. In the case taken there are

two such points, but the two triangles are exactly equal in all respects.

It is to be observed, that no curve whatever is called the locus of a point, unless any point whatever of the curve may be taken as the point in question. Thus, if each of six points should satisfy certain conditions, and all lie upon the circumference of a given circle, and if all other points of that circumference should not satisfy the same conditions, the circle could not, in the technical sense, be called the locus of the six points.

LOG, AND LOG-LINE. A contrivance for determining the velocity of a ship in passing through the water. If at any moment a piece of wood be thrown out of a ship, as soon as it strikes the water it ceases to partake of the ship's motion, the ship goes on and leaves it behind. If then, after a certain interval, say half a minute, the distance from the ship to the floating body be measured, the velocity of the ship may be ascertained. This of course supposes that there are no currents which bear the float along. When such currents exist, the log only shows the relative velocity of the ship with respect to the surface water. Such is the principle of the log and line.

In practice, the log is a flat piece of wood, generally in the shape of a quadrant, loaded with lead at one of its sides, to make it float upright; to this is attached a line about 150 fathoms in length, divided into equal parts by pieces of twine twisted into it. These divisions begin twenty or thirty yards from the log, where a piece of red rag is usually fastened in order to show the place readily. All the line between the rag and the log is called the stray line, and is omitted from the account. When the log is thrown into the sea, which is done from the lee quarter of the vessel, the log-line, by the help of a reel on which it is wound, is immediately veered out as fast as the ship sails; as soon as the red rag leaves the reel, a half-minute glass is turned, and when the sand is all run down, the reel is stopped. Then, by measuring the quantity of line run out, the distance sailed in half a minute becomes known, and thence the distance per hour is readily ascertained.

The most usual way of dividing the line, is into parts of 50 feet each, which will then indicate very nearly the number of miles per hour, by the number of knots run out in a half-minute.

The line should often be remeasured, and to prevent stretching and contracting in consequence of hygrometric changes, it may be soaked in a mixture of three parts of linseed and one part of fish-oil. The log is generally thrown every hour, and the rate recorded in a book, called the log-book.

LOG'A-RITHM [Gr. *λογος*, ratio, and *αριθμος*, number]. The logarithm of a number is the exponent of the power to which it is necessary to raise a fixed number, called the *base*, to produce the given number.

If we suppose a to preserve a constant value in the equation,

$$a^x = N,$$

whilst N is made, in succession, equal to every possible number, it is plain that x will undergo corresponding changes. The values of x, corresponding to the different values of N, are the logarithms of these values of N. In this case, a is the *base* of the system. Any positive number, except 1, may be taken as a base; and if for that particular base, we suppose the logarithms of all numbers to be computed, they constitute what is called a *system* of logarithms. There may be an infinite number of systems, but two only are used: the *common*, and the *Naperian*. In the common system, the base is 10; in the Naperian, it is 2.718281.

A table of logarithms is a table so arranged, that, by means of it, the logarithms of all numbers, within certain limits, may be found. The logarithm of a number is composed of two parts: an *entire* and a *decimal* part; the entire part is called the *characteristic*, the decimal part is sometimes called the *mantissa*. Thus, the common logarithm of 52 is 1.716003, in which 1 is the characteristic, and .716003 the mantissa.

Of the various tables of logarithms, the common table presents the following advantages: 1st, it is not necessary to write the characteristic, it being always equal to 1 plus the number of places of figures in the given number; 2d, the logarithms of decimals can be as readily found from the tables as those of whole numbers. The mantissa is the same as though the decimal were a whole number; the characteristic is negative, and

numerically, 1 greater than the number of 0's, which immediately follow the decimal point. For these reasons, and generally, on account of their greater adaptability to the decimal system, common logarithms are almost exclusively used for purposes of computation.

The following properties of logarithms are common to all systems :

1. *The logarithm of the product of any number of factors, is equal to the sum of the logarithms of the factors, taken separately.*

2. *The logarithm of the quotient of one number by another, is equal to the logarithm of the dividend, minus the logarithm of the divisor.*

3. *The logarithm of any power of a quantity, is equal to the product of the logarithm of the quantity by the exponent of the power.*

4. *The logarithm of any root of a quantity, is equal to the logarithm of the quantity, divided by the index of the root.*

These four principles are used for abbreviating the arithmetical processes of multiplication, division, raising to powers and extracting roots ; and from them we readily deduce the following rules :

1. To multiply one number by another : *find from a table the logarithms of the two numbers, and add them together ; find from the table the number corresponding to this logarithm, and it will be the product required.*

2. To divide one number by another : *find from a table the logarithms of the dividend and divisor, and subtract the latter from the former ; find from the table the number corresponding to this logarithm, and it will be the quotient required.*

3. To raise any number to any power : *find from a table the logarithm of the number. and multiply it by the exponent of the power ; find from the table the number corresponding to this logarithm, and it will be the power required.*

4. To extract any root of a number : *find from a table the logarithm of the number, and divide it by the index of the root ; find from the table the number corresponding to the logarithm, and it will be the root required.*

Besides these principles, which are of every-day use in arithmetical computations, the following are also of much importance in analytical discussions :

1. The logarithm of 1, in any system, is 0.

2. Whatever may be the base of a system, its logarithm, taken in that system, is 1.

3. In any system whose base is greater than 1, the logarithms of all numbers greater than 1 are positive ; of all numbers less than 1, negative : the logarithm of ∞ is $+\infty$; the logarithm of 0 is $-\infty$.

4. In any system of logarithms whose base is less than 1, the logarithms of all numbers greater than 1, are negative ; of all numbers less than 1, positive : the logarithm of ∞ is $-\infty$; and the logarithm of 0 is $+\infty$.

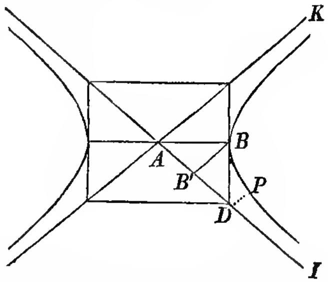
5. No negative number has a real logarithm.

Every logarithm is composed of two factors : one depending upon the base of the system, and constant ; the other upon the number itself, and changing as the number changes.

The first factor is called the *modulus* of the system. The *modulus of a system* of logarithms may be defined to be that number by which it is necessary to multiply the Naperian logarithm of any number, in order to produce the logarithm of the same number in that system.

The modulus of the common system is .434294482 . . . ; and that of the Naperian system is 1. The modulus of any system is equal to the reciprocal of the Naperian logarithm of the base of that system.

The Naperian system is that to which all other systems are referred, and from which any other system may be readily constructed. The Naperian logarithms are sometimes called hyperbolic logarithms, from their relation to certain areas included between the equilateral hyperbola and its asymptotes.



Let BP represent one branch of an equilateral hyperbola, and AK, AL, its asymp-

totes. Assume AB′, the abscissa of the vertex, as 1; then will the number of superficial units, in any area included between the ordinate BB′ and any other ordinate PD, be equal to the Naperian logarithm of the abscissa of the extreme ordinate PD; that is, the area $B'BPD = l\,(AD)$.

There is no reason why the Naperian logarithms should be called hyperbolic, rather than those of any other system; for, the same relation which exists between the Naperian system and the equilateral hyperbola, also exists between other systems and oblique hyperbolas. In the case of oblique hyperbolas, the area is limited by two oblique ordinates, and the modulus of the system is always equal to the sine of the angle between the ordinates.

In a system, whose base is greater than 1, the modulus is *positive*; in one, whose base is less than 1, the modulus is *negative*.

Computations of Tables of Logarithms.

These computations are generally effected by the aid of series, of which the fundamental one, however deduced, is called the logarithmic series; it is the following:

$$\log(1+y) = M\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \&c.\right) (1),$$

in which *log* denotes a logarithm taken in any system, M the modulus of that system, and y any number whatever. If we make $M = 1$, we have the Naperian system, and denoting the logarithms in this system by the symbol l, we have

$$l(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \&c. \ldots (2)\,..$$

This series is not sufficiently converging to be employed in computing tables, but it may readily be transformed so as to be used for that purpose. If in (2) we write $-y$ for y, it becomes

$$l(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \frac{y^5}{5} \cdots (3);$$

and, subtracting (3) from (2), member from member, and writing for

$$l(1+y) - l(1-y), \text{ its value } l\left(\frac{1+y}{1-y}\right),$$

we have

$$l\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \&c.\right) \cdots (4).$$

Making $y = \frac{1}{2z+1}$, and writing for $l\left(\frac{z+1}{z}\right)$, its value $l(z+1) - lz$, we have, after transposition, the formula,

$$l(z+1) = lz + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right) \cdots\cdots (5),$$

which is rapidly converging.

It is to be observed, that in computing tables, it is only requisite to compute the logarithms of prime numbers, because those of composite numbers may be found more easily by the properties of logarithms already explained.

To show the use of formula (5), we shall compute the Naperian logarithms of the first ten numbers.

The logarithm of 1 is 0. If we make $z=1$ in formula (5), we can find from it the logarithm of 2, and so on, as follows:

$$l1 = 0 \quad \ldots\ldots \quad = 0.000000$$

$$l2 = 2\left(\frac{1}{3} + \frac{1}{3.(3)^3} + \frac{1}{5.(3)^5} + \frac{1}{7.(3)^7} + \&c.\right) \quad = 0.693147$$

$$l3 = 0.693147 + 2\left(\frac{1}{5} + \frac{1}{3.(5)^3} + \frac{1}{5.(5)^5} + \frac{1}{7.(5)^7} + \&c.\right) = 1.098612$$

$$l4 = 2 \times l2 \quad \ldots\ldots \quad = 1.386294$$

$$l5 = 1.386294 + 2\left(\frac{1}{9} + \frac{1}{3.(9)^3} + \frac{1}{5.(9)^5} + \frac{1}{7.(9)^7} + \&c.\right) = 1.609437$$

$$l6 = l2 + l3 \quad \ldots\ldots \quad = 1.791759$$

$$l7 = 1.791759 + 2\left(\frac{1}{13} + \frac{1}{3.(13)^3} + \frac{1}{5.(13)^5} + \frac{1}{7.(13)^7} + \&c.\right) = 1.945910$$

$$l8 = l4 + l2 = 3l2 \quad \ldots \quad = 2.079441$$

$$l9 = 2 \times l3 \quad \ldots\ldots \quad = 2.197224$$

$$l10 = l5 + l2 \quad \ldots\ldots \quad = 2.302585$$

&c. &c. &c. &c.

In like manner, the logarithms of the series of natural numbers might be computed to any desired extent in the Naperian system. Then, to get a common table, we need only

multiply each logarithm by the modulus of the common system.

The operation of computing logarithms by the formula given, is extremely tedious, and to facilitate it other series, still more converging, have been deduced.

The following is Borda's series:

$$l(z+2) = 2l(z+1) - 2l(z-1) + l(z-2) + 2\left[\frac{2}{z^3-3z} + \frac{1}{3}\left(\frac{2}{z^3-3z}\right)^3 + \frac{1}{5}\left(\frac{2}{z^3-3z}\right)^5 + \text{\&c.}\right]\cdots$$

From this series, we can compute the logarithm of $z+2$, when we know those of

$$z+1, \quad z-1, \quad \text{and} \quad z-2.$$

This series is rapidly converging, but a still more converging one is that of Haros, as follows:

$$l(z+5) = l(z+4) + l(z+3) - 2lz + l(z-3) + l(z-4) - l(z-5) - 2\left[\frac{72}{z^4-25z^2+72} + \frac{1}{3}\left(\frac{72}{z^4-25z^2+72}\right)^3 + \right].$$

This series requires six logarithms to be known.

When only two logarithms are given, the following series is pretty converging:

$$lz = \tfrac{1}{2}[l(z-1) + l(z+1)] + \frac{1}{2z^2-1} + \frac{1}{3}\left(\frac{1}{2z^2-1}\right)^3 + \text{\&c.}\Big]\cdots$$

No examples of the application of these formulas need be given, as the student who wishes to examine further into the subject, would be likely to consult more extended treatises.

General Logarithms. If we denote the base of the Naperian system by e, we shall have the equation,

$$e^y = x \ldots . (1),$$

in which y is the Naperian logarithm of x for all values of x. Hitherto, we have considered only the real values of y, which correspond to the *arithmetical* logarithms. There is, besides, an infinite number of imaginary values for y that will satisfy the equation, and which constitute what may be called the *algebraic* logarithm of x. The arithmetical and algebraic logarithms, taken together, make up the *general* logarithm, which we shall designate by the symbol L.

Let us assume the equation,

$$e^{\theta\sqrt{-1}} = \cos\theta + \sin\theta\sqrt{-1},$$

and in it make

$$\theta = 2m\pi,$$

m being any whole number, positive or negative. We shall have

$$\cos 2m\pi = 1, \quad \sin 2m\pi = 0;$$

and, consequently,

$$e^{2m\pi\sqrt{-1}} = 1 \ldots . (2).$$

Multiplying equations (1) and (2), member by member, we shall have

$$e^{y+2m\pi\sqrt{-1}} = x \ldots . (3).$$

Hence, if y is the arithmetical logarithm of x, then is

$$y + 2m\pi\sqrt{-1},$$

the general logarithm of x, or denoting the Naperian logarithm by the symbol l, as we have done heretofore, we shall have the equation

$$Lx = lx + 2m\pi\sqrt{-1} \ldots . (4);$$

we have, also,

$$Lz = lz + 2n\pi\sqrt{-1} \ldots . (5).$$

By adding equations (4) and (5), member to member, and subtracting (5) from (4), member from member, we deduce the general formulas,

$$L(xz) = l(xz) + 2(m+n)\sqrt{-1},$$

$$\text{or} \quad L\left(\frac{x}{z}\right) = l\left(\frac{x}{z}\right) + 2(m-n)\sqrt{-1} \ldots (6).$$

It has already been stated that a negative number has no real logarithm; it has, however, a general logarithm, which is imaginary If we make

$$\theta = (2m+1)\pi,$$

we shall have, m being a whole number,

$$e^{(2m+1)\pi\sqrt{-1}} = \cos(2m+1)\pi + \sin(2m+1)\pi\sqrt{-1} = -1.$$

Whence,

$$L(-1) = (2m+1)\pi\sqrt{-1}.$$

Now,

$$L(-x) = Lx + L(-1) = lx + 2n\pi\sqrt{-1} + (2m+1)\pi\sqrt{-1},$$

$$\text{or} \quad L(-x) = lx + (2m+1)\pi\sqrt{-1},$$

since for $2m+1+2n$, we may write $2m+1$.

From the value of $L(-1)$, we deduce the relation,

$$(2m+1)\pi = \frac{L(-1)}{\sqrt{-1}}.$$

If $m = 0$, we have the result,

$$\pi = 3.1416 = \frac{L(-1)}{\sqrt{-1}}.$$

ANTI-LOGARITHMS. It may not be inappropriate in this connection to explain the method of computing a table of anti-logarithms.

Assume the exponential equation,

$$a^x = N,$$

and in it substitute for a, $1+p$, and for x, the fraction $\frac{h}{k}$, h and k being so small that their higher powers may be neglected when added to finite quantities, and we shall have

$$N = (1+p)^{\frac{h}{k}}, \quad \text{or} \quad N^k = (1+p)^h = 1 + hp + \frac{h(h-1)}{1.2}p^2 + \frac{h(h-1)(h-2)}{1.2.3}p^3 + \&c.$$

Neglecting the terms involving the higher powers of h, we have

$$N^k = 1 + h(p - \tfrac{1}{2}p^2 + \tfrac{1}{3}p^3 - \&c.) =$$
$$1 + h[(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.]$$

The quantity within the parenthesis is equal to the reciprocal of the modulus of a system of logarithms, whose base is a; denoting this reciprocal by A, we have

$$N^k = 1 + hA, \quad \text{or} \quad N = (1+hA)^{\frac{1}{k}}.$$

Developing the second member by the exponential the equation, formula becomes

$$N = 1 + \frac{1}{k}(hA) + \frac{1-k}{1.2k^2}(hA)^2$$
$$+ \frac{(1-k)(1-2k)}{1.2.3.k^3}(hA)^3 + \&c.;$$

and, neglecting the powers of k, higher than the first, we have

$$N = 1 + \frac{h}{k}A + \frac{1}{2}\frac{h^2}{k^2}A^2 + \frac{1}{2.3}\frac{h^3}{k^3}A^3 + \&c.$$

replacing x for $\frac{h}{k}$, we get

$$N = 1 + Ax + \frac{1}{2}A^2x^2 + \frac{1}{2.3}A^3x^3 + \&c.,$$

which gives the value of any number, in terms of its logarithm, and the reciprocal of the modulus of the system.

Now, $A = \frac{1}{M}$, and $x = \log N$, M being the modulus; these, substituted in the last equation, give

$$N = 1 + \frac{\log N}{M} + \frac{1}{2}\left(\frac{\log N}{M}\right)^2 + \frac{1}{2.3}\left(\frac{\log N}{M}\right)^3$$
$$+ \frac{1}{2.3.4}\left(\frac{\log N}{M}\right)^4 + \&c. . .$$

This series is not immediately adapted to computation, but can easily be converted into one that is so. Let us assume

$$N = \frac{m}{n}; \quad \text{whence} \quad \log N = \log m - \log n;$$

replacing the second member by d, and passing to the Naperian system, by making $M=1$, we shall have, after reduction,

$$m = n\left(1 + d + \frac{d^2}{1.2} + \frac{d^3}{1.2.3} + \&c.\right) \cdots$$

If, now, we compute the anti-logarithms of a series of logarithms, differing from each other by .00001, we shall have $d = .00001$, which gives

$$m = n\left(1 + .00001 + \frac{(.00001)^2}{1.2}\right.$$
$$\left. + \frac{(.00001)^3}{1.2.3} + \&c.\right) = n(1.00001000005),$$

in which m and n are two numbers, whose logarithms differ by .00001.

If we commence by supposing

$$lm = .00001, \text{ whence } n = 0,$$

we find the anti-logarithm of

.00001 to be 1.00001000005.

Making $lm = .00002$, we have $ln = .00001$, which, substituted above, give the anti-logarithm of .00002 to be $(1.00001000005)^2$.

In like manner, the anti-logarithms of .00003, .00004, &c., may be successively computed.

In the common system, the formula for computation of anti-logarithms is

$$m = n \times (1.000023026116),$$

which gives

$$a = \log(.00002) = (1.000023026116)^2,$$
$$a = \log(.00003) = (1.000023026116)^3,$$
$$a = \log(.00004) = (1.000023026116)^4.$$
&c. &c. &c.

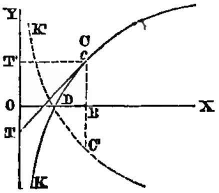
LOGARITHMIC CURVE. A curve that may be referred to a system of rectangular co-ordinate axes, such that the ordinate of any point will be equal to the logarithm of its abscissa. Its equation, when referred to this system, is of the form

$$y = \log x;$$

for any point C, we have the relation

$$CB = \log OB.$$

The axis of X is called the axis of numbers, and the axis of Y the axis of logarithms. The particular curve will depend upon the particular system



in which the logarithms are taken, but in all cases there are certain general properties, which we proceed to enumerate.

The curve always cuts the axis of X at a point D, whose distance from the origin of co-ordinates is equal to 1. If the base of the system of logarithms is greater than 1, the curve takes the position KDC. The axis of Y is an asymptote to the curve, at an infinite distance below the origin; the part DK is convex towards the axis of X, and the part DC concave towards the same line, both being convex towards the axis of Y. If the base of the system is less than 1, the curve assumes the position C′DK′; the axis of Y is an asymptote to the curve above the origin; the part DK′ is convex towards the axis of X; the part DC′ is concave towards the axis, and both are convex towards the axis of Y.

If a tangent be drawn to the curve at any point C, the sub-tangent TT′, taken on the axis of logarithms, is constant, and equal to the modulus of the system.

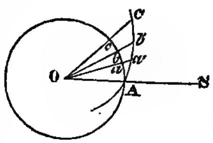
If we suppose the base of the system equal to 1, the curve reduces to a straight line through D and parallel to the axis of Y. This is the limit of the two classes of logarithmic curves, and separates the two systems.

If AB = 2, the area ODCT′ is equal to the entire area between the part DK, the axis of X, and the axis of Y, each being equal to the modulus of the system of logarithms.

Logarithmic Spiral. A spiral whose equation is of the form

$$\log r = v,$$

in which the pole is at the eye of the curve. Let O be the pole, OS the initial line. Then is the origin



of the spiral at A, OA being equal to 1. There are an infinite number of spires outwards from A, and an infinite number of spires inwards towards the eye of the curve O. If any number of radii-vectors be drawn, making equal angles with each other, these radii will be in continued proportion. That is, if

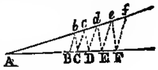
$$AOa = aOb = bOc = \&c.,$$

then will

$$OA : Oa' :: Oa' \quad Ob' :: Ob' : Oc' :: \&c.$$

This property affords the means of making a graphic construction of the curve by points. Suppose that we know the radius vector Oa' and the angle AOa'. Draw any number of lines OA, Oa, Ob, Oc, &c., making angles with each other in succession, equal to AOa'.

Draw any two straight lines, A′F and $A'f$, intersecting each other at A′: make A′B = OA, and $A'b = Oa'$; with A′ as a centre, and a radius $A'b$, describe an arc of a circle cutting AF in C. Draw Cc parallel to the line Bb; with A′ as a centre, and $A'c$ as a radius, describe the arc cD and draw Dd parallel to Bb, and so on: Lay off



$$Ob' = A'c, \ Oc' = A'd, \ \&c.,$$

the points a', b', c', &c., will be points of the curve.

If it is required to find a point of the curve intermediate between any two constructed points, bisect the angle between the radii vectors of these points, and on the bisecting line lay off from O a radius vector equal to a mean proportional between the two radii. If a tangent be drawn to the curve at any point, the angle between this line and the radius vector of the point is constant, and its tangent is equal to the modulus of the system.

If, with O as a centre, and a radius equal to 1, a circumference be described, we may regard this as the directrix of the curve. Now, if distances be laid off from A on this circle, respectively equal to the ordinates of points of the logarithmic curve, and radii be drawn through the extremities of these distances equal to the corresponding abscissas, their extremities will lie in the logarithmic spiral taken in the same system. This shows the intimate connection between the logarithmic curve and the logarithmic spiral. See

Spiral. Both these curves are sometimes called logistic curves.

Newton has shown, that if the intensity of the force of gravity had varied inversely as the cubes of the distances, the planets would have continually receded from the sun in paths which would have been logarithmic spirals. The logarithmic spiral is the equatorial projection of the rhumb line.

LO-GIS'TICS, or, LO-GIS'TIC-AL ARITHMETIC. The same as Sexagesimal arithmetic, that is, that system of arithmetic in which numbers are expressed in the scale of 60. The use of this scale is almost entirely confined to trigonometrical operations for expressing fractional parts of a circumference, or of a right angle.

Logistic Logarithms. The logistic logarithm of a number of seconds is the excess of the logarithm of 3600 over the logarithm of the given number of seconds.

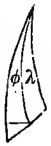
For example, to form the logistic logarithm of 3′ 20″ or 200″, we take the logarithm 2.3010 from 3.5563 and we have 1.2553 for the logistic logarithm of 3′ 20″. Logistic logarithms are tabulated and employed in certain astronomical computations.

LON'GI-TUDE. [L. *longitudo*, from *longus*, long]. The longitude of a place, is the arc of the equator intercepted between the meridian of the place and a meridian passing through some other place from which longitude is reckoned. Longitude, in this country, is most generally reckoned from the meridian of Greenwich. It is also frequently reckoned from the meridian of Washington.

LOSS AND GAIN. A rule of Arithmetic employed by merchants to find the amount lost or gained in the purchase and sale of goods. It is also used to discover the price at which goods must be sold to insure a certain amount of profit. It is merely a practical application of the Rule of Three. See *Cause* and *Effect*, *Rule of Three*, *&c.*

LOX-O-DROM'IC CURVE. [Gr. λοξος, oblique, δρομος, course]. A curve bearing a strong resemblance to the logarithmic spiral. It is traced upon the surface of a sphere by a point moving in such a manner that its path cuts all the meridians at the same angle. In Navigation, the loxodromic curve is the same as the rhumb line, and is the path of a ship sailing always in the same tack. The loxodromic curve turns continually about the pole, but does not reach it till after an infinite number of turns.

To find the equation of a loxodromic curve, let ϕ denote the arc of the meridian intercepted between any point of the curve and the pole, and λ the longitude of the point; then the infinitely small arc of the parallel of latitude corresponding to an increment of the curve is $d\lambda \sin \phi$, and the differential of the co-latitude is $d\phi$; but because the curve cuts all the meridians in the same angle, the variation of the parallel of latitude corresponding to an increment of the curve, is proportional to the variation of the co-latitude; consequently,



$$a d\lambda \sin \phi = d\phi, \quad \text{or,} \quad a d\lambda = \frac{d\phi}{\sin \phi},$$

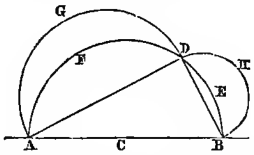
which is the differential equation of the loxodromic curve, a being a constant quantity. See *Navigation*.

LOZ'ENGE. An equilateral rhomboid, or rhombus. See *Rhombus*.

LŪ'MIN-OUS BODY. In Shades and Shadows, a body which emits light, as the sun.

LŪNE. [L. *luna*, the moon]. The area included between the arcs of two circles which intersect each other.

The lune of Hippocrates is famous as having been the first curvilinear space, whose area was exactly determined. The construction of the lune of Hippocrates is as follows:



On the line AB, as a diameter, describe a semicircle, and within it inscribe a right angled triangle ADB; then on each of the sides, AD and DB, as diameters, describe semicircles: the two figures AGFD and DHBE will be lunes, and the sum of their areas will be equivalent to that of the triangle ADB.

For, semicircles are to each other as the

squares of their diameters; but

$$AB^2 = AD^2 + BD^2.$$

Hence, the semicircle ADB is equivalent to the sum of the semicircles AGD and DHB; from these equivalents take away the common segments AFD and DEB, and there remains the right angled triangle ADB equivalent to the sum of the two lunes, AGDF and DHBE.

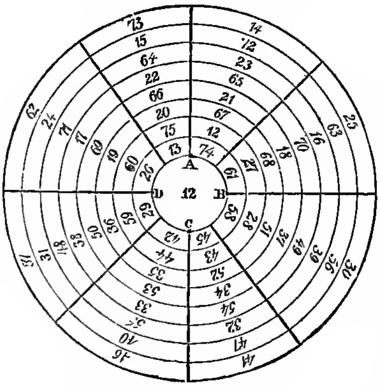
If the two sides AD and DB are equal, the lunes are equal, and each is equivalent to one-fourth of the inscribed square.

This property is called the property of the lune of Hippocrates.

This property has been employed by geometers as a stepping-stone to the solution of the great problem of squaring the circle, but of course, without success. It has, however, led to many interesting results.

M. The thirteenth letter of the English alphabet. It is one of the Roman numeral characters; it stands for 1000. With a dash over it, thus, $\overline{M}$, it stands for 1,000,000.

MAGIC CIRCLE OF CIRCLES. Invented by Dr. Franklin, founded upon the same principles and possessing similar properties with the magic square of squares.



It consists of 8 concentric rings, each divided by radii into 8 equal parts; within the circular spaces thus formed, are written the numbers from 12 to 75, and in the middle the number 12 is also placed.

It possesses the following properties:

1. The sum of all the numbers in any ring, together with the number in the middle, is equal to 360, the number of degrees in a circumference.

2. The sum of the numbers between two consecutive radii, together with the number at the centre, is equal to 360.

3. The sum of the numbers in any half ring, taken either above or below the double horizontal line, with half the number at the centre, is 180.

4. If any four adjoining numbers, as if in a square, be taken, their sum, together with half the central number, is equal to 180.

5. There are also four other sets of circular spaces, bounded by eccentric circles with regard to the primitive centre, each of which set contains 5 spaces, their centres being at A, B, C and D.

These sets intersect the first sets and each other, and yet the sum of the numbers in each of the eccentric spaces taken all around through the 20, which are eccentric, together with the central number, is 360°. Their halves also taken above or below the double central line, together with half the central number, make 180.

MAGIC SQUARE. Several numbers arranged in the form of a square, disposed in parallel and equal ranks, so that the sums of each row taken horizontally, vertically, or diagonally, shall be equal to the same number.

The simplest form of the magic square is formed by arranging the first 9 numbers, giving the square annexed. Here, the sum of any row of three figures, whether horizontal, vertical, or diagonal, is equal to 15.

4	9	2
3	5	7
8	1	6

The magic square formed from the first 25 numbers, is arranged as in the annexed diagram. In it every row of 5 numbers, in whatever direction taken, has for its sum 65. There are many different ways in which numbers may be arranged so as to form magic squares.

16	14	8	2	25
3	22	20	11	9
15	6	4	23	17
24	18	12	10	1
7	5	21	19	13

The following rule enables us to form a magic square of an odd number of terms in geometrical progression.

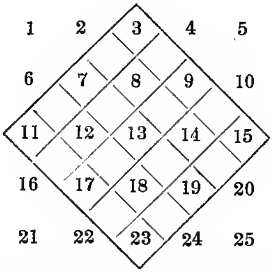
22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

Rule a square, and divide it into the required number of cells; place the number 1 in the cell immediately undor the central one, and the succeeding terms in their natural order in a descending diagonal direction till they run off, either on the bottom or on the side; when they run off at the bottom, carry the next term to the uppermost cell that is not occupied of the same column that it would have occupied below; then proceed as before as far as possible; or till the numbers run off at the bottom, or side, or are interrupted by coming to a cell already filled. Now when a term runs off at the right-hand side, bring it to the furthest cell of the same row it would have fallen in to the right; when the progress diagonal-wise is interrupted by coming to a full cell, descend diagonally to the left till an empty cell is met with, and then enter it, proceeding as before till all the terms are distributed.

The number 4 runs off at the bottom and is carried to the top of the next column; the number 5 runs off at the side and is carried to the left of the next row below; the number 8 falls upon an occupied cell, and is carried diagonally to the left; 10 runs off at the bottom and is carried to the top of the next column; 13 runs off at the side and is carried to the left of the next row below; 15 falls upon an occupied cell and is carried diagonally to the left; 16 runs off at the bottom and is carried to the top of the next column; 21 runs off at the side and is carried to the left of the next row; 22 falls upon an occupied cell, and on being carried diagonally to the left, runs off at the bottom; it is then placed in the highest cell at the top of the column it would have occupied: 29 runs off both at the bottom and side, and is carried to the highest vacant cell in the same column; and so on. These examples, and a little study of the figure, will show how any magic square of an odd number of cells, may be constructed.

There is another and simpler rule than that just given, for forming a magic square. Suppose the first 25 numbers are to be arranged in a magic square. First set down the num-

(1)



bers in the form of a natural square, as shown in diagram (1). Next draw straight lines, cutting off three numbers at each corner, viz: 1, 2, and 6, at the left hand uppei corner; 4, 5, and 10 at the right hand upper corner; 16, 21, 22 at the lower left hand corner; and 20, 24, 25 at the lower right hand corner; these four lines form a square; then draw inner lines parallel to these, as in the

(2.)

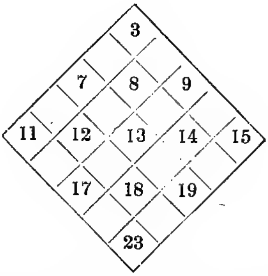
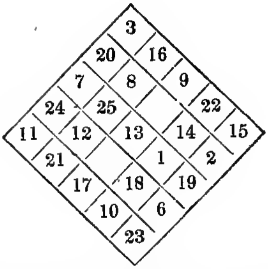


diagram dividing the square into 25 cells; 13 of these cells will be occupied by numbers, as shown in diagram (2). The empty cells are to be filled from the 4 corners as follows each number in the corners is to be carried obliquely up or down along the row where it is found, to the most remote vacant cell, and then written.

The diagram (3) indicates the manner of filling up. The principles explained are

(3).



applicable to any arithmetical progression as well as to the progression of natural numbers.

If the numbers of a geometrical progression be arranged magically, the continued product of the numbers in any row is constant. Let us take the progression

1, 2, 4, 8, 16, 32, 64, 128, 256,

and arrange it magically, we shall have the arrangement exhibited in diagram (4).

(4.)

8	256	2
4	16	64
128	1	32

If the terms of a harmonical progression be arranged magically, the numbers in each row will be in harmonical progression. Such an arrangement is exhibited in diagram (5).

(5.)

1260	840	630
504	420	360
315	280	252

Magic Square of Magic Squares. A magic square of 256 cells filled up by the numbers from 1 to 256, arranged as in diagram (6). It possesses the following properties:

1. The sum of the four numbers in any four cells forming a square, wherever taken, is equal to 514.

MAGIC SQUARE OF SQUARES.

226	255	18	15	194	223	50	47	162	191	82	79	130	159	114	111
32	1	240	241	64	33	208	209	96	65	176	177	128	97	144	145
239	242	31	2	207	210	63	34	175	178	95	66	143	146	127	98
17	16	225	256	49	48	193	224	81	80	161	192	113	112	129	160
228	253	20	13	196	221	52	45	164	189	84	77	132	157	116	109
30	3	238	243	62	35	206	211	94	67	174	179	126	99	142	147
237	244	29	4	205	212	61	36	173	180	93	68	141	148	125	100
19	14	227	254	51	46	195	222	83	78	163	190	115	110	131	158
230	251	22	11	198	219	54	43	166	187	86	75	134	155	118	107
28	5	236	245	60	37	204	213	92	69	172	181	124	101	140	149
235	246	27	6	203	214	59	38	171	182	91	70	139	150	123	102
21	12	229	252	53	44	197	220	85	76	165	188	117	108	133	156
232	249	24	9	200	217	56	41	168	185	88	73	136	153	120	105
26	7	234	247	58	39	202	215	90	71	170	183	122	103	138	151
233	248	25	8	201	216	57	40	169	184	89	72	137	152	121	104
23	10	231	250	55	42	199	218	87	74	167	186	119	106	135	154

2. The sum of the 16 numbers in any 16 cells, forming a square, is equal to $514 \times 4 = 2056$.

3. The sum of the 36 numbers in any 36 cells, forming a square, is equal to $514 \times 9 = 4626$, &c.

4. The sum of the four corner numbers of either of these squares is always equal to 514

5. The sum of the 16 numbers in any bent row, whose halves are parallel to the diagonals, is 2056. Thus, from 26 to 36, and from 173 to 151, is a bent row; also, from 74 to 22, and from 227 to 191, is another bent row.

6. If the square be divided horizontally or vertically through the middle, the halves may change places, and the properties of the square will remain the same as before.

7. If the square be cut into 16 squares, it is manifest that any four of them will make a magic square of 64 cells; any nine of them will make a magic square of 144 cells. Consequently, as many magic squares of 64 cells, and also of 144 cells, may be made of the 16 partial squares, as there are different combinations of 16 quantities taken in sets of four and in sets of nine.

The construction of the great square depends upon that of a magic square of 16 cells having the sum of the four numbers in any square of four cells always the same. To construct such a square from the series 1, 2, 3, ... &c., to 16: *First*,

$$\frac{(16+1)\,8}{4} = 34,$$

the sum of each column or diagonal, or in the four cells. Now arrange the 16 numbers as in diagram (7); then from the nature of the case the sum of the numbers in each diagonal is 34. The excess above 34 in the sum of the 4th column is equal to the defect in the sum of the first column, and the excess of the sum of the lower row is equal to the defect in the sum of the upper row; hence, the corner numbers remaining the same, if 2 and 15, 3 and 14, 5 and 12, 9 and 8, respectively change places, we have the magic square (8). In this square, however, only the middle square of four cells and the four corner ones contain 34 each. But in magic squares varieties are readily obtained by shifting the columns; thus, in (8), the second column from the left may be made the last column on the right, or the third from the left may be made the first, or the two upper or the two lower rows may take the place of each other. Let the third column from the left be brought to the left, and change the two upper horizontal rows one for the other, and we get the magic square (9), having the sum of the four numbers, in any square of four cells, equal to 34. The numbers in (9) consist of pairs situated alternately, the sums being 16 and 18 respectively. Hence, to make the great square with the series 1, 2, . . . to 256, let the numbers be arranged thus:

(7.)

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

(8.)

1	12	8	13
15	6	10	3
14	7	11	2
4	9	5	16

(9.)

10	15	6	3
8	1	12	13
11	14	7	2
5	4	9	16

256	255	254	253	&c.
1	2	3	4	&c.,

the pairs, taken diagonally, making 258, 256, 258, &c., and call the upper numbers complements of the lower numbers; then arrange the first 32 numbers of the lower series, as in diagram (10); next place the 8 upper numbers with their complements in a square of 16 cells, in the same order, from the least to the greatest, as they stand in diagram (9), and we shall have the first or corner square, on the left at the top.

(10).

1	16	17	32
2	15	18	31
3	14	19	30
4	13	20	29
5	12	21	28
6	11	22	27
7	10	23	26
8	9	24	25

The next 8 numbers, or 3, 4, 13, 14, 19, 20, 29, 30, with their complements, make the second square downwards; and for the next four squares, the numbers from 32 to 64 are arranged as above, and two more arrangements, viz: from 64 to 96, and from 96 to 128, in the same manner, will complete the 16 squares.

MAG-NET'IC. The magnetic bearing of a course is the angle included between a course and a magnetic meridian, drawn through the first extremity of the course. See *Bearing*.

MAGNETIC MERIDIAN. If a vertical plane be passed through the axis of a magnetic needle, freely suspended at a point, its intersection with the surface of the earth is called a magnetic meridian of the point. The angle included between this meridian and the true meridian through the point, is called the variation of the needle.

MAGNETIC NEEDLE. A bar of steel in the shape of a needle, magnetized and suspended at its centre upon a sharp-pointed pivot. It is the essential part of the surveyor's compass, and of the mariner's compass. See *Compass*.

MAG'NI-TUDE. [L. *magnitudo*, magnitude]. *Size, extent, quantity*. This term was originally applied to signify the space occupied by a body. As thus used, it applied only to those portions of space which possessed the three attributes of extension: length, breadth, and thickness, or height. By extension of meaning, it has come to signify anything that can be increased, diminished, and measured. Thus, a line or a surface, an angle, or a number, are magnitudes. Time and weight are magnitudes; and, in general, anything of which greater or less can be predicated, is a magnitude. See *Quantity*.

MAN-TIS'SA. [L. *mantissa*, addition]. The decimal part of a logarithm. Thus, the logarithm of 900 being 2.95424, the part .95425 is the mantissa.

MAP. [L. *mappa*, a napkin]. A representation of a portion of the earth's surface upon a plane. A representation of a portion of the heavens upon a plane, is also called a map. There are, therefore, two kinds of maps, *terrestrial* and *celestial*: the former only will be considered.

Terrestrial maps are of two kinds, those which represent portions of land and water together, which are properly called *maps*, and those which represent portions of the ocean, only indicating the directions of currents, soundings, anchorages, rocks, shoals, buoys, lighthouses, &c.; these are called *hydrographical maps* or *charts*.

Maps are *geographical* and *topographical*. *Geographical* maps represent the boundaries of countries, the positions of the principal towns, rivers, lakes, mountain ranges, &c.

Topographical maps represent the minuter features of the surface of the earth, and are more full in detail than geographical maps. On account of the greater minuteness of detail required, topographical maps generally represent smaller portions of the earth's surface than are embraced in geographical maps.

Besides geographical and topographical maps, there are others constructed for a special purpose, such as geological, statistical, historical, ethnological, &c., maps.

It being impossible to represent a spherical surface on a plane, so that the parts shall have to each other their proper relative positions; the representation is, in all cases, conventional. Various devices have been resorted to, each of which has its own peculiar advantages and disadvantages.

A representation of the meridians and circles of latitude, forms, in all cases, the skeleton or basis of every map of an extensive portion of the earth's surface, and it is upon a correct delineation of these that the accuracy of any map depends.

There are five *principal* methods of projection used; 1st. the *orthographic;* 2d. the *stereographic;* 3d. the *globular;* 4th. the *conical;* and 5th. the *cylindrical*, or *Mercator's* projection, besides the various combinations and modifications of these several methods.

In the first three cases, the plane upon which the map is to be drawn is called the primitive plane, and is supposed to be passed through the centre of the earth. The various lines are projected upon this plane, by lines drawn through their different points and some fixed point, called the *point of sight*. Upon the location of the point of sight depends the peculiarities of the three methods of projection.

In the orthographic projection, the point of sight is taken in a line through the centre of the sphere, perpendicular to the primitive plane, and at an infinite distance from the primitive plane. This corresponds to the ordinary orthographic projection of Descriptive Geometry.

In the stereographic projection, the point of sight lies in the same line, at the point where it pierces the surface of the sphere. In the globular projection, the point of sight is in the same line and at a distance from the surface, equal to the radius into the sine of 45°.

In the conical projection, the point of sight is taken at the centre of the sphere, the circles are then projected upon a cone passed tangent to the sphere in some circle of latitude, or cutting the sphere in two parallels of latitude. After the projection is made, the surface of the cone is rolled upon a plane and the developments of the projections determined.

In the cylindrical projection, the point of sight is taken at the centre of the sphere, and the circles projected upon a cylindrical surface passed tangent to the sphere along the equator. After the projection is made, the cylindrical surface is rolled upon a plane and the developments of the projections determined.

Various modifications of these methods have been used, but they involve all the principles employed in projecting maps. For the full explanation of these methods, the reader is referred to the article on *Spherical Projections*.

In the orthographic projection: 1st. circles whose planes are perpendicular to the primitive plane are projected into straight lines, and all other circles into ellipses; 2d. equal spaces or equal distances on the surface are represented in the projection by unequal spaces and distances; 3d. the projections of equal spaces lessen successively from the centre towards the circumference of the projection, so that whilst objects are represented by nearly their proper shapes about the centre, they are very much distorted and crowded together near the circumference.

In the stereographic projection; 1st. all circles which pass through the point of sight are projected into straight lines; all other circles are projected into circles; 2d. equal spaces and equal distances are projected into unequal spaces and distances; 3d. the projections of equal distances increase from the centre towards the circumference, so that maps made by this mode of projection are crowded towards the centre, and scattered towards the circumference; but the forms of the several parts are better preserved in this method than in the orthographic.

In the globular projection: 1st. equal spaces are represented in projection by nearly equal spaces, and the relative dimensions of the countries mapped are more nearly preserved than in either of the other methods. But on account of the projections not being similar in shape to the area projected, there is a distortion in form, which distortion increases the farther we go from the centre. This and the stereographic projections have been the favorite projections, when it has been a question of projecting an entire hemisphere.

When the equatorial regions of the earth are to be mapped, Mercator's projection is a good one: it has many advantages in the construction of sailing charts, and it is frequently employed in projecting that kind of maps. In constructing maps of small portions of the country, the conical method or some of its modifications, presents the most advantages.

The methods of filling in a map, after the skeleton is projected, are infinitely various. Sometimes they are filled in by minute actual survey, all the details are then carefully plotted in accordance with the kind of projection employed. Again, they are filled up from rapid reconnaissance, or sometimes from the mere report of travelers. Of course, the maps resulting from these different methods of proceeding are of very different value. No general principles can be laid down for completing a map after the outline is projected, but each case must be governed by its own particular attendant circumstances. For details on this subject, see *Geodesy*, *Spherical Projections*, *Topography*, *Topographical Maps*, *&c.*

MAR'I-NER'S COMPASS. See *Compass*.

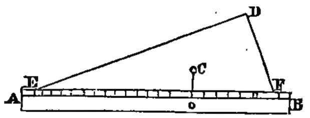
MAR'I-TIME SURVEYING. See *Hydrography*.

MARQUOI'S RULERS. A set of rulers devised by an artist named Marquoi, for the purpose of facilitating the operations of plotting and plan drawing. The set consists of a triangular ruler, whose hypothenuse is three times as long as the shorter side of the triangle, and several rectangular rulers, graduated into equal parts, according to different scales. The rulers are made of hard wood, ivory, or metal, and the graduation lines are cut close to the edges of the rectangular rulers for facility of application.

The scales on the rectangular rulers are of two kinds: 1st. the natural scales, varying from 20 to 60 parts to the inch; 2d. artificial scales, each division of which is equal to 3 divisions of the natural scale. The former are used for ordinary plotting, and the latter for drawing parallels by means of the ruler. The former needs no explanation, the latter may easily be understood by the following description:

The triangular ruler has an index C, about the middle of its hypothenusal edge, which, as in the figure, we will suppose to be set at

the 0 of the rectangular scale. Now, if a straight line is drawn along the edge ED, or if the combination be placed so that the edge



ED coincides with a given line, and whilst the rectangular ruler is held fast, if the triangular one is slid along till the index stands at the 10th division of the scale, and a second line be drawn along the edge ED, then will this be parallel to the first and at a distance from it equal to 10 parts of the natural scale, or the scale of the plot. This arrangement requires as many rules as there are scales employed, but when a great number of lines have to be drawn to a scale, as in architectural plans, the rulers will be found of use. Instead of having one triangular ruler and several rectangular ones, the case might be reversed, having one rectangular ruler and several triangular rulers, whose hypothenusal and shorter edges have different ratios to each other, to correspond to different scales. This method is not so good as the former.

MATH-E-MAT'IC-AL. [L. *mathematicus*]. Pertaining to mathematics; as, mathematical instruments, mathematical demonstration, &c.

MATH-E-MAT'IC-AL-LY. According to the principles of mathematics: with mathematical certainty.

MATH-E-MA-TĬ''CIAN. One skilled in the science of mathematics.

MATH-E-MAT'ICS. [L. *mathematica*; Gr. $\mu\alpha\theta\eta\mu\alpha\tau\iota\kappa\eta$, from $\mu\alpha\nu\theta\alpha\nu\omega$, to learn]. That science which treats *primarily* of the relations and measurement of quantities, and secondarily of the operations and processes, by means of which these relations are ascertained.

The science is based upon a few self-evident and universally admitted relations of quantities. From these relations, by a course of logical argument, the most complicated results are obtained. As the science becomes extended, old processes are found inconvenient and cumbersome, and a want of new ones begins to be felt. This want leads to the discovery or invention of new methods, better adapted to the attainment of the desired end. Hence it happens, that a thorough examination of the *philosophy of mathematical operations* comes to be a most important branch of the science. A careful study of the philosophy of *operation* has done much to perfect and advance the science, and to it we are to look for still further advancement.

Mathematics, considered as the science of exact relation, is divided into three branches: 1. Arithmetic. 2. Geometry. 3. Analysis.

1. *Arithmetic* is that branch which treats of the relations of numbers, expressed by the aid of figures and combinations of figures.

It is divided into two parts. The *first* treats of the methods of representing and reading numbers by means of figures, together with the fundamental operations. These embrace Notation and Numeration, Addition, Subtraction, Multiplication, Division, Raising to powers, and Extracting roots of numbers, whose units are either entire or fractional. It also treats of the transformation of numbers from one scale to another, in which the fundamental unit may be different, or in which the scale of places may be different. It also treats of the theory of the construction of scales, and of the general relations existing amongst all numbers. This makes up what may be called the *science* of Arithmetic.

The *second* part treats of the applications of the principles of the first part, to the practical wants of life. It embraces the Rule of Three, Percentage, Interest, Practice, Fellowship, and a variety of other rules.

2. *Geometry* has for its object the investigation of the properties and relations of magnitudes, by reasoning directly upon the magnitudes themselves, or upon their pictorial representatives. The magnitudes, considered as this branch of mathematics, are simply *lines*, *surfaces*, *volumes* and *angles*.

Geometry is divided into two parts:

1st. *Elementary Geometry*, which treats of those magnitudes whose elements are the right line and circle.

It embraces all propositions relating to figures bounded by straight lines, circles, or portions of circles, together with the surfaces of the sphere, cylinder, and cone. It

treats of the properties of all volumes bounded by plane faces, together with the three round bodies, the sphere, the cylinder, and the cone.

An immediate application of this part of Geometry is to Plane Trigonometry, which treats of the relations between the sides and angles of plane triangles.

It also embraces the construction of all problems, which can be performed by the aid of the circle and straight line alone.

2d. *Higher Geometry* embraces all propositions appertaining to magnitudes, whose elements are more complex lines than the straight line and circle; such as the Conic Sections, &c. It includes the higher investigations of the ancient geometers, many of which are, in a great measure, superseded by the recent improvements in analysis. Of this class, are the famous isoperimetrical problems, from which originated the Calculus of Variations, as well as the noted problems of the duplication of the cube, and the trisection of an angle. It includes the solution of all geometrical problems, which cannot be solved by the circle and straight line.

The direct application of both branches of Geometry are

1st. Descriptive Geometry, which has for its object the graphical solution of all problems involving three dimensions. In this branch of construction, lines are given by their projections upon two planes of reference, taken at right angles to each other; planes are given by traces upon these planes of reference; and surfaces by projections of certain of their elements. The principles of both Elementary and Higher Geometry are employed in the solutions of Descriptive Geometry. A very extensive and useful branch of Descriptive Geometry is found in solving problems of Shades, Shadows, Stone-cutting, and Architecture.

2. A second application of Descriptive Geometry is found in Perspective, in which objects are represented upon a single plane, by means of projections made by drawing straight lines through a particular point, called the *point of sight*, and through points of the lines to be projected. The plane upon which the projections are made, is called the perspective plane.

Spherical projections are but modifications of perspective; the perspective plane being in general, taken through the centre of the sphere.

3. *Analysis* embraces all that part of mathematics, in which the quantities considered are represented by letters, and the operations to be performed are indicated by means of signs or conventional symbols. Analysis is generally treated of under the heads of *Algebra, Analytical Geometry,* and *Calculus.*

1st. *Algebra* investigates the relations and properties of numbers analytically. It consists of two parts: Elementary, and Higher, or, as it is sometimes called, Transcendental Algebra.

Elementary Algebra explains the nature of the symbols employed, develops the nature of the operations indicated by the signs, and teaches the method of interpreting results. It investigates the methods and principles of performing what are called the ordinary operations of algebra; that is, addition, subtraction, multiplication, division, raising to powers denoted by constant exponents, and the extraction of roots indicated by constant indices. It also embraces the investigation of the nature and properties of all equations, in which the relation between the known and unknown quantities is expressed by the ordinary operations of algebra; which equations are called algebraic equations.

Higher, or Transcendental Algebra, treats of those quantities which cannot be expressed by a finite number of algebraic terms. It also investigates the nature and properties of transcendental equations, that is, all equations which are not algebraic. Under this part of algebra, comes the investigation of logarithmic and trigonometric formulas; and series of all kinds having an infinite number of terms.

2d. *Analytical Geometry,* is that part of analysis which has for its object the analytical investigation of the properties and relations of geometrical magnitudes. In Analytical Geometry, points, lines and surfaces are referred to fixed objects by means of certain elements of reference, which elements vary from point to point, and are called co-ordinates. The equation which expresses a relation between the co-ordinates of every point of a magnitude, is called the equation of the magnitude. All lines whose equations can be expressed, are called mathe-

matical lines. By proper transformations and combinations of the equations of magnitudes, and a judicious interpretation of the results, all the properties of magnitudes may be deduced.

Analytical Geometry is divided into two parts—*Determinate* and *Indeterminate*.

Determinate Geometry has for its object the solution of determinate problems, that is, those problems in which the given conditions limit the number and afford the means of determining the values of the required parts. This part includes the entire subject of the application of algebra to the solution of geometrical problems.

Indeterminate Geometry investigates the general relations of lines and surfaces. Indeterminate Geometry has two branches. The *first* embraces all investigations when the relations of the co-ordinates of points of magnitudes can be expressed by the ordinary operations of algebra. This is called Elementary Analytical Geometry. The *second* embraces those investigations in which the relations between the co-ordinates cannot be expressed by the ordinary operations of algebra. This is called Transcendental Analytical Geometry.

The first embraces a complete discussion of the straight line, and the conic sections and all surfaces of the first and second orders: the second embraces a discussion of a great variety of curves, such as the cycloid, logarithmic curve, curve of sines, tangents, &c., spirals of all kinds, together with the corresponding surfaces of which these lines form elements. A complete theory of the latter magnitudes is more readily formed by the aid of calculus.

3d. *Calculus*. A name originally applied to any operation involving computation or calculation, is now, by universal consent, applied solely to the highest branch of mathematics, viz.: that which treats of the nature and forms of functions. It is divided into three principal parts—*Differential Calculus*, *Integral Calculus* and the *Calculus of Variations*.

Differential Calculus explains,

1st. The relations which functions bear to certain derived functions, called their differential co-efficients; and, 2d., it explains the method of applying them in the discussions of the higher branches of Analytical Geometry. Hence, we see that there are two divisions of Differential Calculus. The first relates to the method of finding the differential co-efficients of all kinds of functions; the second to the methods of applying them in the processes of Analytical Geometry, or in the various branches of Mathematical Philosophy.

Integral Calculus is the inverse of Differential Calculus, and like it, may be divided into two parts. The object of the first part is to show how to pass from any function, regarded as a differential co-efficient, to the function from which it might have been derived. The object of the second part is to show the various applications of the principles deduced in the first part, in the investigations of Analytical Geometry and Physical Science.

One great advantage of the differential and integral calculus consists in this, viz.: that we are often able by the aid of the first to find the differential co-efficient of a function without knowing the form of the function, and then, by the aid of the second, to deduce from this differential co-efficient the form of the desired function, which might not have been reached by any other known process. Hence, the great use of the calculus in finding expressions for the lengths of curves, the measures of curvilinear areas, and of curved surfaces, the volumes of solids, &c. It is of still more importance in the investigation of Physics.

The *Calculus of Variations* is the highest branch of mathematics, and treats of the lsw of forms of functions. The first part of this branch treats of the methods of deducing the variations of functions, which variations are but mathematical expressions for the law of variation in the forms of the functions: the second part explains the method of spplying these principles to transcendental problems, and to the more complicated investigations of physical science.

Such is a rapid outline of the great divisions, and the most important sub-divisions of the science of mathematics.

It will be observed that throughout this sketch, in every division, there are two parts, the first having for its object the investigation of general principles and rules, whilst the

second explains their application. The first parts constitute the *science* of pure mathematics, whilst the latter may with propriety be called the *art of mathematics*. The first part is often called pure mathematics, the second, mixed mathematics.

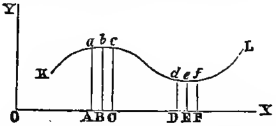
The science of mathematics forms an important element of a liberal education. It impresses the mind with clear and distinct ideas ; cultivates habits of close and accurate discrimination ; gives, in an eminent degree, the power of abstraction ; sharpens and strengthens all the faculties, and develops to their highest range the reasoning powers. The tendency of this study is to raise the mind from the servile habit of imitation to the dignity of self-reliance and self-action. It arms it with the inherent energies of its own elastic nature, and urges it out on the great ocean of thought, to make new discoveries, and enlarge the boundaries of mental effort.

In its practical applications, Mathematics aids in developing the physical sciences, and contributes directly and indirectly to the rapid progress of the race in the advancement of every science and art. It is at once the guide and support of the astronomer. The laws of nature eluded the researches of the philosopher, until he brought to his aid the irresistible power of mathematical science ; and, without this, the chemist could never have reached the inner laboratory of the material world.

The rules and practice of all the mechanic arts are but applications of mathematical science. The mason computes the quantity of his materials by the principles of geometry and the rules of arithmetic. The carpenter frames his building, and adjusts all its parts, each to the others, by the rules of practical geometry. The mill-wright computes the pressure of the water, and adjusts the driving to the driven wheel, by rules evolved from the formulas of analysis.

Workshops and factories afford marked illustrations of the utility and value of practical science. They are the embodiments, by intelligent labor, of the most difficult investigations of mathematical science.

MAX'I-MA AND MIN'I-MA. [L.] A function of a single variable is at a maximum state, when it is greater than both the state which immediately precedes and the state which immediately follows it ; and it is at a minimum state, when it is less than both the state which immediately precedes and the state which immediately follows it.



Thus, if we regard the ordinate of any point of the curve KL as a function of the corresponding abscissa, we shall have Bb a maximum, because $Aa < Bb > Cc$; and Ee a minimum, because $Dd > Ee < Ff$.

In speaking of preceding and succeeding states, reference is had to the order of increase of the variable, so that one state of a function precedes another, when it corresponds to a less value of the independent variable ; and one state succeeds or follows another, when it corresponds to a greater value of the independent variable.

A function of one variable may have any number of maximum and minimum states : but if the function is continuous, there must be a maximum state between any two minimum states, and a minimum between any two maximum states. For it is evident that, after a maximum, the function decreases as the variable increases ; and since, before it reaches a second maximum state, the function must again increase, it follows that there is some intermediate state at which the function ceases to decrease, and begins to increase : that state is a minimum. In like manner, it may be shown, that between each two minimum states there must be a maximum state. Hence, the number of maximum states of a function is either equal to the number of minimum states, or, at most, differs from it by 1.

Just before reaching a maximum state, the function increases as the variable increases, hence, its first differential co-efficient is positive : just after passing the maximum state, the function decreases as the variable increases, and consequently its differential co-efficient is negative : this shows that the sign of the differential co-efficient must change from plus to minus, in passing through a maximum state. Just before reaching a minimum state, the function diminishes as

the variable increases, and therefore its differential co-efficient is negative: just after passing the minimum state, the function increases as the variable increases, and consequently its differential co-efficient is positive: this shows, that the sign of the differential co-efficient must change from minus to plus, in passing through a minimum state. Here we see that a change of sign of the first differential co-efficient is the analytical characteristic of either a maximum or minimum state of a function of one variable.

But a continuous function cannot change its sign except by becoming either 0 or ∞. These principles indicate a general rule for finding all those values of the variable, which can possibly make a function of one variable either a maximum or minimum.

Differentiate the function, and find its first differential co-efficient, and place it equal to 0, and also equal to ∞. Solve the resulting equations, and find the values of the variable; these values will be the only ones that can possibly make the given function either a maximum or a minimum; there may be amongst them some values that do not correspond either to a maximum or a minimum. It therefore becomes necessary to introduce some test to separate those which correspond to maxima, from those which correspond to minima states. There are two such tests:

First. Substitute one of the roots minus an infinitely small quantity, for the variable in the given function, and set the result aside; next, substitute the root itself, and then the root plus an infinitely small quantity, and set the results aside. If the second result is greater than both the first and third results, this is a maximum state, and the root is the corresponding value of the variable; if it is less than both these states, it is a minimum, and the root is the corresponding value of the variable; if it is greater than one, and less than the other, it is neither a maximum nor a minimum, and the root is to be rejected. Test each root in this way in succession, rejecting all that do not correspond to maxima or minima.

Second. Substitute the root, minus an infinitely small quantity, and then plus an infinitely small quantity, for the variable in the expression for the first differential co-efficient, and note the signs of the result. If the first is positive and the second is negative, the root corresponds to a maximum; if the first is negative and the second positive, the root corresponds to a minimum; if they are both alike, the root corresponds neither to a maximum nor a minimum, and is to be rejected. The maxima and minima may be found by substituting the corresponding values of the variable in the function.

The second test will, in general, be found most convenient.

To illustrate the preceding rule, let it be required to find the maximum and minimum values of the function,

$$u = a - bx + x^2.$$

Differentiating, we find $\dfrac{du}{dx} = -b + 2x$; and, by the rule, $-b + 2x = 0$; whence, $x = \dfrac{b}{2}$, and $-b + 2x = \infty$; which gives no finite value for x.

Applying the first test to the root, $x = \dfrac{b}{2}$, we find the relations:

for $x = \dfrac{b}{2} - h$, we have $a - b\left(\dfrac{b}{2} - h\right) + \left(\dfrac{b}{2} - h\right)^2 = a - \dfrac{b^2}{4} + h^2$ . . . . . 1st result;

for $x = \dfrac{b}{2}$, we have $a - \dfrac{b^2}{4}$ . . . 2d result;

for $x = \dfrac{b}{2} + h$, we have $a - b\left(\dfrac{b}{2} + h\right) + \left(\dfrac{b}{2} + h\right)^2 = a^2 - \dfrac{b^2}{4} + h^2$ . . . . . 3d result.

Whence,

$$a - \frac{b^2}{4} + h^2 > a - \frac{b^2}{4} < a - \frac{b^2}{4} + h^2;$$

which shows, that $x = \dfrac{b}{2}$ corresponds to a minimum, which minimum is $a - \dfrac{b^2}{4}$.

Applying the second test, we find the following results:

for $x = \dfrac{b}{2} - h$, we have $-b + 2\left(\dfrac{b}{2} - h\right)$
$= -2h$, a negative result;

for $x = \dfrac{b}{2} + h$, we have $-b + 2\left(\dfrac{b}{2} + h\right)$
$= +2h$, a positive result.

These results indicate, that $x = \frac{b}{2}$ corresponds to a minimum, which may be found by making $x = \frac{b}{2}$, in the given function. It is found equal to $a + \frac{b^2}{4}$, as before indicated.

There is a practical rule which corresponds to those cases, in which the first differential co-efficient is equal to 0, and which applies to a great majority of cases, founded on the form of the development of a function of one variable and its increment, according to the ascending powers of the increment. It is as follows:

Differentiate the given function, find its first differential co-efficient, and place it equal to 0. Solve the resulting equation, and find the roots. Substitute each root in succession, in the second, third, fourth, &c. differential co-efficients, till one is found, which does not reduce to 0 or ∞. If the first one, which does not reduce to 0 or ∞, is of an odd order, the root does not correspond to either a maximum or minimum; but if the first one, which does not reduce to 0 or ∞, is of an even order, and becomes negative, the root corresponds to a maximum; if positive, to a minimum: all the roots which do not correspond to maxima or minima, are to be rejected. Those which correspond to maxima and minima, are to be substituted in succession in the function, and the corresponding results will be the maxima and minima required. It will not, in general, be found necessary to carry the substitution further than the second differential co-efficient.

To illustrate the rule, let it be required to find the maxima and minima values of the function,

$$u = 3a^2x^2 - b^4x + c^5.$$

Differentiating twice, we find,

$$\frac{du}{dx} = 9a^2x^2 - b^4, \quad \text{and} \quad \frac{d^2u}{dx^2} = 18a^2x.$$

From the equation $9a^2x^2 = b^4$, we find

$$x = \pm \frac{b^2}{3a}.$$

The plus root substituted in $18a^2x$ gives $+ 6ab^2$ and indicates a minimum. The minus root substituted in $18a^2x$ gives $-6ab^2$, and indicates a maximum. The roots in the function give

$$c^5 - \frac{2b^6}{9a} \text{ for the minimum, and}$$

$$c^5 + \frac{2b^6}{9a} \text{ for the maximum.}$$

It often happens that an important simplification can be made, in finding the value of the second differential co-efficient corresponding to a root which is to be tested. This happens when the first differential co-efficient is composed of two factors, one of which placed equal to 0, gives the root in question. The simplification is this; differentiate the factor corresponding to the root, multiply its differential co-efficient by the other factor, and in this product make the required substitution, the result will be the same as though the substitution had been made in the second differential co-efficient itself. This principle may be extended to the case where it is necessary to substitute in the successive differential co-efficients.

To illustrate the method of making the simplification, let it be required to divide a quantity into two parts, such that the n^{th} power of one of them multiplied by the n^{th} power of the other shall be a maximum or minimum.

Denote the given quantity by a, and one of the parts by x, the other one will be denoted by $a - x$, and we shall have for the function,

$$u = x^m (a - x)^n,$$

whence,

$$\frac{du}{dx} = mx^{m-1} (a - x)^n - nx^m (a - x)^{n-1}$$

$$= (ma - mx - nx)x^{n-1} (a - x)^{n-1},$$

and by placing each factor separately equal to 0, we have,

$$x = \frac{ma}{m + n}, \quad x = 0 \quad \text{and} \quad x = a.$$

The differential co-efficient of the first factor multiplied by the remaining factors, is

$$-(m + n)\, x^{n-1} (a - x)^{n-1}; \text{ for } x = \frac{ma}{m - n}$$

it reduces to

$$\frac{-m^{m-1}\, n^{n-1}\, a^{m+n-2}}{(m + n)^{m+n-3}},$$

a negative result; hence, this value corresponds to a maximum. The other values satisfy the equation of the problem, but do not conform to the conditions of it and need not be considered.

In seeking for the values of the variable which correspond to maxima and minima, any positive constant factor of the function may be omitted without changing the final result. We may also throw off a radical sign, and be sure to find all the values of the variable, but in this case we may get some values that will make the power a maximum or minimum, but which will not make the root a maximum or minimum.

A function of two variables is at a maximum state, when it is greater than all the consecutive states, and it is at a minimum state, when it is less than all the consecutive states.

Every function of two variables may be regarded as one of the rectangular co-ordinates of a point of a surface, of which the two variables are the other two co-ordinates; it is generally taken as the vertical one. We may conceive the idea of a maximum ordinate, if we consider a sphere lying upon a plane. The ordinate of the highest point is a maximum, and that of the lowest point a minimum.

The practical rule for finding the maximum and minimum states of a function of two variables is as follows:

Differentiate the function, and find the partial differential co-efficients of the first order, and also, the partial differential co-efficients of the second order. Place the partial differential co-efficients of the first order separately equal to 0: combine the resulting equations, and find the values of the variables. Substitute these in each of the three partial differential co-efficients of the second order, and find the results.

Multiply the first and third results together, and square the second; then, if the product of the first and third is greater than, or equal to, the square of the second, there will be either a maximum or minimum; a maximum when the first result is negative, a minimum when it is positive.

For example, let it be required to find the maxima and minima of the function

$$u = x^3y^2(a - x - y).$$

Differentiating, we have

$$\frac{du}{dx} = x^2y^2(3a - 3y - 4x), \text{ and}$$

$$\frac{du}{dy} = x^3y(2a - 3y - 2x).$$

Placing these separately equal to 0,

$$3a - 3y - 4x = 0, \quad \text{and} \quad 2a - 3y - 2x = 0;$$

whence, by combination,

$$x = \frac{a}{2}, \quad y = \frac{a}{3}.$$

We have, also,

$$\frac{d^2u}{dx^2} = 2xy^2(3a - 3y - 6x),$$

$$\frac{d^2u}{dxdy} = x^2y(6a - 9y - 8x), \text{ and}$$

$$\frac{d^2u}{dy^2} = x^3(2a - 6y - 2x).$$

Substituting in these the values of x and y deduced above, and applying the rule, we get

$-\frac{a^4}{9}$ ·· *first* result; $-\frac{a^4}{12}$ ·· *second* result; $-\frac{a^4}{8}$ ·· *third* result; and since

$$\left(-\frac{a^4}{9}\right)\left(-\frac{a^4}{8}\right) > \left(-\frac{a^4}{12}\right)^2, \text{ or, } \frac{a^8}{72} > \frac{a^8}{144},$$

the deduced values correspond to either a maximum or a minimum, and since the first result is negative, it is a maximum. Substituting these values in the function, the maximum value is found equal to

$$\frac{a^6}{432}.$$

MEAN. The mean of two quantities is a quantity lying between them and connected with them by some mathematical law.

There are several kinds of *means*, the principal ones being the *Arithmetical* and the *Geometrical mean*.

The *Arithmetical mean*, or average of several quantities of the same kind, is their sum divided by their number. Thus, the arithmetical mean of 10, 12, 17, and 25, is $\frac{64}{4}$ or 16. The arithmetical mean is understood when the word mean is used alone.

The *Geometrical mean* of two quantities, is the square root of their product: thus, the geometrical mean of 2 and 8 is $\sqrt{16} = 4$. The *greater* of the given quantities is as *many times* greater than the *mean*, as the *mean* is greater than the *less* quantity. Such is the idea of the geometrical mean. In a geometrical progression, each term is a geometrical mean between the preceding and succeeding terms; in an arithmetical progres-

sion each term is an arithmetical mean between the preceding and succeeding terms.

The practical applications of the principles of *means*, is in determining the most probable amongst several discordant results obtained by measurement, or by experiment. Suppose that by different measurements, all conducted with equal care, we find that the length of a given line is 23.021, 23.012 23.101, and 23.010 inches, then it is probable, from the data, that the true length is a mean of the measured lengths, or 23.036 inches. This rule supposes that all the measurements are equally trustworthy, which is not always the case. When some of the observations are better than others, they are said to have greater weight; thus, suppose three observations to give 26, 28 and 29, and that it is thought by the observer, that the observation giving 26 is as good as a mean of four observations, that giving 28 is as good as a mean of eight observations, and that giving 29 is as good as a mean of six observations, we may regard the whole system as composed of 18 observations, 4 of which give the result 26, 8 the result 28, and 6 the result 29. The numbers 4, 8, and 6, are called the weights of the observations respectively.

To find the mean or probable result, multiply each result by its weight, and take the sum of the products; divide this by the sum of the weights, and the quotient will be the value sought. Thus,

$$8 \times 28 + 6 \times 29 + 4 \times 26 = 502,$$

which, divided by 18, gives 27.89 as the most probable result.

The constant use of the principle above explained, in all branches of experimental philosophy, and particularly in astronomy, has given rise to much research, in order to find the most probable result in any given chain of observations.

Omitting the analysis, which has led to the subjoined rule, we simply annex a table, which has been computed in accordance with the principle of least squares, and an example to show its use.

Suppose that we have a number of observations on a given phenomenon, and wish to determine the probability that the truth lies within a given degree of exactness to the average of them.

The following is the rule: Let M denote the average of the observations, and let

$$M + m \quad \text{and} \quad M - m$$

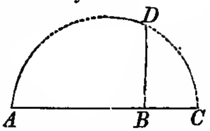
be two limits. Let it be required to ascertain the probability of the truth lying between them. Take the difference between M and each of the results of observation, and add together the squares of these differences. Multiply 100 times the number of observations by m, and divide by the square root of twice the sum just found; take the number nearest to the result in the column marked A, and opposite to it, in the column marked B, will be found the number of chances out of 10,000 for the degree of nearness required.

TABLE.

A	*B*	*A*	*B*	*A*	*B*	*A*	*B*
1	113	24	2657	47	4937	69	6708
2	226	25	2763	48	5027	70	6778
3	338	26	2869	49	5117	71	6847
4	451	27	2974	50	5205	72	6914
5	564	28	3079	51	5292	73	6981
6	676	29	3183	52	5379	74	7047
7	789	30	3286	53	5465	75	7112
8	901	31	3389	54	5549	76	7175
9	1013	32	3491	55	5633	77	7238
10	1125	33	3593	56	5716	78	7300
11	1236	34	3694	57	5798	79	7361
12	1348	35	3794	58	5879	80	7421
13	1459	36	3893	59	5959	81	7480
14	1569	37	3992	60	6039	82	7538
15	1680	38	4090	61	6117	83	7595
16	1790	39	4187	62	6194	84	7651
17	1900	40	4284	63	6270	85	7707
18	2009	41	4380	64	6346	86	7761
19	2118	42	4475	65	6420	87	7814
20	2227	43	4569	66	6494	88	7867
21	2335	44	4662	67	6566	89	7918
22	2443	45	4755	68	6638	90	7969
23	2550	46	4847				

Suppose, for example, that seven observations give 10.03, 10.71, 10.98, 10.26, 10.30, 10.72, 10.81, the average of which is 10.54, differing from the respective observations by .51, .17, .44, .28, .24, .18 and .27. The sum of the squares of these is .7239, and twice the sum is 1.14478, the square root of which is 1.203. Let it be required to find the chance of the truth lying between 10.48 and 10.60. We multiply .06 by 700, which gives 42, and dividing by 1.203 we find 34.9: opposite to 35, in the column B we find 3794, so that $\frac{3794}{10000}$ is the chance of the truth lying within the limits assigned.

To construct a mean proportional between two given lines, geometrically.



Draw an indefinite straight line AC, and lay off on it a distance AB, equal to one of the lines, and from B lay off the distance BC, equal to the second line; construct a circle on AC as a diameter, and draw the line BD through the point B perpendicular to AC, till it meets the curve in D: BD is the mean proportional required.

To find two mean proportionals between two quantities, that is, to insert two quantities between them, so that the four shall be in geometrical proportion.

Multiply each quantity by the square of the other, and extract the cube roots of the product; these will be the means required.

Let it be required to insert two means between 2 and 16. By the rule, we find

$$\sqrt[3]{2^2 \times 16} = 4, \quad \textit{first mean},$$

and $$\sqrt[3]{16^2 \times 4} = 8, \quad \textit{second mean};$$

hence, the progression is

$$2 : 4 . 8 \cdot 16.$$

The following table exhibits the method of proceeding to insert any number of means between any two given quantities. Let the quantities be denoted by a and b:

1 mean, $a : \sqrt{ab} : b.$

2 means, $a : \sqrt[3]{a^2b} : \sqrt[3]{ab^2} : b.$

3 means, $a : \sqrt[4]{a^3b} : \sqrt[4]{a^2b^2} : \sqrt[4]{ab^3} : b.$

4 means, $a : \sqrt[5]{a^4b} : \sqrt[5]{a^3b^2} : \sqrt[5]{a^2b^3} : \sqrt[5]{ab^4} : b.$

....

$(n-1)$ means

$a : \sqrt[n]{a^{n-1}b} : \sqrt[n]{a^{n-2}b^2} : \sqrt[n]{a^{n-3}b^3} .. \sqrt[n]{ab^{n-1}} : b.$

The geometrical construction of $(n-1)$ geometrical means is impossible by the aid of elementary geometry, but it can be effected by means of the higher geometry. The logarithmic spiral affords the readiest means of effecting the construction.

With the first quantity as a radius, describe a directing circle, and through its centre draw a radius; lay off from this, as an initial, in succession $(n-1)$ equal angles of any given value; on the last side of the last angle, lay off from the centre a distance equal to the second quantity; through the extremities of the first and last distances laid off, draw a logarithmic spiral; this can always be done; the distances cut off from the successive sides of the intermediate angles will be the required means.

Harmonical Mean between two Quantities. The reciprocal of the arithmetical means of the reciprocals of the two quantities. Let a and b denote the quantities; then will their reciprocals be denoted by $\frac{1}{a}$ and $\frac{1}{b}$, and the mean of these reciprocals will be denoted by

$$\frac{1}{2a} + \frac{1}{2b} = \frac{a+b}{2ab}; \text{ hence}$$

$$\frac{2ab}{a+b}$$

is the harmonical mean between a and b. The harmonical mean is a third proportional to the arithmetical and geometrical means; that is,

$$\frac{a+b}{2} : \sqrt{ab} :: \sqrt{ab} : \frac{2ab}{a+b}.$$

Mean Diameter. In Gauging, a mean between the head diameter and the bung diameter.

MĔAS'URE. [L. *mensura*, a measuring]. The measure of a quantity is its extent, or its value, in terms of some other quantity of the same kind, taken as a unit of measure. The measure of a line is the number of linear units, as feet, yards, miles, &c, which it contains.

The measure of a surface is the number of square units of surface which it contains. The measure of a volume is the number of cubic units which it contains. The measure of an angle is the number of angular units which it contains, whether the angular unit be a right angle or a degree. A measure of a quantity is always expressed by means of some number and the unit of measure. The measure of a ratio is the numerical value of the ratio.

Measure or Unit of Measure. A given quantity, used as a standard of comparison in measuring a quantity of the same kind. Every kind of quantity has its own unit of measure, and, under different circumstances, the same kind of quantity may have different

units of measure. Thus, in measuring distances, we may compare them with a foot, a yard, a rod, a mile, a league, &c., but in all such cases, these different units are so related to each other, that the measurement can always be reduced to a comparison with some one fixed unit; so that, in fact, though different units may have been used, there is virtually but a single unit, that unit being, in any given instance, arbitrary. Since every kind of magnitude has its own unit of measure, we have units of distance, units of surface, units of volume, units of weight, units of force, units of temperature, in short, units of every kind of quantity. There appears to be an exception to the general principle laid down, that the unit of measure is always of the same kind as the quantity measured, in the case of the measurement of angles, when the angle is measured in terms of the arc of a circle. The exception in this case is only apparent, for it is shown that the measure of the relation between two angles is the same as that between two arcs of circles, whose centre is at the vertex of the angle, and the arcs thus intercepted between the sides of the angle. So that, in this case, we replace both the unit and the quantity measured by other quantities, which bear the same relation to each other as the given quantities. In all cases, we again assert, the unit of measure is of the same kind as the thing measured, no matter what may be the form of expression used in giving the result of the measurement.

Units of Measure for Distances. One of the most important units of measure is that for distances or measures of length. A practical want has ever been felt of some fixed and invariable standard by means of which all distances may at once be compared, and such fixed standard has been sought for in nature. There are two natural laws, either of which afford this desired natural element. Upon one of them, the English have founded their system of measures, and upon the other the French have based their system. These two systems being the only ones of importance, we shall only consider them, disregarding the various suggestions that have been made, looking to the adoption of some other element as a standard of comparison.

First. The English system of measures, to which our own system conforms, is based upon the law of nature that the force of gravity is constant at the same point of the earth's surface, and consequently that the length of a pendulum which oscillates a certain number of times, in a given period, is also constant. It is accordingly decreed by the English law that the $\frac{1}{3\cdot26159}$th part of the length of a simple second's pendulum at the Tower of London shall be regarded as a standard English *foot*, and from this, by multiplication and division, the entire system of linear measures is established.

Second. The French system of measures is founded upon the principle of the invariability of the length of an arc of the same meridian between two fixed points. By a very minute survey of the length of an arc of the meridian from Dunkirk to Barcelona, the length of a quadrant of the meridian was computed, and it has been decreed by French law that the *ten millionth* part of this length shall be regarded as a standard French *metre*, and from this, by multiplication and division, the entire system of linear measures has been established. On comparing two accurate scales, Capt. Kater found that the French metre was equal to 3.280899 English feet, or 39.37079 English inches. This relation enables us to convert all measures in either system into the corresponding measures of the other system.

Other Units of Measure. The unit of length having been established, the *unit of surface* is taken, equal to the area of a square, one of whose sides is the unit of length. The *unit of volume* is taken equal to the volume of a cube, one of whose edges is equal to the linear unit. The cubic unit, or unit of volume being established, it affords the means of fixing a convenient *unit of weight*.

It has been agreed that a cubic foot of distilled water, at the temperature of 39.83°F., shall be regarded as weighing 1000 ounces. This fixes the standard ounce, and all other weights are then determined by being referred to this as a standard. See *Weights*.

The following statements show the relations between the measures of the United States, England and France;

1. *Weights and Measures of the United*

States. The unit of length is the same as the English unit. The comparison is made by means of a scale 82 inches in length, now in the possession of the Treasury Department, and manufactured by Troughton in London.

The standard unit of weight is the Troy pound, copied in 1827 by Capt. Kater, from the imperial pound Troy of England. This standard is to be used at a standard height of the barometer equal to 30 inches, and a temperature of 62° Fahrenheit. The standard is at present kept at the mint of the United States at Philadelphia.

The standard of liquid measure is the gallon, a vessel which contains 58372.2 grains, or 8.3389 pounds, avoirdupois weight, of water, when at a temperature of 39°.83 Fahrenheit; the water to be weighed in air when the barometer stands at 30 inches, the temperature being 62° Fahr. This gallon is the wine gallon, nearly, and contains about 231 cubic inches.

The standard of dry measure is the bushel, which holds 543391.89 grains, or 77.6274 pounds avoirdupois weight of distilled water, determined under the same conditions as in the preceding case. This coincides nearly with the English bushel.

The avoirdupois pound is equal to $\frac{175}{144}$ times the Troy pound.

2. *Linear Measures. English Measures.* The unit of linear measure is the yard, equal to 3 feet. The following table shows the relation between the different linear units often used:

Inches.	Feet.	Yards.	Poles.	Furlongs.	Miles.
1.	0.083	0.028	0.00505	0.00012626	0.0000157828
12.	1.	0.3333	0.06060	0.00151515	0.00018939
36.	3.	1.	0.1818	0.00454545	0.00056818
198.	16.5	5.5	1.	0.025	0.003125
7920.	660.	220.	40.	1.	0.125
63360.	5280.	1760.	320.	8.	1.

Measures of Surface. The unit of measure is the square yard. The units employed in land measure are the perch, rood and acre. The following table shows the relations between these various units:

Square Feet.	Square Yards	Perches.	Roods.	Acres.
1.	0.1111	0.00367309	0.000091827	0.000022957
9.	1.	0.0330579	0.000826448	0.000206612
272.25	30.25	1.	0.025	0.00625
10890.	1210.	40.	1.	0.25
43560.	4840.	160.	4.	1.

Measures of Volume. Solids are estimated in cubic yards, feet and inches. 1728 cubic inches make a cubic foot, and 27 cubic feet make a cubic yard.

The contents of the imperial standard gallon are about 277.274 cubic inches. The parts of a gallon are quarts, and pints. The multiples of a gallon are pecks, bushels and quarters. Their relations are shown in the following table:

Pints.	Quarts.	Gallons.	Pecks.	Bushels.	Quarters.
1.	0.5	0.125	0.0625	0.015625	0.001953125
2.	1.	0.25	0.125	0.03125	0.00390625
8.	4.	1.	0.5	0.125	0.015625
16.	8.	2.	1.	0.25	0.03125
64.	32.	8.	4.	1.	0.125
512.	256.	64.	32.	8.	1.

French Measures. The unit of length is the *metre* The superficial unit is the *are*, a surface containing 100 square metres. The unit of volume is the *litre*, the cube of the

decimetre. The standard temperature being 32° Fahr.

The following tables show the relations existing between the several units in the French system:

Measures of Length.

Myriametre,	equal to	10000	metres.
Kilometre,	"	1000	"
Hectometre,	"	100	"
Metre,	"	1	"
Decimetre,	"	0.1	"
Centimetre,	"	0.01	"
Millimetre,	"	0.001	"

Measures of Surface.

Hectare,	equal to	10,000	square metres.
Are,	"	100	"
Centiare,	"	1	"

Measures of Capacity.

Kilolitre,	containing	1000	litres.
Hectolitre,	"	100	"
Decalitre,	"	10	"
Litre,	"	1	"
Decilitre,	"	0.1	"
Centilitre,	"	0.01	"

The unit of volume for solids is the *stere*, or the cube of the metre, which is equivalent to 35.31658 cubic feet. No system of metrology has equaled the French in simplicity, nevertheless, it is not in general use, even in France, for every day purposes. There is in fact, at present, three system of measures more or less used.

The following table shows the relations between the standard foot, in some of the principal countries of Europe.

	Russian.	Prussian.	Bavarian.	Hanoverian.	Saxon.	Austrian.
English feet.	1.	1.065765	0.957561	0.958333	0.929118	1.037128

Measure of Angles. The right angle being taken as the angular unit, its subdivisions are degrees, minutes, and seconds. The right angle contains 90 degrees, the degree 60 minutes, and the minute 60 seconds. All smaller fractions are expressed decimally in terms of the second. The French have proposed to divide the right angle into 100 equal parts, called *grades*, but the suggestion has not been extensively adopted. For measures of weight. see *Weight*.

Measure, Common. The same as common divisor. Any quantity which will exactly divide two quantities, is said to measure them both, or is their common measure.

Measures, Line of. The line of intersection of the primitive plane, with a plane passing through the axis of the primitive circle and the axis of the circle to be projected. See *Line*.

ME-CHAN'IC-AL CURVE. The same as transcendental curve. See *Line*.

MĒ'DI-AL ALLIGATION. See *Alligation*.

MEM'BER. [L. *membrum*]. Every equation is made up of two parts, connected by the sign of equality. These parts are called *members*; the one on the left is called the first member, and the one on the right, the second member. It often happens that the second member is 0; indeed, in all general discussions, with respect to equations, we suppose them reduced to such a form that the second member shall be 0.

MEN-SU-RĀ'TION. [L. *mensura*, measure]. That branch of applied geometry which gives the rules for finding the lengths of lines, the areas of surfaces, and the volumes of solids. The following are some of the most important formulas:

I. Length of Lines.

1. *Circumference of Circle.*

$$s = 2\pi r \cdots\cdots\cdots (1)$$

in which r denotes the radius, s the length of the circumference, and $\pi = 3.14159$.

$$s' = \frac{a\pi r}{180} \cdots\cdots\cdots (2).$$

in which s' denotes the length of any arc, a the number of degrees in the arc.

$$s' = \frac{8c' - c}{3} \quad nearly \cdots (3);$$

in which s' is the length of any arc, c the chord of the arc, and c' the chord of half the arc, or

$$c' = \sqrt{\tfrac{1}{4}c^2 + \text{ver-sin}^2}.$$

2. *Circumference of Ellipse.*

$$s = \frac{199}{200}\,\pi\,\sqrt{\tfrac{1}{2}(a^2 + b^2)}, nearly\ (4),$$

in which s denotes the length of the circumference, a and b the semi-axes.

3. *Arc of Parabola from the vertex.*

$$s = \sqrt{\frac{4a^2}{3} + \sqrt{b}}, \textit{ nearly}; \cdots (5);$$

in which s denotes the length of the arc, a the abscissa, and b the ordinate of the extreme point.

4. *Arc of any Plane Curve.*

$$s = \int_a^b \sqrt{dx^2 + dy^2} \cdots\cdots (6);$$

in which s denotes the length of the arc, a and b the abscissas of the extreme points, dy is to be determined in terms of x and dx, from the equation and differential equation of the curve.

II. AREAS OF SURFACES.

1. *Plane Triangle.*

$$A = \frac{bh}{2} \cdots\cdots \cdots\cdots (7);$$

in which A denotes the area, b the base, and h the altitude

$$A = \frac{ab \sin C}{2} \cdots\cdots\cdots (8);$$

in which a and b are adjacent sides, and C their included angle.

$$A = \sqrt{s(s-a)(s-b)(s-c)} \cdot (9);$$

in which a, b and c, are the three sides, and

$$s = \frac{a+b+c}{2}.$$

2. *Parallelogram.*

$$A = bh \cdots\cdots\cdots (10);$$

in which b is the base, and h the altitude.

$$A = ab \sin C \cdots\cdots\cdots (11);$$

in which a and b are adjacent sides, and C their included angle.

$$A = 2\sqrt{s(s-c)(s-a)(s-b)} \cdot (12);$$

in which a and b are adjacent sides, c the diagonal of the parallelogram joining their extremities, and

$$s = \frac{a+b+c}{2}$$

3. *Trapezium.*

$$A = \frac{b+b'}{2} h \cdots\cdots (13);$$

in which b and b' are the parallel bases, and h the altitude of the trapezium.

$$A = \frac{b + b}{2} l \sin C \cdots\cdots (14);$$

in which l is the length of one of the oblique sides, and C the angle between it and one of the parallel bases, b and b' being the same as before.

4. *Any Quadrilateral.*

$$A = \frac{dd'}{2} \sin C \cdots\cdots (15);$$

in which d and d' are its diagonals, and C their included angle.

5. *Any Regular Polygon.*

$$A = \frac{n\left(\frac{a}{2}\right)^2}{\tan\left(\frac{180°}{n}\right)} \cdots\cdots (16);$$

in which n is the number of sides, and a the length of one of them.

6. *Circle.*

$$A = \pi r^2 \cdots\cdots\cdots (17);$$

in which r is the radius.

$$A' = \frac{n}{360} \pi r^2 \cdots\cdots (18);$$

in which A' denotes the area of a circular sector, n being the number of degrees in the sector.

$$A'' = \left(c + \frac{4}{3}\sqrt{\frac{c^2}{4} + b^2}\right) \frac{4}{10} v, \textit{ nearly } (19);$$

in which A'' denotes the arc of a circular segment, c its chord, and v its height, or the ver-sin of half the arc of the segment.

7. *Ellipse.*

$$A = \pi ab \cdots\cdots\cdots (20);$$

in which a and b are the semi-axes.

8. *Parabola.*

$$A = \frac{2}{3} ab \cdots\cdots\cdots (21);$$

in which a is the abscissa, and b the ordinate of the extreme point.

9. *Surface of Cylinder, exclusive of bases.*

$$A = 2\pi rh \cdots\cdots\cdots (22);$$

in which r is the radius of the base, and h the altitude.

10. *Surface of Cone, exclusive of base.*

$$A = \pi ra \cdots\cdots\cdots (23);$$

in which r is the radius of base, and a the slant height.

$$A' = \pi (r + r') a \ldots\ldots. (24);$$

in which A' denotes the area of the surface of a conic frustum, r and r' radii of the upper and lower bases, and a the slant height.

11. *Surface of Sphere.*

$$A = 4\pi r^2 \ldots\ldots\ldots (25);$$

in which r is the radius of the sphere.

$$A' = 2\pi rh \ldots\ldots\ldots (26);$$

in which A' represents the area of a zone, r the radius of the sphere, and h the altitude of the zone.

$$A' = 4\pi r^2 \sin \tfrac{1}{2} (l' - l) \cos \tfrac{1}{2} (l' + l) \ldots (27);$$

in which A' denotes the area of the surface of a zone whose bases are parallel to the equator; r the radius of the sphere; l and l' the latitudes of the bases of the zone, $+$ when north, $-$ when south.

$$A'' = \frac{\pi}{90^\circ} (m' - m) r^2 \sin \tfrac{1}{2} (l' - l) \times \cos \tfrac{1}{2} (l' + l) \ldots\ldots\ldots\ldots (28);$$

in which A'' denotes the area of a spherical quadrilateral bounded by two parallels of latitude and two meridians; m and m' the longitude of the extreme meridians in degrees; l and l' the latitudes of the bounding parallels, $+$ when north, $-$ when south; and r the radius of the sphere.

$$A''' = \left(\frac{A + B + C - 180^\circ}{90^\circ}\right) T \ \ldots . (29);$$

in which A''' is the area of a spherical triangle; A, B and C three angles in degrees; and T is the area of the trirectangular triangle, or equal to $\frac{1}{2}\pi r^2$.

$$A^{\text{iv}} = (s - 2n + 4) \times T \ldots\ldots\ldots (30);$$

in which A^{iv} denotes the area of a spherical polygon; s the sum of all the angles in degrees, divided by 90°; n the number of sides; and T the area of the trirectangular triangle, or equal to $\frac{1}{2}\pi r^2$.

12. *Area of any Plane Curve.*

$$A = \int_a^b y dx \ \ldots\ldots\ldots (31);$$

in which a and b are the abscissas of the extreme ordinates; and y must be deduced by solving the equation of the curve with respect to y.

13. *Area of any Surface of revolution.*

$$A = \int_a^b 2\pi y \sqrt{dx^2 + dy^2} \ldots\ldots\ldots (32);$$

in which y and dy are to be found from the equation and differential equation of the meridian curve referred to the axis of revolution; a and b denote the abscissas corresponding to the limiting circles of the surface of revolution.

14. *Area of any Surface.*

$$A = \int_a^b \int_c^d dxdy \sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}} \ \ldots (33);$$

in which $\frac{dz}{dx}$, $\frac{dz}{dy}$, are to be found from the equation and partial differential equations of the surface, in terms of x and y; a and b the ordinates of the extreme limits, in the direction of the axis of X; c and d the ordinates of the extreme limits, in the direction of the axis of Y.

III. Volumes of Solids.

1. *Parallelopipedon.*

$$V = abh \ldots\ldots\ldots (34);$$

in which V denotes the volume, a the length of the base, b the breadth of the base, and h the altitude.

2. *Prism and Cylinder.*

$$V = Bh \ldots\ldots\ldots\ldots (35);$$

in which B denotes the area of the base, and h the altitude.

3. *Pyramid and Cone.*

$$V = \frac{Bh}{3} \ldots\ldots\ldots\ldots (36);$$

in which B denotes the area of the base, and h the altitude.

$$V' = (A + B + \sqrt{AB}) \frac{h}{3} \ \ldots (37);$$

in which V' denotes the volume of a frustum, A the area of the upper, B the area of the lower base, and h the altitude of the frustum.

4. *Sphere.*

$$V = \tfrac{4}{3}\pi r^3 \ \ldots\ldots\ldots (38);$$

in which r is the radius of the sphere.

$$V' = \tfrac{2}{3}\pi r^2 h \ \ldots\ldots\ldots (39)$$

in which V' denotes the volume of a spherical sector, r the radius of the sphere, and h the altitude of the zone that forms the base of the sector.

$$V'' = \frac{A + B}{2} h + \frac{1}{6}\pi h^3 \ \ldots (40);$$

in which V'' denotes the volume of a spher-

ical segment, A and B the areas of its parallel bases, and h the altitude of the segment. Either A or B may become 0; in which case the segment has but one base.

5. *Prismoid.* A solid similar to that formed in rail-road cuttings, terminated by parallel cross-sections perpendicular to the axis of the road-way. The volume is equal to the sum of the end-section plus four times a section midway between them, multiplied by one-sixth of the length of the volume, in the direction of the axis of the road-way; or,

$$V = \left[(b + rh')h' + (b + rh)h + 4\left(b + r\frac{h + h'}{2}\right)\frac{h + h'}{2}\right]\frac{l}{6} \quad \ldots (41);$$

in which b denotes the breadth of the cutting at the bottom; h the perpendicular height of the cutting, at the upper end; h' the height, at the lower end; l the length of the volume, and r the tangent of the angle which the slope of the side makes with the vertical; h and h' the mean heights, at the two ends.

6. *Any Solid of revolution.*

$$V = \int_a^b \pi y^2 dx \ldots\ldots (42);$$

in which y is to be deduced from the equation of the curve, in terms of x; a and b are the abscissas of the limiting planes perpendicular to the axis of revolution.

7. *Any Solid, the equation of whose bounding surface is given.*

$$V = \int_a^b \int_c^d z dx dy \ldots (43);$$

in which z is to be deduced from the equation of the bounding surface, in terms of x and y; a and b are the ordinates of the limiting planes perpendicular to the axis of X; c and d the ordinates of the limiting planes perpendicular to the axis of Y.

The following principles of mensuration are due to GULDINUS:

1st. The area of any surface of revolution, generated by a plane curve revolving about a straight line, as an axis, is equal to the length of the curve multiplied by the circumference described by the centre of gravity of the arc.

2d. The volume of a solid, generated by revolving a plane curve about a straight line in its own plane, as an axis, is equal to the area of the curve multiplied by the circumference of the circle described by the centre of gravity of the revolving circle.

MERCATOR'S CHART. A representation of a portion of the surface of the earth upon a plane, in which the meridians are represented by equi-distant parallel straight lines, and the parallels of latitude by straight lines perpendicular to them.

This chart is particularly adapted to the purposes of navigation, inasmuch as the plot of a ship's course, or a rhumb line between two points upon it, is represented by a straight line. On this account, as well as on account of the facilities which it affords for making calculations necessary in navigation, Mercator's Chart is now almost universally adopted for sailing purposes.

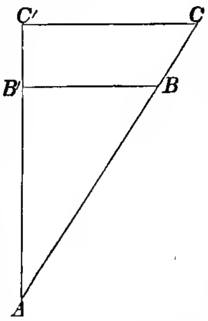
The principle on which the projection is made, is this: The projection of the meridians being assumed as equi-distant parallel straight lines, it is plain that as we recede from the equator, the scale on which a degree of longitude is represented will continually increase. In order, therefore, that the chart may fulfill the required conditions, the scale of latitudes is made to increase in the same proportion.

Now the length of a degree of longitnde in any degree of latitude, is equal to the length of a degree of longitude at the equator multiplied by the cosine of the latitude, but as the length of a degree of longitude is in every latitude represented by a constant distance, it follows that the scale increases inversely as the cosine of the latitude, or, what is the same thing, as the secant of the latitude; hence, the scale of latitudes must increase as the secant of the latitude, that is, if a given line represents the length of a degree of latitude at the equator, then will that line, multiplied by the secant of the latitude l, represent the length of a degree of latitude at the point whose latitude is l.

If a minute of the equator, or a nautical mile, be taken as the unit of measure, and that unit be taken as the radius of the tables of natural sines, &c., then will the representation of a minute of latitude, at any point, be represented by the number which is found

from the tables for the secant of the latitude of that point. A table of results formed by the successive additions of the secants corresponding to each minute of latitude, is given in every work on navigation, under the head of meridional parts, and by its aid Mercator's Chart may easily be projected. See *Spherical Projection.*

MERCATOR'S SAILING. The method of computing the cases of sailing in accordance with the principles of Mercator's Chart.



In the right angled triangle ABB′, let AB′ represent the true difference of latitude between two places, the angle BAB′ the angle which the course sailed makes with the meridian, and AB the true distance sailed; then is BB′ what is called the departure, as in plane sailing. Produce AB′ till AC′ is equal to the meridional difference of latitude, and draw CC′ parallel to BB′, then will CC′ represent the difference in longitude. If we denote the angle BAB′ by ϕ, we shall have, from the right angled triangle of the figure, the proportions,

1 : $\sin\phi$: : distance : departure,
1 : $\cos\phi$: : distance : diff. of latitude,
1 $\tan\phi$: : mer. diff. of lat. : diff. of long.

By means of these formulas, all cases of Mercator's Sailing may be solved.

ME-RID′I-AN OF A PLACE. [L. *meridies*, noon]. The intersection of the surface of the earth, with a plane passing through the axis of the earth and the place. The meridian is the same as a north and south line. The meridian, as defined, is called the true meridian.

MAGNETIC MERIDIAN of a place, is the intersection of the surface of the earth, with a vertical plane through the axis of a magnetic needle suspended freely at the place. The magnetic meridian of a place is continually changing. The angle which it makes with the true meridian, is called the variation of the needle, consequently the variation of the needle at any place, is continually changing.

MERIDIAN CURVE OF A SURFACE OF REVOLUTION. The section of the surface made by a plane passing through the axis of revolution.

MERIDIAN DISTANCE OF A POINT. In Surveying, the distance from the point to some assumed meridian, generally the one drawn through the extreme east or west point of the survey. The meridian distance of a course is the meridian distance of its middle point, or it is the arithmetical mean of the meridian distances of all of its points

MERIDIAN DISTANCE IN NAVIGATION. The same as departure or easting and westing, or the distance between two meridians, one drawn through each of the points, whose meridian distance apart is considered.

MERIDIAN PLANE OF A SURFACE OF REVOLUTION. Any plane passed through the axis of revolution.

ME-RID′I-ON-AL PARTS. Parts of the projected meridian, according to Mercator's system, corresponding to each minute of latitude, from the equator up to some fixed limit, usually 80°.

These parts are tabulated, and the tables are of use in projecting charts, and in solving cases in Mercator's Sailing. The theory of the construction of the table of meridional parts is very simple. If we take the length of one equatorial minute as the unit of measure, and as the radius of a system of natural secants, then will the length of a minute of the meridian in any latitude be represented by the natural secant of that latitude, and the distance of the projection of any parallel of latitude from the equator, will be equal to the sum of all the secants of the arcs from 0 up to the given latitude. For most purposes, the sum of the secants corresponding to each minute, will be sufficiently accurate.

The construction of the table is as follows:

For 1′ mer. part $= \sec 1'$,
" 2′ " $= \sec 1' + \sec 2'$,
" 3′ " $= \sec 1' + \sec 2' + \sec 3'$,
. . . .
. . . .
. . . .
. . . .
" n' " $= \sec 1' + \sec 2' + \sec 3' + \dots + \sec n'$.

The second member of each of the above formulas, when reduced, gives the distance of the projection of the parallel from the equator.

The table as above constructed is only approximately true, and is, besides, somewhat tedious to compute. Other methods have been invented, both more accurate and of easier application, but the method above given shows more clearly than any other the nature of the tables. Of the other methods of computing tables the best is founded on the property that the scale of latitudes, in Mercator's projection, is analogous to the scale of logarithmic tangents of half the complements of the latitudes.

MERIDIONAL PARTS ON THE SPHEROID. Denote the eccentricity of the ellipsoidal meridian by e; denote the latitude for which the meridional part is required by l, and denote the natural sine of the latitude by s. Find the arc whose sine is se, and call it l'; take the logarithmic tangent of half the complement of l' from the common tables, and subtract it from 10; multiply the remainder by 7915.7044679, and that product by e; this result taken from the meridional part found from the table of meridional parts for the latitude l, will be the meridional part for the same latitude on the surface of an oblate spheroid.

MES'O-LABE. [Gr. $\mu\epsilon\varsigma o\varsigma$, middle, and $\lambda\alpha\mu\beta\alpha\nu\omega$, to take]. An instrument invented for constructing two mean proportionals between two given straight lines mechanically. It was used in solving the problem of the duplication of the cube.

ME'TRE. A unit of measure in the French decimal system. It is equivalent to the ten millionth part of the distance from the equator to the north pole, or about 39.37 inches. See *Measure*.

ME-TROL'O-GY. [Gr. $\mu\epsilon\tau\rho o\nu$, measure, and $\lambda o\gamma o\varsigma$, discourse]. The art and science of mensuration. See *Mensuration*.

MID-DLE. [L. *medius*, in the midst]. Equidistant from the extremes. The middle point of a limited straight line is the point which is at the same distance from each extremity.

MID-DLE LATITUDE. The middle latitude of two points on the surface of a sphere or spheroid, is the half sum of the two latitudes when both are of the same name, or the half difference of the latitudes when both are not of the same name. The middle latitude is affected with the name of the greater. If we agree to call north latitudes positive, and south latitudes negative, the middle latitude in all cases is equal to half the algebraic sum of the two latitudes.

MIDDLE LATITUDE SAILING. The method of computing cases in sailing, by means of the middle latitude, by a combination of the principles of plane and parallel sailing. This method is only approximately correct. The computations are made on the following principle:

The departure is considered as the meridional distance for the middle latitude of the place sailed from and the place sailed to. The results are the more accurate as the two places are near the equator.

The following proportions serve to solve all cases of middle latitude sailing. Calling the latitude of the two places l and l' we have

$$\cos\left(\frac{l + l'}{2}\right) : 1 :: \text{departure} : \text{difference of longitude.}$$

$$\cos\left(\frac{l + l'}{2}\right) : \text{tan of course} :: \text{difference of latitude} : \text{difference of longitude.}$$

MILE. [L. *mille passus*, a thousand paces; *passus* has been dropped by usage, and the numeral *mille* has acquired a substantive signification]. A unit of measure equivalent to 5280 feet, or 1760 yards. The mile of different countries is somewhat different. See *Measure*. The Roman mile contained 1000 paces of 5 feet, each foot being equivalent to 11.62 inches; whence the name *mile*.

A square mile contains 640 acres, and has been adopted by the United States government as a unit of surface in dividing the public lands. A square mile of land is called a *section*.

MILL. [L. *mille*, a thousand]. The tenth part of a cent, or the thousandth part of a dollar. A unit of money in the United States of the lowest denomination

MILL-ION. [L. *mille*, one thousand]. A thousand thousand, or 1,000,000.

MIN'I-MUM. [L. *minimus*, smallest]. See *Maxima* and *Minima*.

MIN'U-END. [L. *minuendus*, from *minuo*, to lessen]. That quantity from which another is to be subtracted. The second quantity is called the subtrahend.

MĪ'NUS. [L. *minus*, less]. The name of the sign of subtraction; it is a simple horizontal mark, thus —. See *Symbols*.

MIN'UTE. [L. *minutum*, a small portion]. The 60th part of an hour, or the 1440th part of a day. See *Day*.

In angular measure, the 60th part of a degree, or the 5400th part of a right angle. See *Degree*.

MIXED. [L. *misceo*, to mix]. Composed of heterogeneous elements.

Mixed Mathematics. The application of mathematical principles to practical constructions, or to the investigations of general science. The term is used in contradistinction to the term pure mathematics, which implies the investigation of the purely scientific principles of mathematical science.

Mixed Number. A number expressed by the aid of both integral and fractional parts; thus, $2\frac{1}{2}$ is a mixed number, or mixed fraction; so also is 2.5. This is often called a mixed decimal.

Mixed Quantity. A quantity composed of entire and fractional parts; thus, $a + \frac{c}{d}$ is a mixed quantity. All mixed quantities can be reduced to the form of simple fractions. See *Fraction*.

MOD'U-LUS. [L. *modulus*, a measure]. A constant factor of a variable function which serves to connect the function with a particular system or base.

The modulus of a system of logarithms is a constant factor, by which, if the Naperian logarithm of any number be multiplied, the product will be the logarithm of the same number in that system.

The modulus of any system of logarithms is always equal to the reciprocal of the Naperian logarithm of the base of the system; it is also equal to the logarithm of the Naperian base,

$$e = 2.718281828,$$

taken in that system. The modulus of the Naperian system is 1, the base of that system being 2.718281828. The modulus of the common system is .434294482, and the base of that system is 10. From the definition of the modulus of a system of logarithms, it follows that the logarithms of the same number, in different systems, are to each other as the moduli of those systems.

Modulus of a Number or Quantity. M. Mourey has shown that every quantity can always be reduced to the general form

$$a + b\sqrt{-1},$$

in which a and b are always real, but may be entire or fractional, positive or negative rational or irrational. When $b = 0$, the quantity is real; when b is not 0, the quantity is imaginary. He proposes to call

$$\sqrt{a^2 + b^2}$$

the modulus of the quantity, and in his geometrical interpretation of imaginary results, he shows that this modulus represents the length of a straight line, whilst the relation between a and b determines the direction of the line with respect to a fixed initial line.

MO-NŌ'MI-AL. [Gr. μονος, sole, and ονομα, name]. A single algebraic expression; that is, an expression unconnected with any other by the signs of addition, subtraction, equality, or inequality.

MONTH. The twelfth part of a year. Months are variously distinguished. The *calendar* months are named January, February, March, April, May, June, July, August, September, October, November and December. The first, third, fifth, seventh, eighth, tenth and twelvth, have each 31 days, all the rest have 30, except February, which has 28 in ordinary years and 29 in leap years. A *lunar* month embraces the period between two consecutive new moons. It is about $29\frac{1}{2}$ days in length, so that there are nearly 13 lunar months in a year.

The civil lunar month is a period alternately of 29 and 30 days. The civil solar month is a period alternately of 30 and 31 days, except one month, which consists only of 29 days, which in leap year has 30 days. These distinctions are not now regarded.

MULT-AN''GU-LAR. [L. *multus*, many, and *angulus*, angle]. Many angled, polygonal. See *Polygon*.

MUL-TI-NŌ'MI-AL. [L. *multus* and *nomen*, name]. An expression composed of two or more monomials, connected by the signs *plus* or *minus* (+ or −). See *Polynomial.*

MULTINOMIAL THEOREM. A theorem of Algebra, which has for its object to deduce a formula for developing any power of a polynomial. This formula is called the multinomial formula.

MULTINOMIAL FORMULA. A formula for developing any power of a polynomial without performing the successive multiplications. The formula is as follows:

$$(a + bx + cx^2 + \ldots + px^n)^m = B$$

$$+ mbB\frac{x}{-} + \left.\begin{array}{l} 2mcB \\ + (m-1)\,bB' \end{array}\right| \frac{x^2}{2a} + \left.\begin{array}{l} 3mdB \\ +(2m-1)cB' \\ +(m-2)bB'' \end{array}\right| \frac{x^3}{3a}$$

$$\left.\begin{array}{l} +\ 4meB \\ + (3m-1)\,dB' \\ + (2m-2)\,cB'' \\ + (\ m-3)\,bB''' \end{array}\right| \frac{x^4}{4a} + \text{\&c.}$$

in which $B = a^m$; and B', B'', B''', &c., represent the co-efficients of the terms immediately preceding those in which they first appear.

1. Let it be proposed to find the cube of the polynomial

$$1 + x + x^2 + x^3 + \text{\&c.};$$

here,

$$a = b = c = d = \text{\&c.} = 1,\ m = 3;$$

hence,

$$a^m = B = 1,\ mbB = 3 \times 1 \times 1 = 3 = B',$$

$$\frac{2mcB+(m-1)bB'}{2a} = \frac{6+2\times 3}{2} = 6 = B'',$$

$$\frac{3mdB + (2m-1)\,cB' + (m-2)\,bB''}{3a}$$

$$= \frac{9 + 15 + 6}{3} = B'''; \text{ \&c., \&c.}$$

Substituting these, in the formula, we find

$$(1 + x + x^2 + x^3 + \ldots + x^n)^3 = 1 + 3x$$
$$+ 6x^2 + 10x^3 + 15x^4 + \text{\&c.}$$

2. Again let it be required to find the cube root of the series

$$1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \tfrac{1}{4}x^3 + \text{\&c.};$$

here,

$$a = 1,\ b = \tfrac{1}{2},\ c = \tfrac{1}{3},\ d = \tfrac{1}{4}, \text{ \&c., and } m = \tfrac{1}{3},$$

$$a^m = 1^{\frac{1}{3}} = 1 = B;\ mbB = \tfrac{1}{6} = B';$$

$$\frac{2mcB+(m-1)bB'}{2a} = \tfrac{1}{2}(\tfrac{2}{9} - \tfrac{1}{18}) = \tfrac{1}{12} = B'';$$

$$\frac{3mdB + (2m-1)\,cB' + (m-2)\,bB''}{3a}$$

$$= \frac{35}{648} = B'''; \text{ \&c., \&c.}$$

Hence, from the formula,

$$(1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \tfrac{1}{4}x^3 + \text{\&c.} \ldots)^{\frac{1}{3}}$$
$$= 1 + \tfrac{1}{6}x + \tfrac{1}{12}x^2 + \tfrac{35}{648}x^3 + \text{\&c., \&c.}$$

MUL'TI-PLE. The multiple of two or more quantities, is a quantity which they will separately divide without a remainder.

Thus, 12 is a multiple of 2, 3 and 4; 24 is also a multiple of 2, 3 and 4; hence, there may be more than one multiple of any given set of quantities.

LEAST COMMON MULTIPLE. The least common multiple of two or more quantities, is the least quantity which they will separately divide without a remainder. Thus, 12 is the least common multiple of 2, 3, 4 and 6. The least common multiple of several quantities must contain all the prime factors which enter each quantity, and it must contain each factor raised to the highest power to which it enters any one of the numbers, nor will it contain any other factors. Hence, to find the least common multiple of several quantities.

Resolve them into their prime factors; form the continued product of all the prime factors of the quantities, each raised to the highest power to which it enters any of the quantities, and this will be the least common multiple sought.

Let it be required to find the least common multiple of

$$98,\ 72,\ 63 \text{ and } 49:$$

resolving them into prime factors, the numbers take the respective forms,

$$2 \times 7^2,\ 2^3 \times 3^2,\ 3^2 \times 7 \text{ and } 7^2;$$

hence, the least common multiple is

$$3528 = 2^3 \times 3^2 \times 7^2.$$

The least common multiple of

$$8a^3b,\ 4ab^2c \text{ and } 72, \text{ is}$$

$$72a^3b^2c = 2^3 \times 3^2a^3b^2c.$$

It has been proposed to call the least com-

mon multiple of several quantities, their least common dividend. Hence, we should define the least common dividend of several quantities, to be the least quantity that they will separately divide without a remainder. The term *least*, as used above, refers only to the numerical value of the quantities to which it is applied.

Multiple Point of a Curve. In Analysis, is a point in which two or more branches of a curve intersect each other. The analytical characteristic of a multiple point of a curve is that at it the first differential co-efficient of the ordinate must have two or more values. Hence, to find whether a given curve has any multiple points, differentiate its equation, and from the equation of the curve and its differential equation, find an expression for the first differential co-efficient of the ordinate. See whether there are any real values of the variables which will give to the first differential co-efficient found, two or more values, and at the same time satisfy the equation of the curve; if so, the corresponding points are multiple points. The number of multiple points will be determined by the number of sets of real values of the variables which fulfill the required conditions.

If the differential co-efficient has two values at a point, the point is a double multiple point; if three, a triple multiple point; if four, a quadruple multiple point, and so on. See *Singular Points*.

MUL-TI-PLI-CAND′. [L. *multiplico*, to multiply]. That quantity which is to be repeated, or which is to be multiplied.

MUL-TI-PLI-CA′TION. [L. *multiplicatio*, increasing]. The operation of finding the product of two quantities. The *product* is the result obtained by taking one of the quantities as many times as there are units in the other. The quantity to be multiplied or taken is called the *multiplicand*, the quantity by which it is to be multiplied is called the *multiplier*, and the result of the operation is called the *product*. Both multiplicand and multipliers are called *factors* of the product.

I. Arithmetical Multiplication.

1. *To multiply any number by a multiplier less than* 10.

Write the multiplier under the right hand figure of the multiplicand. Multiply in succession each digit of the multiplicand by the multiplier, beginning at the right hand; if any product is expressed by more than one figure, set down the right hand figure under the digit multiplied and add the number expressed by the remaining figure or figures to the next product, and so on to the last figure of the multiplicand, when the entire product is set down.

Multiplicand,	3896439
Multiplier,	8
Product,	31171512

2. *To multiply any number by a multiplier greater than* 10.

Write down the multiplier under the multiplicand, so that units of the same order shall fall in the same column. Multiply the entire multiplicand by each digit of the multiplier, and write down these partial products so that units of the same order shall fall in the same column, and take their sum, which will be the product required.

Multiplicand,	45684
Multiplier,	4374
Partial products,	182736
	319788
	137052
	182736
Product,	199821816

The preceding rule is equally applicable to decimal fractions, and to mixed decimals.

3. *To multiply one vulgar fraction by another.*

Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.

This rule applies when either factor is a whole number or a mixed number. In the former case, the denominator is 1, and in the latter case, the mixed number must be transformed into an equivalent fraction by the rule. See *Fraction*.

II. Algebraic Multiplication.

1. *To multiply one monomial by another.*

Multiply the co-efficients together for a new co-efficient, after this write all the letters that enter both factors, giving to each an expo-

nent equal to the sum of its exponents in both factors.

The rule of signs, applicable in all cases of algebraic multiplication, is that the product of two terms, preceded by like signs, is affected with the sign $+$; and the product of two terms, preceded by unlike signs, is affected with the sign $-$.

$$\begin{array}{ll} \text{Multiplicand} & -2abc^3f \\ \text{Multiplier} & \underline{4c^3bf^5} \\ \text{Product} & -8ab^2c^6f^6. \end{array}$$

2. *To multiply a polynomial by a monomial.*

Multiply each term of the polynomial by the monomial, and connect the results by their respective signs; the final result will be the product.

$$\begin{array}{lr} \text{Multiplicand} & 8ab - cx^2 + 2cf - g \\ \text{Multiplier} & 3cg^2 \\ \hline \text{Product} & 24abcg^2 - 3c^2g^2x^2 + 6c^2fg^2 - 3cg^3 \end{array}$$

3. *To multiply one polynomial by another.*

Write down the multiplier under the multiplicand, arranging both with reference to the same leading letter. Multiply all the terms of the multiplicand by each term of the multiplier in succession, placing similar terms of the product, if there are any. in the same column; then reduce the polynomial result to its simplest form, and it will be the required product.

$$\begin{array}{ll} \text{Multiplicand} & 3a^2 + 4ab + b^2 \\ \text{Multiplier} & 2a + 5b \\ \hline & 6a^3 + 8a^2b + 2ab^2 \\ & 15a^2b + 20ab^2 + 5b^3 \\ \hline \text{Product} & 6a^3 + 23a^2b + 22ab^2 + 5b^3. \end{array}$$

If both factors are homogeneous, the product will be homogeneous, and its degree will be expressed by the sum of the numbers which express the degrees of the factors taken separately.

If no two terms of the partial products are similar, the number of terms in the product will be equal to the number of terms in the multiplicand multiplied by the number of terms in the multiplier.

There are always at least two terms of the partial products which cannot be reduced with any others. 1st, the term arising from the multiplication of those terms of the multiplicand and multiplier, which are of the highest degree with respect to the leading letter; 2d, that which arises from the multiplication of the two terms, which are of the lowest degree with respect to the same letter.

Algebraic fractions are multiplied by the same rule as arithmetical fractions.

4. *To multiply one radical by another.*

Reduce the radicals to equivalent ones of the same degree; multiply the co-efficients together for a new co-efficient, after which write the radical sign with the common index, and under it place the product of the quantities under the radical sign in both radicals.

Let it be required to find the product of

$$2\sqrt{6a^3b} \quad \text{and} \quad 3\sqrt[3]{81a^2b};$$

reducing them to equivalent radicals of the 6th degree, they become, respectively,

$$8\sqrt[6]{216a^9b^3} \quad \text{and} \quad 9\sqrt[6]{6561a^4b^2}.$$

Then $\quad 8\sqrt[6]{216a^9b^3} \times 9\sqrt[6]{6561a^4b^2}$

$$= 72\sqrt[6]{216 \times 6561a^{13}b^5}.$$

Where radicals are expressed by means of fractional exponents, they are multiplied by the same rules that are applicable to quantities affected with entire exponents.

5. *Multiplication by detached co-efficients.*

When the multiplicand and multiplier are both homogeneous, and contain but two letters, if both be arranged according to the same leading letter, the literal part of the several terms of the product may be written immediately, since the exponents of the leading letter will go on decreasing from left to right by a constant difference, in each term, and the sum of the exponents of both letters in each term is constantly the same.

Hence, the product may be obtained by writing down the co-efficients alone, and multiplying by the general rule, after which the literal parts are annexed in accordance with the law above indicated. It must be observed that when any power of the leading letter does not enter, the corresponding co-efficient must be taken equal to 0.

1. Let it be required to multiply

$$2a^2 - 3ab^2 + 5b^3 \quad \text{by} \quad 2a^2 - 5b^2.$$

$$\begin{array}{lll} \text{Multip'd} & 2+0-3+5 & \text{, co-ef's of multip'd.} \\ \text{Multip'r} & 2+0-5 & \text{, co-ef's of multipl'r.} \\ \hline & 4+0-6+10 & \\ & -10-0+15-25 & \\ \hline \text{Co-ef's.} & 4+0-16+10+15-25 & \\ \text{Product} & 4a^5-16a^3b^2+10a^2b^3+15ab^4-25b^5 & \end{array}$$

This method by detached co-efficients is applicable when the multiplicand and multiplier contain but a single letter.

6. *Multiplication by means of logarithms.*

Find, from a table, the logarithms of both factors, and take their sum; find from the table the number corresponding to this logarithm, and it will be the product of the two factors.

1. Let it be required to multiply 7843 by 6328.

Log. 7843	3.8944822
Log. 6328	3.8012665
Log. 49630504	7.6957487

Hence, the product is 49630504. See *Logarithms.*

MULTIPLICATION TABLE. A table showing the product of factors, taken in pairs, up to some assumed limit. The ordinary tables run both ways from 1 to 12. A more useful table would extend to 9 in one series of numbers, and to 25 in the other. There is no reason why the operation of multiplying a number by a number less than 25, should not be performed directly, instead of by following the rule and making two partial multiplications, as is generally done for numbers between 12 and 25. The subjoined table is easily committed to memory, and will be found of great utility in performing multiplications.

Multiplication Table, from 1 to 9, and from 1 to 25.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81
10	20	30	40	50	60	70	80	90
11	22	33	44	55	66	77	88	99
12	24	36	48	60	72	84	96	108
13	26	39	52	65	78	91	104	117
14	28	42	56	70	84	98	112	126
15	30	45	60	75	90	105	120	135
16	32	48	64	80	96	112	128	144
17	34	51	68	85	102	119	136	153
18	36	54	72	90	108	126	144	162
19	38	57	76	95	114	133	152	171
20	40	60	80	100	120	140	160	180
21	42	63	84	105	126	147	168	189
22	44	66	88	110	132	154	176	198
23	46	69	92	115	138	161	184	207
24	48	72	96	120	144	168	192	216
25	50	75	100	125	150	175	200	225

To use the table: Look for the least factor at the top of the table, follow down the table till the number is found, which stands opposite the other factor; this number is the product of the two factors which is sought.

Multiplication tables have been formed extending much farther in each direction, but such tables are intended for reference simply, and not to be committed to memory for everyday use. The famous table formed by Pythagoras, and called from its inventor, the abacus Pythagoricus, was of this kind, and extended to 60 in both directions.

MUL'TI-PLĪ-ER. That factor of a product which indicates the number of times which the other factor is to be taken. See *Multiplication.*

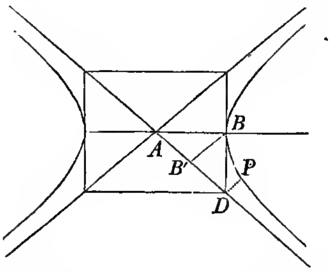
MŪ'TU-AL. [L. *mutuus*; from *muto*, to change]. A species of relation in which two quantities are similarly affected with respect to each other. Thus, two straight lines, as the diagonals of a parallelogram, are said to bisect each other mutually; that is, the first bisects the second, and the second bisects the first. The three angles in each of two triangles, are mutually equal when taken in the same order: the first angle of the first triangle is equal to the first angle of the second triangle, the second angle of the first triangle equal to the second angle of the second triangle, and the third angle of the first triangle, equal to the third angle of the second triangle.

N. The fourteenth letter of the English alphabet. In surveying, it stands as an abbreviation for North. In Analysis, n is generally taken as a symbol to represent any number. As a numeral, it stood for 900; with a dash over it, thus, $\overline{N}$, it represented 900,000.

NĀ'DIR. The point of the celestial sphere directly opposite to the zenith. If at any

place a vertical line be erected to the earth's surface, and prolonged indefinitely in both directions, the point above, in which it pierces the celestial sphere is the zenith, and the point below, in which it pierces the celestial sphere, is the Nadir. Every point on the earth's surface has a different nadir point.

NAPERIAN LOGARITHMS. A system of logarithms whose base is 2.718281828, and whose modulus is 1. The system was thus named from its discoverer, Baron Napier. They are sometimes called hyperbolic logarithms, for if BP represents one branch of an equilateral hyperbola, AK and AL its asymptotes at right angles to each other,



B its principal vertex, and BB′ the ordinate through the vertex; then if AB′ is taken as the unit, or 1, the curvilineal area B′BPD will be expressed by the logarithm of AD, the abscissa of its extreme ordinate DP.

NAPIER'S RODS OR BONES. A set of rods contrived by Baron Napier, for the purpose of facilitating the numerical operations of multiplication and division. They consist of pieces of bone, or ivory, in the shape of a parallelopipedon, about 3 inches long and $\frac{3}{10}$ of an inch in width, the faces of each being

1	1	2	3	4	5	6	7	8	9	0
2	/2	/4	/6	/8	1/0	1/2	1/4	1/6	1/8	/0
3	/3	/6	/9	1/2	1/5	1/8	2/1	2/4	2/7	/0
4	/4	/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6	/0
5	/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5	/0
6	/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4	/0
7	/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3	/0
8	/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2	/0
9	/9	1/8	2/7	3/6	4/5	6/4	6/3	7/2	8/1	/0

divided into squares, which are again subdivided on ten of the rods by diagonals into triangles, except the squares at the upper ends of the rods. These spaces are numbered as shown in the diagram.

The analogy betwen the numbering and the multiplication table will be perceived on inspection. The rods, in fact, constitute a sort of movable or portable multiplication table.

To show the manner of performing multiplication by means of the rods, let it be required to multiply 5978 by 937. Select the proper rods, and dispose them in such a manner that the numbers at the top shall exhibit the multiplicand, as in the figure, and on their left place the rod of units. In the rod of units seek the right hand figure of the multiplier, which, in this case, is 7, and the numbers corresponding to it on the other rods. Beginning on the left

1	5	9	7	8
2	1/0	1/8	1/4	1/6
3	1/5	2/7	2/1	2/4
4	2/0	3/6	2/8	3/2
5	2/5	4/5	3/5	4/0
6	3/0	5/4	4/2	4/8
7	3/5	6/3	4/9	5/6
8	4/0	7/2	5/6	6/4
9	4/5	8/1	6/3	7/2

add the digits in each parallelogram, formed by triangles of adjacent rods, and write them down as in ordinary multiplication; then take the sum of the several products as in ordinary multiplication, and it will be the product required.

$$\begin{array}{r} 41846 \\ 17934 \\ 53802 \\ \hline 5601386 \end{array}$$

From the outermost triangle on the line with 7, write out the number there found, 6; in the next parallelogram on the left add 9 and 5 there found; their sum being 14, set down the 4 and carry the 1 to be added to 3, and 4 found in the next parallelogram on the left; this sum being 8, set it down; in the next parallelogram on the left occur the numbers 5 and 6, their sum being 11, set down 1, and carry 1 to the next number on the left; the number 3 found in the triangle on the left of the row, increased by 1, gives 4, which set down; proceed in like manner till all of the partial products are found, and take their sum as in the example. This contrivance is rather curious than useful.

NAPPE. One of the two parts of a conic surface, which meet at the vertex. If a

straight line be moved in such a manner as to pass through a fixed point, and continually touch a given curve, the surface generated is that of a cone, the fixed point being the *vertex*. If the straight line be prolonged in both directions, through the vertex, the surface generated will consist of two parts, meeting at the vertex, and symmetrically disposed with respect to the vertex: these are called nappes. The nappe on which the directrix lies, is called the lower, and the other one, the upper nappe of the cone.

Nappe of an Hyperboloid. One of the branches, of which the surface is composed. If an hyperbola be revolved about its transverse axis, as an axis of revolution, each branch of the hyperbola will generate a separate branch of the hyperboloid of revolution, each of which is called a nappe. If the hyperbola be revolved about its conjugate axis, as an axis of revolution, both branches will generate the same branch, and the hyperboloid has but one nappe. Hence, hyperboloids are of one or two nappes. Those of one nappe are warped surfaces; those of two nappes are double curved surfaces.

NAT'U-RAL. [L. *naturalis*, natural]. A term used in mathematics to indicate, that a function is taken in, or referred to, some system, in which the base is 1. Natural numbers are those commencing at 1; each being equal to the preceding, plus 1. Natural sines, cosines, tangents, cotangents, &c., are the sines, cosines, tangents, cotangents, &c., taken in arcs, whose radii are 1. Natural logarithms, or Naperian logarithms, are those taken in a system, whose modulus is 1.

NAU'TIC-AL. [L. *nauticus*; from *nauta*, a seaman]. Appertaining to navigation. A nautical mile is the 60th part of a degree of latitude; 60 nautical miles make about 69½ English miles. A nautical chart is a chart constructed for the use of navigators.

NAV-I-GA'TION. [L. *navigatio*; from *navis*, a ship]. The art of conducting vessels from one port to another, on the ocean, by the best route. It embraces all the rules and principles necessary to determine the position of the vessel at any moment, and also to determine the direction and distance of the destined port. There are two methods of determining the position of a ship at sea: the *first* is by means of the reckoning; that is, from a record which is kept, of the courses sailed, and distances made, on each course; the *second* is, by means of observations made on the heavenly bodies, and the aid of Spherical Trigonometry. The first method gives only approximate results; the second admits of great accuracy. The position of the vessel being known at any moment, the direction and distance of any other point may be determined, either by the aid of a chart, or by the application of the principles of Trigonometry.

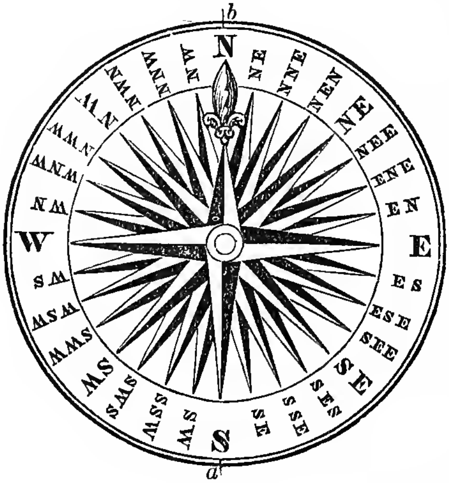
To understand the principles of navigation, it is necessary to know the form and magnitude of the earth, the relative positions and the forms of lines drawn upon its surface; together with the relative positions of places on the earth's surface, as well as their positions with respect to certain fixed lines on the earth's surface. The positions of places on the earth's surface, with respect to certain fixed lines of reference, are given by means of charts. Besides these elements of knowledge, we must know how to obtain and use the data, from which the position of the ship's place, at any time, can be ascertained. This involves a knowledge of the instruments used, the method of using them, and the methods of making the necessary computations required to deduce, from the observations made, the ship's position. The method of making the necessary observations does not properly fall within the limits of a mathematical discussion of the subject of navigation, and will not therefore be treated of in this article.

The form of the earth's surface is that of an ellipsoid of revolution, the axis of revolution coinciding with the shortest diameter of the surface. The length of the longest, or equatorial diameter of the earth is nearly 7925 English miles, and that of the shortest or polar diameter is about 7898 miles. The mean radius is about 3956 miles. For the purposes of navigation, the earth may be regarded as a sphere, having the radius equal to the mean radius of the real form, and will be so regarded in this article.

Every plane through the axis is called a *meridian plane*, and its intersection with the surface of the earth is a *meridian curve*, or simply a *meridian*. The plane through the

centre, and perpendicular to the axis, is called the *plane of the equator*, and the circle cut from the surface is the *equator*. Every plane parallel to the plane of the equator, cuts from the surface a circle, called a circle or *parallel of latitude*. A curve passing through any two places on the surface of the earth, and making the same angle with every meridian which it crosses, is called a *rhumb line*. The angle which a *rhumb line*, between any two places, makes with any meridian which it crosses, is called the course from one place to the other.

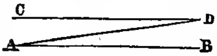
If the two places lie on the same meridian, the rhumb is that meridian, and the course is either north or south, according as the ship sails from or towards the North pole. In this case, the distance of the places, estimated on the rhumb, is the same as their distance on the great circle passing through them. If



the two places lie on the same parallel of latitude, the rhumb line coincides with the parallel, and the course between them is east or west, according as the ship sails in the same or in a contrary direction with the motion of the earth on its axis. If the places are both on the equator, the *rhumb-line* distance, or, as it is called, their *nautical distance*, is the same as their distance reckoned on the great circle through the places. In all other cases, the *rhumb line*, or nautical distance of two places, is greater than their distance reckoned on the great circle through them. If the ship makes both northing and easting, the course is lettered N. and E., thus, N. 15° E.; if it makes northing and westing, the course is lettered N. and W., thus, N. 15° W.; if it makes both southing and easting, the course is lettered S. and E., thus, S. 15° E.; if it makes both southing and westing, the course is lettered S. and W., thus, S. 15° W. This system of notation embraces every possible case. But it is customary to designate courses so as to correspond with the lettering of the Mariner's Compass. In the Mariner's Compass, a card is attached to the needle, whose circumference is divided into 32 equal parts, called *points*, each point being sub-divided into 4 equal parts, called *quarter points*. The direction of the needle coincides with the line NS, the head being towards N. The points of the compass beginning at the point N, are read around towards the east; thus, north, north and by east, north northeast, northeast and by north, northeast; and so on around. The needle bearing the card being poised freely upon a pivot, will indicate the magnetic meridian. On the compass-box are marked two points, *a* and *b*, which lie on a line passing through the centre of the card, and the compass-box is so placed, that the points *a* and *b* shall lie on a line parallel to the keel of the ship, *b* being placed towards the bow of the vessel; the point of the card which is opposite *b* will show the magnetic course of the ship, which on being corrected for variation of the needle, gives the true course. The course is generally read to quarter points, which by means of a table given in all works on Navigation, can be converted into degrees and minutes, or the reading may be converted into degrees and minutes, by recollecting that each point is equal to 11° 15′, and consequently, each quarter point is equal to 2° 48′ 45″

Before being used, the course has to be corrected for *leeway*. The leeway is the deviation of the course actually run, from that steered upon, in consequence of winds, cur-

rents, or other causes, or it is the angle formed by the ship's keel with the line that she actually describes in passing through the water. Let AB be the direction steered upon, or the direction of the ship's keel, and AD the line upon which the ship actually runs in consequence of the action of the wind, currents, &c.; then is the angle DAB the leeway, and must be allowed for, by being added to the leeward. To obviate the effect of making leeway, the ship is steered that much nearer the wind.



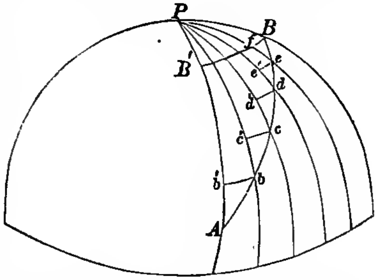
The nautical distance, is the distance sailed in the direction of the rhumb line on the course, and is practically determined by the *log* and *line*, which is thrown every hour. See *Log*. The elements of course and distance, are the data for determining the ship's place, by reckoning, at any time.

The *difference of latitude* of any two places, is the arc of a meridian intercepted between the parallels of latitude passing through the places, expressed in degrees. If both places are on the same side of the equator, the numerical value of the difference of latitude is equal to the difference of the latitude of the two places; if on opposite sides, it is equal to the sum of their latitudes.

The *difference of longitude* of two places, is the arc of the equator intercepted between two meridians, one passing through each place, and expressed in degrees. Longitudes are all reckoned from some fixed meridian, called the principal meridian, which we shall suppose to be that of Washington. If the two places lie on the same side of the principal meridian, the numerical value of the difference of longitude is equal to the differences of the longitudes of the two places; if they lie on opposite sides of the principal meridian, it is equal to their sum. The longitudes are supposed to be estimated from the principal meridian in both directions to 180°. If, however, we were only to estimate in one direction, the difference of longitude would, in all cases, be equal to the difference of the longitudes of the places.

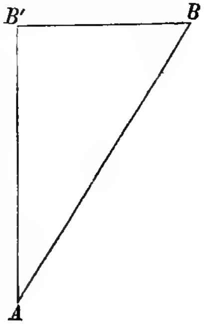
When a ship, sailing on a rhumb line between two places, arrives at the second place, the arc of the parallel through the second place, intercepted between the meridians of the two places, is called the meridian distance which the ship has made, and the sum of all the intermediate meridian distances corresponding to infinitely small portions of the rhumb line, is called the departure.

Let P be the pole of the earth, A and B any two places, PA and PB meridians through



them; and suppose that Ab, bc, cd, &c., are indefinitely small portions of the rhumb line through A and B. Let BB′ be a circle of latitude through B, bb', cc', dd'. &c., be circles of latitude through the points of division, and Pb, Pc, &c., meridians through the same points. Then is AB′ the difference of latitude of A and B, BB′ is the meridian distance made, and the sum of the arcs bb', cc', dd', &c., to B, is the *departure*. Now the angles bAb', cbc', &c., being all equal, and the triangles so small that they may be regarded as rectilinear, the *difference of latitude*, the *distance sailed*, (equal to $Ab + bc + cd +$ &c.), and the departure, may be represented by the sides of a plane right angled triangle, and these with the course form four elements of the triangle, any two of which being given, the remaining ones may be found. Upon these principles depend what is called

Plane Sailing.



Let ABB′ be a right angled triangle, right angled at B′, the angle B′AB being the *course* sailed, AB the *distance*, BB′ the *departure*, and AB′ the *difference of latitude*. Denote these quantities respectively, by v, c, d and

l, then from the right angled triangle, we have the relations

$$d = c \sin v = l \tan v = \sqrt{c^2 - l^2} \quad \cdot \cdot \ (1).$$

$$l = c \cos v = d \cot v = \sqrt{c^2 - d^2} \quad \cdot \ (2).$$

$$c = \sqrt{d^2 + l^2} = d \text{ co-sec } v = l \sec v \ . \ (3).$$

$$\sin v = \frac{d}{c} \cos v = \frac{l}{c} \tan v = \frac{d}{l} \quad \cdot \cdot \ (4).$$

By means of these formulas any problem in plane sailing may be solved.

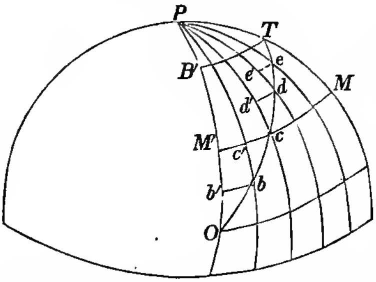
1. A ship leaving a point in lat. 47° 30′ N., sails S. W. by S. 98 nautical miles. What latitude is she in, and what departure has she made?

From the table of rhumbs we find S. W. by S. corresponds to a course of S. 33° 45′ W., hence, $v = 33° 45'$, $c = 98$ miles. From formula (1), we find $l = 81^m.48$, or 81′.48, or 1° 21′ 29″; hence, she is in latitude 46° 08′ 31″ N. We find from formula (1) $d = 54^m.45$.

This method gives the departure, but does not give the difference of longitude, except when sailing on a parallel of latitude. In this case the departure in miles can be converted into difference of longitude in minutes, by multiplying by the cosine of the latitude. When the ship sails on a rhumb line the difference of longitude may be found by the *middle latitude* method, or by *Mercator's* method.

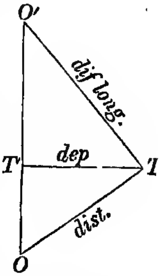
Middle Latitude Sailing.

Middle latitude sailing is based on the principle that the sum of all the meridian distances bb', cc', dd', &c., is equal to the distance MM′, the arc of a parallel of latitude



equidistant from the parallels through T and O. This supposition is only approximately true, but it serves to make a very close approximation in most cases. This method is very inaccurate when the course is small and the nautical distance is great, but it is very correct when the course is great and the distance small. The method is, however, rendered applicable in most cases by a table of corrections computed by Workman, called Workman's tables, which may be found in every work on navigation.

To find formulas for solving the problems of middle latitude sailing, let us take the triangle T′OT.



If the ship sails on a rhumb line from O to T, the hypothenuse OT will represent the *distance*, OT′ the *difference of latitude*, and T′T the *departure*. Now by the hypothesis made in this kind of sailing, the departure T′T is equal to the middle parallel intercepted between the meridians of the two places; so that the difference of longitude between the two places can be had by multiplying T′T by the cosine of the middle latitude. Draw TO′, making the angle O′TT′ equal to the middle latitude of the two places, denoted by m. Then in the triangle O′TT′, having adopted the previous notation,

$$\cos m : d :: 1 : \text{diff. of longitude} \cdot \cdot \cdot (5).$$

In the triangle O′TO, we have

$$\cos m : c :: \sin v : \text{diff. of longitude} \cdot \cdot (6).$$

From the triangle OTT′, we have

$$l : \text{diff. of longitude} :: \cos m : \tan v \cdot \cdot (7).$$

By the aid of these three, and the previous formulas, every problem in middle latitude sailing may be solved.

To use Workman's table in solving problems of middle latitude sailing, enter the table with the middle latitude and the value of d, take out the corresponding correction, and add it to the middle latitude; this sum will be the latitude in which the meridian distance is exactly equal to the departure.

Mercator's Sailing.

For an account of this method of computing the elements used in navigation, see Mercator's chart and Mercator's sailing.

It only remains to indicate the method of

determining the difference of latitude and longitude when the ship sails successively on different courses. The broken line which the ship sails under these circumstances, is called a *traverse*, and the method of making the computations is called

Traverse Sailing.

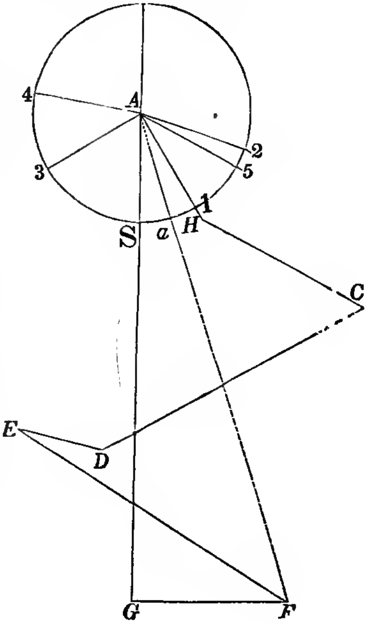
In this method the operation of finding a single course and distance equivalent to the aggregate of the partial courses, is called resolving, or working the traverse. The method of working a traverse is best shown by an example.

Rule a table with eight columns, as shown below. In the first, enter the number of each course; in the second, the course as taken from the log-book; in the third, the same course reduced to degrees by means of the table of rhumbs, and in the fourth column, the distance in nautical miles sailed on each course. The remaining four columns are headed N. S. E. W. From a table of latitudes and departures, the latitude and departure of each course is taken and entered under their proper headings. Then will the algebraic sum of the northings and southings give the northing or southing of the single course, and the algebraic sum of the eastings and westings will give the easting or westing of the single course.

Courses.			Dist.	Diff. of Latitude.		Departure.	
No.		Angle.	Miles.	N.	S.	E.	W.
1	S. S. E. $\frac{1}{4}$ E.	25° 18′	16		14.47	6.83	
2	E. S. E.	67 30	23		8.80	21.25	
3	S. W. by W. $\frac{1}{2}$ W.	61 52	36		17 04		31.71
4	W. $\frac{3}{4}$ N.	81 33	12	1.77			11.87
5	S. E. by E. $\frac{1}{4}$ E.	59 03	41		21.12	35.14	
				1.77	61.43	63.22	43.58
					1.77	43.58	
				Diff.	59.66	19.64	

From formulas (1), we find $v = 18° 13'$, and $c = 62.83$. Hence, the bearing from the starting point to the last point is S. 18° 13′ E., and the distance 62.83 miles.

The traverse may be wrought by means of a plot, as shown in the figure. The plot is made by means of what is called a mariner's scale and a pair of dividers. One line on the mariner's scale, is a scale of chords to every quarter point of the compass.



To construct the transverse: with any point A as a centre, and with a chord of 60 degrees as a radius, describe the circle NS, and draw the meridian NS. Take from the scale the chord of $2\frac{1}{2}$ points, and apply it from S to 1, to the right of S, since the first course is southeasterly, and draw A1. Lay off AH, equal to 16, from the scale of equal parts. Lay off the chord of 6 points from S to 2, and draw A2; then through H draw HC parallel to A2, and from the scale of equal parts lay off HC equal to 23, and in like manner continue the plot till the last point F is reached; then draw AF. Take AF in the dividers and apply it to the scale of equal parts, it will indicate the length of the single course

required, and the measure of the angle dAS will indicate the angle which the course makes with the meridian.

All matters relating to the navigation of a ship are entered in a book called a log book, and the record thus made, is called the ship's journal. The principal columns in the log book are, for the hour of the day, the course, the rate of sailing, leeway, and winds. There are also columns for entering general remarks, the results of astronomical observations, for notes on the weather, and notes relating to the most important points of duty attended to on ship-board. To this is daily appended the latitude and longitude of the ship at noon, both as determined in the manner already explained, and by astronomical observation. The place of the ship determined by the principles explained in this article, is called the place by *dead reckoning*.

To the approximate methods of determining a ship's position already given, it is necessary to add frequent checks by astronomical observations. The principal objects to be attained by astronomical observations are, to ascertain the latitude, the longitude, and the variation of the needle, for correcting the dead reckoning. For a full explanation of these principles the reader is referred to works on practical astronomy.

NEE'DLE, MAGNETIC. A bar of steel in the shape of a needle, magnetized, and freely suspended upon a pivot, so that it may yield freely to the directive force of the earth's magnetism, by virtue of which it takes the direction of the magnetic meridian. It forms the principal part of the Surveyor's and Mariner's Compass. In these instruments, the needle has sometimes a hard stone set in its under face, into which a hole is drilled, to receive the pivot of hardened steel upon which the needle turns. The object of this arrangement is to diminish, as much as possible, the friction between the needle and pivot. The needle should be well balanced. See *Compass*.

NEG'A-TĬVE SIGN. [L. *negativus*, from *nego*, to deny]. The algebraic sign, —, also called *minus*.

Negative Quantity. Any quantity preceded by the sign —, is called a *negative quantity*.

Negative Result. Whenever the result of any analytical operation appears, preceded by the sign —, it is called a *negative result*.

The negative sign, regarded as an algebraic symbol, may be considered in two different points of view.

First. It may be regarded as a *symbol of operation*. In this sense, when written between two quantities, it indicates that the one on the right is to be subtracted from the one on the left. Here, we understand the term subtraction in its most general sense, that is, the operation of finding such a quantity as being added to, or aggregated with, a second quantity, will produce the first. Thus, in the expression, $7 - 9$, the symbol — is one of pure operation, and the subtraction indicated gives the result -2, which being added to, or aggregated with 9, produces 7. In this point of view, the negative sign presents no difficulties. It has been said that 9 cannot be subtracted from 7, and therefore, the result -2 is absurd. This is true, if the term subtraction be regarded in the limited point of view, which the original derivation of the term would indicate. But the algebraic language is infinitely more general and comprehensive than our ordinary language, and the absurdity consists in attempting to use terms of ordinary language, as synonymous with the more comprehensive ones of the algebraic language. Subtraction, as used in Arithmetic, implies the operation of taking a less quantity from a greater; in Algebra, there is no such limitation in regard to the relative values of the quantities, the idea being simply to find a third quantity, which united with a second, by aggregation, or algebraic addition, the result shall be the first quantity. Of course, results of operations conducted under these different views, though known by the same names, will present points of difference, but, as in the present case, none that are not perfectly consistent with the arbitrary definitions of the same terms in the two different cases.

Secondly. We may regard the negative sign as a symbol of interpretation. In this case, a result affected with the negative sign is to be interpreted in a sense exactly contrary to what it would have been interpreted, had it been affected with the positive sign. This view embraces the whole subject of the inter-

pretation of negative quantities. From the very nature of the case, the operations indicated by the signs + and − are diametrically opposed to each other, and it is natural to infer that, if we agree to consider a quantity in any particular sense as positive, a quantity considered in an opposite sense should be regarded as negative. Hence, if we regard certain quantities in an analytical investigation, as positive, and then operate upon them by correct rules of analysis, properly applied, and find a negative result, that result indicates that the quantity sought is to be taken in a sense directly contrary to the quantities that were taken as positive. For example, let the element sought be the period of some event. Suppose that we agree to consider time following some fixed epoch, as positive. Now, if by algebraic investigation, the value of the time sought comes out with a negative sign, the only interpretation that can be given to the result is, that the time of the event was before the fixed epoch. In like manner, every negative result may be interpreted from a knowledge of the nature of the assumed elements of the problem in question.

NĪNE. The number 9 possesses some remarkable properties, a few of which it is proposed to develop and explain in the present article. If any number be divided by 9. the remainder is called the excess of 9's. If in performing the division we neglect the quotient, the operation is called *casting out the 9's*.

The number expressed by 1, followed by any number of 0's, may be written

$$1000 \ldots 000 = 999 \ldots 999 + 1 \ \ldots (1).$$

Now, if we multiply both members of equation (1) successively by 2, 3, 4, &c., up to 9, we shall deduce the group of equations,

$$\left.\begin{array}{l} 2000 \ldots 000 = 2 \times 999 \ldots 999 + 2 \\ 3000 \ldots 000 = 3 \times 999 \ldots 999 + 3 \\ 4000 \ldots 000 = 4 \times 999 \ldots 999 + 4 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 8000 \ldots 000 = 8 \times 999 \ldots 999 + 8 \\ 9000 \ldots 000 = 9 \times 999 \ldots 999 + 9 \end{array}\right\} (2).$$

If now we examine the second member of each equation of the group, we see that it is composed of two terms, the first of which is exactly divisible by 9, and the second is therefore the excess of 9's of the first member, except in the last equation, where the excess is 0. Hence we conclude, that the excess of 9's in a number, expressed by a digit followed by any number of 0's, is denoted by that digit.

If now we have any number, as, for example, 3865, it may be written

$$3000 + 800 + 60 + 5\,;$$

or, from (2),

$$3 \times 999 + 3 + 8 \times 99 + 8 + 6 \times 9 + 6 + 5\,;$$

or,

$$9\,(333 + 88 + 6) + 3 + 8 + 6 + 5.$$

Now, since the first term of the result is divisible by 9, it follows that the excess of 9's in the given number is equal to the excess of 9's in the sum of its digits. This principle is general, and serves, as a basis, for the deduction of several practical rules.

1. The excess of 9's, in any number, is equal to the excess of 9's in the aggregate of its several parts. Hence, to prove addition, take the excess of the 9's of the sum of the digits, in each of the added numbers, and take the excess of 9's in the sum of these excesses. Then take the excess of 9's in the sum found; if these results are equal, the work is probably right; if they are not equal, the work is certainly wrong.

EXAMPLE.

	Excess of 9's.
4567	4
8903	2
3245	5
5887	1
22602	3

The excess of 9's in the first number, is 4; in the second, 2; in the third, 5; and in the fourth, 1: the excess of 9's in the sum of these, is 3. But the excess of 9's in the sum of the four numbers, as found, is also 3; hence, the work is probably right.

2. The excess of 9's, in the difference of two numbers is equal to the difference of the excesses of 9's in the two numbers; that is, the excess of 9's, in the sum of the remainder and subtrahend, is equal to the excess of 9's in the minuend. Hence, to prove subtraction by casting out the 9's:

Find the excess of 9's in the subtrahend and in the minuend, and take their sum, from which cast out the nines, and find the excess. Find the excess of 9's in the minuend, and if these results are equal, the work is probably right.

		Excess of 9's.
Minuend	8713864	1
Subtrahend	223568	8
Remainder	8490296	2.

Here, the excess of 9's in the subtrahend is 8; in the remainder it is 2; and in the sum of 8 and 2 it is 1: this is also the excess of 9's in the minuend; the work is therefore presumed to be correct.

3. The excess of 9's, in the product of two numbers, is equal to the excess of 9's in the product of the excess of 9's in the two factors. Hence, to prove multiplication:

Find the excess of 9's in both multiplicand and multiplier; multiply these excesses together, and cast out the 9's from the product, finding the excess. Find the excess of 9's in the product found; if these results are equal, the work is probably correct.

		Excess of 9's
Multiplicand	818327	2
Multiplier	9874	1
Product	8080160798	2.

Here, the excess of 9's in the multiplicand is 2; in the multiplier, 1; and in their product, 2. The excess of 9's, in the product, is also 2; hence, the work is probably right.

4. Since the dividend, in division, is the product of the divisor and quotient, the rule for proving division comes at once from the preceding:

Cast out the 9's of the divisor and quotient, multiply the excess together, and find the excess of 9's in this product. Find the excess of 9's in the dividend; then, if these results are equal, the work is probably right.

	Divisor.	Dividend.	Quotient.
	87603	864203595	9865
Excess of 9's	6	6	1

Here, the excess of 9's, in the divisor, is 6; in the quotient, 1; and in their product, 6. The excess of 9's, in the dividend, is also 6; hence, the work is probably right. These rules are of little use in practice.

From the rule for proving subtraction, it at once follows—that, if two numbers are expressed by the same digits, no matter how taken, their difference will always be divisible by 9.

Thus,

$$\begin{array}{r|r} & 8436832 \\ & 2386348 \\ 9 & 6050484 \\ & 672276 \end{array}$$

Since the excess of 9's, in the minuend, is the same as the excess of 9's in the sum of its digits; and since the excess of 9's, in the subtrahend, is equal to the excess of 9's in the sum of its digits, and since the digits are the same in each case,—it follows, that the excess of 9's in the remainder, must be 0, or the remainder must be divisible by 9. It is plain that any number of digits may be introduced into either number, provided their sum is divisible by 9, and the property enunciated will remain true.

The number 3 possesses properties analogous to those of the number 9.

NOR'MAL. [L. *normalis*; from *norma*, a square, a rule]. A normal line to a plane curve, is a straight line in the plane of the curve, perpendicular to the tangent at the point of contact. If we denote the co-ordinates of the point of contact, and normalcy, by x'' and y'', the equation of the tangent is,

$$y - y'' = \frac{dy''}{dx''}(x - x'').$$

Now, since the normal must pass through the point of contact, and be perpendicular to the tangent, its equation is

$$y - y'' = -\frac{dx''}{dy''}(x - x'').$$

The name, normal, is given to that portion of the normal lying between the point of contact and the point in which the normal cuts the axis of X. The general formula for the length of the normal, with respect to the axis of X, is

$$N = y''\sqrt{1 + \frac{dy''^2}{dx''^2}}.$$

The term, normal, is sometimes used to denote the distance from the point of contact to the centre of the osculatory circle, at the point of contact.

This appears to be the correct and only definite view to be taken of the normal; and when we come to curves of double curvature, the appropriateness of this view will be manifest. In this case, the formula for the normal is,

$$N = \frac{(dx''^2 + dy''^2)^{\frac{3}{2}}}{dx''d^2y''} = \frac{\left[1 + \left(\frac{dy''}{dx''}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y''}{dx''^2}}.$$

In applying either of these formulas, the values of

$$\frac{dy''}{dx''} \quad \text{and} \quad \frac{d^2y''}{dx''^2}$$

are to be found by combining the equation and differential equations of the curve, and substituting in the results x'' and y'' for x and y.

The normal to a curve of double curvature, is a straight line lying in the osculatory plane, and perpendicular to the tangent at the point of contact. The length of the normal, in this case, will be equal to the distance from the point of contact to the centre of the osculatory circle at the point. The formula for the normal in this case is

$$N = \frac{ds''^2}{\sqrt{(d^2x'')^2 + (d^2y'')^2 + (d^2z'')^2 - (d^2s'')^2}}.$$

A normal plane to a curve is a plane through the normal line, perpendicular to the tangent at the point of contact.

A normal line to a surface is a straight line perpendicular to the tangent plane at the point of contact.

The length of the normal is the distance from the point of contact to the centre of the osculatory sphere at the point.

A normal plane to a surface is any plane passed through a normal line to the surface.

NORTH. One of the four cardinal points of the compass. True north is the direction of the true meridian from the equator to the north pole. Magnetic north is the direction of the magnetic meridian towards the north magnetic pole.

NORTH'ING. In Surveying, the distance between two east and west lines, one through each extremity of the course. See *Course*, *Latitude*, and *Navigation*.

NO-TĀ'TION. [L. *notatio*, from, *noto*, to mark]. The conventional method of representing mathematical quantities and operations by means of symbols.

A complete analysis of this method embraces the entire science of *mathematical language*, including not only an account of the symbols employed, but also the methods of combining them so as to express, in the simplest manner, every mathematical operation.

A correct system of notation is of the utmost importance in every branch of science: it facilitates the acquirement of truths already established, and serves to impress them more deeply upon the memory, and is a powerful instrument in the development and discovery of new principles. In no branch of science is a perfect system more necessary than in that of mathematics, and in no branch has there been a greater diversity of systems proposed by different writers. The present state of the mathematical language is due to the labors of many men, living in different ages, speaking different languages, and of different habits of thought; from these elements a language has sprung up, defective in many respects, but nevertheless sufficiently copious for most of the purposes of analysis and investigation.

It is not our purpose in this article to give an account of the origin and progress of this language, or to attempt any detailed account of its many mutations and dialects; but we shall simply endeavor to explain the meaning and use of those symbols which have stood the test of time, and which have been adopted by the best mathematical writers. To arrive at this result, we shall endeavor to analyze the notation of each branch, separately, as far as possible, without repetition.

I. Arithmetical Notation.

The principal part of arithmetical notation consists in representing numbers by means of characters.

Only two methods of expressing numbers are at present in use—the Roman and the Arabic.

1. *Roman Method.*

1. In the Roman method, seven characters are employed, called numeral letters. The letters, separately, stand for the following numbers, viz.: I for *one*. V for *five*, X for *ten*, L for *fifty*, C for *one hundred*, D for *five hundred*, and M for *one thousand*. By combining these characters in accordance with the following principles, every number may be expressed:

1. When a letter stands alone, it represents the number above given; thus, I stands for *one*.

2. When a letter is repeated, the combination stands for the product of the number denoted by the letter by the number of times which it is taken; thus, III stands for *three*, XX for *twenty*, &c.

3. When a letter precedes another, taken in the above order, the combination stands for the number denoted by the greater diminished by that denoted by the less; thus, IV stands for *four*, IX for *nine*, &c.

4. When a letter, taken in the above order, follows another, the combination stands for the sum of the numbers denoted by the letters taken separately; thus, VI stands for *six*, LV for *fifty-five*, CIV for *one hundred and four*.

In accordance with these principles, we have the following table, exhibiting some of the most important combinations:

TABLE OF ROMAN NUMERALS.

I	1	one,
II	2	two,
III	3	three,
IV	4	four,
V	5	five,
VI	6	six,
VII	7	seven,
VIII	8	eight,
IX	9	nine,
X	10	ten,
XI	11	eleven,
XII	12	twelve,
...		
XL	40	forty,
L	50	fifty,
LV	55	fifty-five,
LX	60	sixty,
XC	90	ninety,
C	100	one hundred,
CX	110	one hundred and ten.
CD	400	four hundred,
D	500	five hundred,
DC	600	six hundred,
M	1000	one thousand,
MD	1500	fifteen hundred,
MDC	1600	sixteen hundred,
MDCCCLIV	1854	one thousand eight hundred and fifty-four.

This method of notation is now only used for dates, headings of chapters, &c.

2. *Arabic Method.*

In the Arabic method, numbers are represented by the symbols,

0, 1, 2, 3, 4, 5, 6, 7, 8, 9,

and by their combinations, according to certain conventional rules. These characters are called *figures*, and, taken in their order, stand for naught, one, two, three, four, five, six, seven, eight, nine; the value of the unit depends upon the place which the figure occupies in the scale adopted. In the ordinary method of numbering by tens, numbers are arranged in groups, so that ten units of any group or *order* make one unit of the next higher group or order.

The scale of place indicates the value of the unit, and the digit standing in any place indicates the number of units of that order which are taken.

Ascending Scale. *Descending Scale.*

&c.	&c. Billions.	hundreds of millions.	tens of millions.	millions.	hundreds of thousands.	tens of thousands.	thousands.	hundreds.	tens.	units.	Decimal point.	Tenths.	hundredths.	thousandths.	ten thousandths.	hundred thousandths.	millionths.	ten millionths.	hundred millionths.	billionths.	&c. &c.
..	0	0	0	0	0	0	0	0	0	0	.	0	0	0	0	0	0	0	0	0	.

In the above blank scale, if a digit be written in the place of any 0, its unit will correspond to that order, and it will stand for as many units of the kind named as the digit indicates. Units of the first order always occupy the first place on the left of the decimal point; those of the second order, the second place; those of the third order, the third place, and so on.

The descending scale is called the scale of decimals, and the relation of the successive orders of units is the same as in the ascending scale. The orders of decimal units are reckoned from the decimal point to the right, and any digit written in the place of a 0 in this scale, indicates as many decimal units of the kind named as the digit stands for.

In general, in either branch of the scale of tens, a unit of any order is equal to ten units of the next order on the right. To write any number in this scale, write a digit in each order expressing the number of units of that order in the given number; if there are no units of a particular order, the 0 of the scale is allowed to occupy the place, to indicate that fact. Let it be required to write the number three thousand four hundred and five, and seventy-five thousandths; it will be expressed thus:

thousands.	hundreds.	tens.	units.		tenths	hundredths.	thousandths.
3	4	0	5	.	0	7	5

In this way any number in the scale of tens may be written. It is not necessary to write the value of a unit in each place, but it is done in the above example to indicate more fully the method of writing numbers, and to show the law of the scale.

Numbers may be expressed in any other scale in which the values of a unit in the suc-

cessive places are in geometrical progression, but such scales are not in common use

There is a kind of scale often used. called the varying scale. In these cases, the value of a unit of each order is connected with that of the succeeding order by some conventional law.

Thus the scale of long measure is written as shown below:

leagues.	miles.	furlongs.	rods.	fathoms.	yards.	feet.	inches.
0	0	0	0	0	0	0	0

In it, 12 units of the first order are equal to 1 of the second; 3 of the second to 1 of the third; 2 of the third to 1 of the fourth; $2\frac{3}{4}$ of the fourth to 1 of the fifth; 40 of the 5th to 1 of the sixth; 8 of the sixth to 1 of the seventh; and 3 of the seventh to 1 of the eighth.

There is a great variety of these varying scales used in arithmetic; in each case the law of the scale is given in a small table, constructed for the purpose.

Fractions.

To represent a vulgar fraction, it has been agreed to write one number over another, with a horizontal line between them. The upper number is called the numerator; the lower one, the denominator. The denominator indicates the number of equal parts into which 1 is divided to produce the unit of the fraction, and the numerator indicates the number of these units which are taken to constitute the fraction. The numerator is written as any other number in the scale of tens. The ordinary algebraic symbols of operation are employed to indicate the operations of addition, subtraction, multiplication, division, raising to powers, extracting roots, equality, proportion, and so on. See *Algebraic Notation.*

The *sign of cancellation* is simply an oblique stroke drawn across the factors canceled. These, with the conventional abbreviations for the names of things, such as £, *s*, *d*., °, ′, ″, &c., make up the system of arithmetical notation.

II. Algebraic Notation.

There are four kinds of symbols used in algebra: 1st, the symbols of quantity; 2d, those of operation; 3d, those of relation; and 4th, those of abbreviation.

1st. *Symbols of Quantity.* Quantities are generally represented by letters. *Known* quantities are represented by the leading letters of the alphabet, or by the final letters with one or more accents, thus: x', x''', y'', &c. *Unknown* quantities are represented by the final letters of the alphabet, as x, y, z, &c. Besides the letters of the English alphabet, those of the Greek alphabet are often made use of. Certain letters have come to represent certain quantities. Thus, π generally stands for the ratio of the diameter to the circumference of a circle, or the number 3.1416; e denotes the base of the Naperian system of logarithms, or the number 2.718281828; M denotes the modulus of any system of logarithms. In the common system, it is 0.434294482. In choosing letters to denote particular quantities, attention should be directed to such a selection as will suggest the nature of the quantity, as the initial letter, or something of the kind; thus, the letters r, R, ρ, &c., may be taken to denote radii of circles; h, H, &c., to denote altitudes of triangles, pyramids, &c.; B, b, to denote the base of a magnitude; L, l, λ, to denote latitude, &c. The leading letters of the Greek alphabet are generally used to denote known angles; the final ones to denote unknown angles. When several quantities of the same kind are involved in an investigation, they may be designated by the same letter, differently accented, as, a, a', a'', a_1, a_2, &c. The symbol ∞ denotes an infinitely great quantity.

2. *Symbols of Operation.* The sign $+$, *plus*, when written between two quantities, signifies that the second is to be added to the first; as, $a + b$. The sign $-$, *minus*, when placed between two quantities, denotes that the one on the right is to be subtracted from the one on the left; as, $a - b$. The sign $\times$, when placed between two quantities, denotes that the one on the left is to be multiplied by the one on the right; as, $a \times b$. Multiplication may be indicated by placing a point between the factors when they are both expressed by letters; as, $a \,.\, b$: This method of notation is not applicable when the factors are numbers, because in that case the indicated product would be confounded with a mixed decimal fraction; thus, 5 . 6, instead

of being read, product of 5 by 6, would be read 5 and 6 tenths. There are cases, however, where the sign is used as a sign of multiplication between numerical factors, as in series where the factors follow a law which it is desirable to keep before the eye: thus, the general term of the binomial formula is,

$$\frac{m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots \cdots \cdots n.} a^n x^{m-n}$$

The sign $\div$, placed between two quantities, indicates that the one on the left is to be divided by the one on the right; as, $a \div b$. Division may also be indicated by writing the quantities in place of the points; as, $\frac{a}{b}$. It may also be indicated thus: $\underline{a\,|}\,b \ldots$

The sign $\sim$ denotes the difference between two quantities, without implying which is to be subtracted from the other; as, $a \sim b$.

A number written before a letter, or combination of letters, is called a *co-efficient*, and indicates the number of times that the quantity before which it is placed is to be taken additively; as, $6\,a$. A number written to the right, and a little above a quantity, is called an *exponent*, and denotes the number of times that the quantity is taken as a factor; as, a^5. The sign $\sqrt{}$ is called the radical sign, and when placed over a quantity, indicates that its root is to be taken; as, $\sqrt{a}$: the degree of the root is indicated by a number written over the sign, which is called the index of the *root* or *radical*; thus, $\sqrt[3]{a}$, $\sqrt[4]{a}$, &c. The sign $\sqrt{}$, indicates the square root.

Radical quantities and reciprocals are also indicated by means of fractional and negative exponents, in accordance with the following principles: When a quantity is affected with a fractional exponent, the numerator indicates the degree of the power to which the quantity is to be raised, and the denominator indicates the degree of the root of that result, which is to be extracted. When a quantity is affected with a negative exponent, whether entire or fractional, it indicates the reciprocal of the same quantity with the sign of the exponent changed. From these principles we have the following equivalent expressions:

$a^{\frac{1}{n}}$	equivalent to	$\sqrt[n]{a}$
$a^{\frac{m}{n}}$	" "	$\sqrt[n]{a^m}$
a^{-n}	equivalent to	$\frac{1}{a^n}$
$a^{-\frac{1}{n}}$	" "	$\frac{1}{a^{\frac{1}{n}}}$ or $\frac{1}{\sqrt[n]{a}}$
$a^{-\frac{m}{n}}$	" "	$\frac{1}{a^{\frac{m}{n}}}$ $\frac{1}{\sqrt[n]{a^m}}$

A vinculum ——, bar $|$, brackets [ ], { }, parenthesis ( ), &c., indicate that the quantities enclosed by them are to be regarded together; as, $(a+b)\,x$, $\left.\begin{matrix}a\\+b\end{matrix}\right|x$, &c.

The symbol Σ denotes that the algebraic sum of several quantities of the same nature as that to which the symbol is prefixed, is to be taken; thus,

$$\Sigma \frac{q}{n(n+p)}, = \frac{1}{p}\left[\Sigma\left(\frac{q}{n}\right) - \Sigma\left(\frac{q}{n+p}\right)\right]$$

is a formula, in which p being constant and q and n arbitrary, signifies that of the algebraic sum of any number of terms deduced by attributing values to q and n, is equal to $\frac{1}{p}$ multiplied by the difference of the algebraic sums of the terms, which are deduced by attributing the same values to q and n in the expressions $\frac{q}{n}$ and $\frac{q}{n+p}$. The expression

$$\left(\frac{z^n - y^n}{z - y}\right)_{z=y}$$

denotes the values which the quantity within the parenthesis reduces to, when z is made equal to y.

3. *Symbols of relation.*

The letters f, F, ϕ, written before any quantity, or quantities, separated by commas, as

$$F(x),\ f(x, y),\ \phi(x, y, z), \&c.,$$

denote quantities depending upon the quantity or quantities within the parenthesis, without designating the nature of the relation.

The sign of equality, $=$, between two quantities, denotes that those quantities are equal to each other.

The sign of inequality, $>$, placed between two quantities, denotes that the one placed at the opening of the sign is greater than the one placed at the vertex of the sign; thus, $a > b$, a greater than b.

The signs of proportion

$$:, \quad ::, \quad =,$$

when placed between quantities, taken two and two, show that the quantities are in proportion; thus,

$$a : b :: c : d,$$

is read, *a is to b. as c is to d.* The first and third are signs of ratio, and the second the sign of equality, so that the above proportion might be written

$$\frac{b}{a} = \frac{d}{c}.$$

4. *Symbols of abbreviation.*

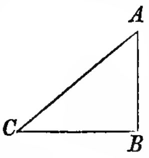
The sign $\therefore$ stands for *hence*, or consequently.

The sign $\because$ stands for *because.*

The symbol $y = f(x)$ is a general sign, which indicates that there is a general relation between y and x; that is, that they are so connected that x cannot change without y changing at the same time. The symbol $F(x, y, z) = 0$, implies that there is a general relation between x, y, and z, without specifying the nature of the relation.

III. Geometrical Notation.

The notation of Geometry borrows most of its elements from the algebraic notation just explained. Magnitudes are represented pictorially. A line is designated by the two letters standing at its two extremities. Angles are denoted by the three letters at the two extremities of the sides of the angles, the letter at the vertex being in the middle: thus, ACB, or sometimes, when there can be no doubt as to the meaning, the letter at the vertex alone is used.



The symbol $\angle$, is sometimes used as an abbreviation, or pictorial symbol for *angle.*

IV. Trigonometrical Notation.

In Trigonometry, besides the notation of Algebra and Geometry, the symbols *sin, cos, tan, co-tan, sec, co-sec, ver-sin*, and *co-ver-sin*, are used as abbreviations for the words, sine, co-sine, tangent, co-tangent, secant, co-secant, versed-sine, and co-versed-sine. When the arc varies, these several quantities vary to correspond with it, and we call them *direct* trigonometric functions. When the arc is supposed to depend for its value, upon any of the trigonometric lines, the function is called an inverse trigonometrical function. The following symbols are used to denote this kind of relation: $\sin^{-1}y$, $\cos^{-1}y$, $\tan^{-1}y$, $\cot^{-1}y$, $\sec^{-1}y$, co-$\sec^{-1}y$, ver-$\sin^{-1}y$, co-ver-$\sin^{-1}y$, which stand respectively for the arc upon sine, co-sine, tangent, co-tangent, secant, co-secant, versed-sine and co-versed-sine, is y. This principle of notation has been extended to all inverse functions; thus,

$$\log^{-1}y,\ d^{-1}(xdx), \text{ \&c.};$$

which stand respectively for the quantity whose logarithm is y, the quantity whose differential is xdx, &c.

V. Notation of Analysis and Calculus.

Rectilineal co-ordinates of points in a plane, are represented by x and y, x denoting the abscissa, and y the ordinate. Rectilineal co-ordinates of points in space, are denoted by x, y and z, z denoting the vertical ordinate, and x and y the horizontal co-ordinates. Polar co-ordinates of points in a plane are denoted by r and v, r representing the radius-vector, and v the angle which it makes with the initial line. Polar co-ordinates of points in space, are represented by r, v and u, r denoting the radius-vector, v the angle between it and the initial plane, and u the angle which the projection on the initial plane makes with the initial line in that plane.

Lines and surfaces are given by equations which express the relations between the co-ordinates, either rectilinear or polar, of every point of the lines or surfaces.

The differential of a function, or an independent variable, is denoted by the letter d; thus,

$$d(y^2) = 2ydy.$$

Differentials of functions of the second, third, &c., orders, are designated thus,

$$d^2u,\ d^3y,\ d^4z, \text{ \&c.},$$

in which u, y and z, are symbols standing for the functions. If we suppose x to be the independent variable, the second, third, fourth, &c., differential co-efficients are thus expressed,

$$\frac{d^2u}{dx^2},\ \frac{d^3y}{dx^3},\ \frac{d^4z}{dx^4}, \text{ \&c.}$$

A partial differential co-efficient of a func-

tion of any order taken with respect to one variable, is expressed thus,

$$\frac{d^m u}{dx^m} dx^m ;$$

a partial differential co-efficient taken with respect to several variables is thus expressed;

$$\frac{d^m u}{dx^p \, dy^q \, dz^r \ldots} dx^p \, dy^q \, dz^r \ldots$$

If we suppose the form of the function to vary, the symbol employed to denote the variation is δ, thus: δu, δy, δx, &c. If both the form of the function and the independent variables of the function vary together, the resulting variation is denoted by the symbol D: thus, $Df(x, y)$ . .

The differential is the difference between two consecutive states of the quantity differentiated. If it is desired to represent the difference between two states of a function which are not consecutive, the symbol Δ is employed: thus,

$\Delta(fx)$ is the same as $f(x+h) - fx$;

h being the increment of the independent variable x; h itself may be denoted by the symbol Δx.

The ratio of an increment of a variable to the corresponding increment of the function may be written $\frac{\Delta u}{\Delta x}$. The limit of this ratio may be designated symbolically by the sign $L\left(\frac{\Delta u}{\Delta x}\right)$ and is equal to $\frac{du}{dx}$. The symbol L is generally used by itself, to denote this ratio.

Successive finite differences are represented by the symbols,

$$\Delta u, \ \Delta^2 u, \ \Delta^3 u, \ \Delta^4 u, \ \text{\&c.}$$

In the same manner, all other successive operations are denoted. Thus, if D stands for an operation to be performed upon any function, and if the same operation is to be performed, in succession, upon the result of each preceding operation, these successive operations will be indicated by the symbols,

Du, D^2u, D^3u, and so on.

These symbols, being set apart to denote operations, ought not to be taken as the representatives of quantities, for fear of confusion.

The symbol Σ is used, as in Algebra, to denote an algebraical sum, but its use is principally restricted, in Calculus, to the denotation of the sum of the finite differences of a function.

The symbol $\int$ denotes an integration to be performed, thus,

$\int dx$ is the same as $d^{-1}(dx)$.

When several successive integrations are to be performed, the symbol $\int^m$ is used, in which m denotes the number of times that the operation is to be successively performed. The symbol $\int_a^b$ is used to denote a definite integral taken between the limits a and b. When a quantity is to be integrated, successively, with respect to different variables, the symbol employed is

$$\int_a^b \int_c^d u dx dy ;$$

this symbol implies that the quantity $udxdy$ is to be integrated, first with respect to y, between the limits c and d, and that result with respect to x, between the limits a and b.

The symbol, $\Gamma(x+1)$, stands for the integral $\int e^{-v} v^x dv$.

$\Gamma(x) = \int e^{-v} v^{x-1} dv$, and $\Gamma(x+1) = x\Gamma(x)$ is a functional equation.

The foregoing embrace nearly all the symbols employed by American writers. Others are sometimes adopted, for explanations of which, the reader is referred to the works where they occur.

NOTHING. A term sometimes employed as synonymous with *zero*, (0), but when so employed the idea conveyed is generally erroneous. Zero stands for a quantity less than any assignable quantity, and sometimes for *no quantity*. Thus, in the fraction $\frac{a}{b}$, if, while a remains the same, b continually increases, the value of the fraction continually diminishes. When b becomes exceedingly great, with reference to a, the fraction becomes exceedingly small, and when b becomes greater than any assignable quantity, the fraction becomes less than any assignable quantity, and is then called *zero*. This is the true mathematical idea of zero, in almost every case in which it is used. Again, if a be subtracted from a, the result is also denoted by the symbol 0, and is called zero. The remainder, in this case, is synonymous with *nothing*, but in the former case, zero and

nothing are far from being synonymous terms. *Nothing*, is fast falling into disuse as a mathematical term, and the proper term, zero, is as rapidly acquiring its true place in the mathematical vocabulary. See *Zero*.

NUM'BER. Abstractly considered, the measure of the relation between quantities or things of the same kind. We can form no conception of the absolute magnitude of any quantity, and can only acquire a relative conception of it, by comparing it with some other quantity of the same kind, assumed as a standard of comparison. The comparison is made by seeking how many times the standard is contained in the quantity measured. The result of this comparison is a *number*. The quantity compared with the standard may be equal to it, or it may not: in the first case, the resulting relation is one of *equality*, and the measure of that relation is called *one* (1). To this result, all other like comparisons are referred by a natural process of the mind, and hence it is, that the unit 1 becomes the base of all numbers.

If the relation is one of inequality, we have different measures, consequently, different results; and since the relations may be infinitely various, the results must be equally so: these several results are called *numbers*.

When the relation is that of inequality, there may be two cases; *first*, when the quantity measured is made up of parts, each equal to the standard, in which case the numbers are called, *whole*, or *integral* numbers; *secondly*, when it is not thus made up, in which case, the numbers are *fractional*

If the quantity measured is double the standard, the resulting number is called *two* (2). Two is then equal to *one*, and *one* more. If the quantity considered is triple the standard, the resulting number is called *three* (3). Three is therefore *two* and *one* more. If the quantity is quadruple the standard, the resulting number is called *four* (4). Four is therefore equal to *three* and *one* more; and so on through the series of natural numbers, each of which is equal to the preceding one, and *one* more. Here. then, we acquire the idea of *collection*, and by analyzing the process of arriving at numbers, we see that all whole numbers are collections of *ones*, and we may show that all fractional numbers are collections of equal parts of *one*. Numbers, thus considered, are purely *abstract*, and have no reference to the nature of the quantities or things compared. But it sometimes happens that we name the unit of comparison, as when we speak of *seven feet*, *nine pounds*, *&c.*, in which cases, from analogy and common custom, we come to regard these quantities as numbers, and we call them *concrete* or *denominate numbers*. In these cases, numbers are collections of units of the kind named; thus, *seven feet* is a collection of feet, *seven* in number; the unit is a *foot*, and the number of times that it is taken is *seven*.

We have, therefore, the ordinary definition of numbers, viz.: "a collection of things of the same kind." *One of these things* forms the base of the number, and is called a *unit*.

It has been a question, whether, in accordance with this definition, the unit 1 is a number. Since 1 cannot be regarded as a *collection* in the ordinary sense of that term, it has been urged that it ought not to be considered as a number. If, however, we go back to the abstract idea of number, viz.: that it is "the measure of the relation between quantities or things of the same kind," we see that it is not only a number, but is also the base of *all* numbers. It is evident, therefore, that the term collection, as used in the common definition of number, is technical, and by convention, is made to cover the case of a single thing of the kind collected. We therefore regard 1 as a number, falling under the definition last given.

When the unit of a number is abstract, the number itself is *abstract*, when it is concrete or denominate, the number is *concrete* or *denominate*. Thus, *seven pounds*, *seven feet*, *seven hours*, are all concrete numbers, in which the numerical idea is the same, but which differ from each other, in the fact, that the kind of quantity collected is different in each case.

So far as arithmetical operations are concerned, there is no difference between abstract and denominate numbers, provided we reject the name of the denominate unit. The only difference in the final result is one of interpretation. (See *Interpretation*). If we multiply 7 feet by 5 feet, we neglect the name of the unit and multiply 7 by 5, but in interpreting the product, we take into account the nature of the concrete factors, and pronounce

the result to be 35 *square feet*; and in like manner for all other similar cases.

We have seen that the unit of comparison is arbitrary, so that we may, if we please, refer the same number to different units, in succession. In fact, a good share of the science of arithmetic consists in transforming numbers from one unit to another. If we have the number 300 we may regard it in several points of view. 1st. We may regard it as a collection of hundreds, and write it 3 *hundreds*; here, the base or unit is 100, and it is taken three times. 2d. We may regard it as a collection of tens, and write it 30 *tens*; here, the unit is 10, and it is taken thirty times. 3d. We may regard it as a collection of ones, and write it 300; in this case, the unit is not named, it being understood to be 1, or the primary base of all numbers.

If we analyze the denominate number

cwt.	*qr.*	*lb.*	*oz.*	*dr.*
13	2	20	12	4,

we see that 1 cwt. is the unit or base of 13 cwt.; 1 qr. the base of 2 qrs.; 1 lb. the base of 20 lbs.; 1 oz. the base of 12 ozs, and 1 dr. the base of 4 drs Here is a complex denominate number, but all the bases may, by transformation, be referred to 1 dram as a base; and since the same may be done in all cases, we see that all complex concrete numbers can be referred to the primary base 1.

We come next to consider *fractional numbers*, or *fractions*, as they are most commonly called, and we shall endeavor to show their connection with the primary base 1.

A fractional number may be defined to be "a collection of equal parts of 1." The word collection is used here, in its technical sense, to include the case of 1 of the equal parts. If we suppose the number 1 to be divided into any number of equal parts, as b, one of these parts is called a base, or a *fractional unit*, and may be written $\frac{1}{b}$, which is a fraction. If a certain number of these units, as a, be taken, the collection is written $\frac{a}{b}$. Here we see that the fraction $\frac{a}{b}$ differs in no respect from a whole number, except in the value of the unit. If we have the fraction $\frac{7}{8}$, we see that the unit or base is $\frac{1}{8}$, and the collection is made by taking 7 of these units. The unit 1 is still the primary base of the fraction, whilst $\frac{1}{8}$ is the *fractional unit*, just as we regarded 300 as 3 hundreds, or 30 tens, whilst the primary base 1 remained unaltered.

From the preceding discussion we see that the measure of the relation of equality is called 1; that this unit 1 is the base of all numbers, *integral*, *fractional*, and *concrete*, and that numbers of all kinds are merely collections of *ones* or *equal parts of one*, called *units*. It may be added that every arithmetical rule, and every analytical process has a direct reference to the unit 1, and that a great share of the science of mathematics consists in transforming numbers from one unit to another.

The symbols, by means of which numbers are most usually denoted, are the Arabic characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 Taken separately they are called *figures*, but when grouped according to the rules of notation, they form the *names of numbers*. These names are often used for the numbers themselves, so that, instead of regarding the combination 29, as the symbolical name of the number *twenty-nine*, it is regarded as the number itself. For all practical purposes this conventional term, when well understood, is amply sufficient.

NUMBERS, APPELLATIONS OF. Various names have been given to classes of whole numbers which are expressions of some property or properties common to the whole class. The following are some of the appellations employed:

1. The series of whole numbers, 1, 2, 3, 4, 5, &c., is called the series of *natural numbers*; it is subdivided into the series of *odd numbers*, 1, 3, 5, 7, &c, and the series of *even numbers*, 2, 4, 6, 8, &c. The odd numbers are again subdivided into the *oddly odd numbers*, 3, 7, 11, 15, &c., and the *evenly odd numbers*, 1, 5, 9, 13, &c. The even numbers are subdivided into the *oddly even numbers*, 2, 6, 10, 14, &c., and the *evenly even numbers*, 4, 8, 12, 16, &c.

2. The series of *square numbers* is made up of the squares of the natural numbers; it is 1, 4, 9, 16, 25, &c. The series of *cube numbers* constituted in like manner, 1, 8, 27. 64, 125, &c. The series of *fourth powers*

1, 16, 81, 256, 625, &c., and so on to any extent.

3. *Prime numbers* are those that cannot be exactly divided by any other number except 1; thus, 3, 5, 7, 11, 13, &c. *Composite numbers* are those which may be resolved into factors differing from 1.

4. *Figurate numbers* are deduced from a formula which will be understood from an examination of the subjoined table:

	1	2	3	4	5	6	7	8 &c.
I	1	3	6	10	15	21	28	36 &c.
II	1	4	10	20	35	56	84	120 &c.
III	1	5	15	35	70	126	210	330 &c.
IV	1	6	21	56	126	252	462	792 &c.
V	1	7	28	84	210	462	924	1716 &c.
&c.	&c.		&c.		&c.		&c.	&c.

The figurate series are marked I. II. III. &c., and are deduced as follows. The numbers in the series marked I. are deduced by taking the sum of all the numbers in the preceding row up to the rank of the required number. Thus, the number

$$10 = 1 + 2 + 3 + 4,$$
$$15 = 1 + 2 + 3 + 4 + 5,$$

and so on. The second figurate series is deduced from the first by the same law; thus,

$$35 = 1 + 3 + 6 + 10 + 15.$$

The third series is deduced from the second by the same law; thus,

$$70 = 1 + 4 + 10 + 20 + 35,$$

and so on for all the subsequent series. The general formulas are as follows:

1. For the n^{th} term of the first series,

$$\frac{n(n+1)}{1 \cdot 2}.$$

2. For the n^{th} term of the second series,

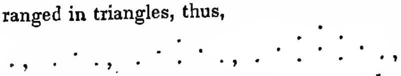
$$\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

3. For the n^{th} term of the third series,

$$\frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}.$$

and so on.

5. Polygonal Numbers, so called on account of their relation to polygons. The numbers of balls which can be regularly arranged in triangles, thus,



are called *triangular numbers;* they are denoted by 1, 3, 6, 10, 15, &c., the first series of figurate numbers.

The numbers of balls which can be regularly arranged in squares, thus,

are called *square numbers*, or *quadrangular numbers;* they are

1, 4, 9, 16, 25, &c.

In like manner we have *pentagonal numbers*,

1, 5, 12, 22, 35, &c.,

hexagonal numbers,

1, 6, 15, 28, 45, &c.,

and so on.

Pyramidal Numbers, are the numbers of balls which can be arranged in pyramids; they are formed by summing polygonal numbers. The *triangular pyramidal numbers* are 1, 4, 10, 20, &c., the second series of figurate numbers. The pentangular pyramidal numbers are 1, 6, 18. 40, 75, 125, &c.

6. Numbers Redundant, *perfect* and *defective*. A *redundant* number is one in which the sum of all its divisors, except itself, exceeds the number; 12 is a redundant number, because

$$1 + 2 + 3 + 4 + 6 > 12.$$

A *perfect* number is one in which this sum equals the number; 6 is a perfect number, because

$$1 + 2 + 3 = 6.$$

A *defective* number is one in which this sum is less than the number; 10 is defective, because

$$1 + 2 + 5 < 10.$$

Whenever $(2^n - 1)$ is a prime number, then $2^{n-1}(2^n - 1)$ is perfect; thus, $2^7 - 1$ or 127 is prime; hence $2^6(2^7 - 1)$, that is, 8128, is a perfect number.

Amicable Numbers are those, each of which is equal to the sum of all the divisors of the other; such are 284 and 220; 17296 and 18416; 9363583 and 9437056. Other appellations have been given to numbers, which are alluded to in their proper places, but these are some of the most important.

Numbers of Bernouilli. This name is given to certain numbers. first made use of by James Bernouilli. They are, in fact, the co-efficients of the powers of x in the differ-

ent terms of the series, obtained by developing the expression $\frac{1}{e^x - 1}$. The form of the development is

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \frac{B'}{1 \cdot 2} x - \frac{B''' x^3}{1 \cdot 2 \cdot 3}.$$

The numbers B', B''' &c., are Bernouilli's numbers. The following table contains some of these numbers:

No.	Numerator.	Denominator.
1	1	6
3	1	30
5	1	42
7	1	30
9	5	66
11	691	2730
13	7	6
15	3617	510
17	43867	798
19	174611	330
21	854513	138
23	236364091	2730
25	8553103	2
27	23749461029	870
29	8615841276005	14322

The first column gives the exponents of x, the second the numerators, and the third the denominators.

Thus the co-efficients of $\frac{x^{15}}{1 \cdot 2 \cdot 3 \cdots \cdot 15}$ is $\frac{3617}{510}$. These numbers are used in the higher branches of mathematics in developing series.

NŪ′MER-ALS. The characters, by means of which numbers are expressed. In the Arabic system, they are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. In the Roman system, they are I, V, X, L, C, D and M.

NU-MER-Ā′TION. [L. *numeratio*, a counting]. The art of reading numbers, when expressed by means of numerals. The term is almost exclusively applied to the art of reading numbers, written in the scale of tens, by the Arabic method.

For the convenience of reading numbers, they are separated into periods of three figures each. The units of the first order are read simply *units:* those of the second order, *tens;* those of the third order, *hundreds;* and so on, according to the following

NUMERATION TABLE.

Period of Septillio's.			Period of Sextillio's.			Period of Quintill's.			Period of Quadrill's.			Period of Trillions.			Period of Billions.			Period of Millions.			Period of Thousa'ds.			Period of Units.			&c.
Hundreds of Septillions	Tens of Septillions	Septillions	Hundreds of Sextillions	Tens of Sextillions	Sextillions	Hundreds of Quintillions	Tens of Quintillions	Quintillions	Hundreds of Quadrillions	Tens of Quadrillions	Quadrillions	Hundreds of Trillions	Tens of Trillions	Trillions	Hundreds of Billions	Tens of Billions	Billions	Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thousands	Tens of Thousands	Thousands	Hundreds	Tens	Units	&c., &c.,
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

The table may be continued to any extent; the next higher periods are octillions, nonillions, decillions, undecillions, duodecillions, &c. The table may be continued to the right, giving the numeration-table for decimal fractions. *(See the Table opposite.)*

Period of Thous'th's			Period of Millionths.			Period of Billionths.			&c.
Tenths	Hundredths	Thousandths	Tens of Thousandths	Hundreds of Thousandths	Millionths	Tens of Millionths	Hundreds of Millionths	Thousands of Millionths	&c., &c.,
. 0	0	0	0	0	0	0	0	0	

NŪ′MER-Ā-TOR. That term of a fraction which indicates the number of fractional units that are taken. It is the term written above the horizontal line. In the fraction $\frac{a}{b}$, a is the numerator. In a decimal fraction, the numerator is the number following the

decimal point, the denominator not being written: thus, in the decimal fraction .764, the numerator is 764, and the denominator, which is understood, is 1000. In general, the denominator is expressed by 1 followed by as many 0's as there are places of figures in the numerator. See *Fraction*.

NU-MER'IC-AL. A term which stands opposed to literal, and implies that the numbers entering a given expression are expressed by figures, and not by letters. A numerical equation is an equation, in which all the quantities, except the unknown or variable quantities, are numbers. Numerical, as opposed to algebraical, is applied to the values of quantities; thus we say, that -5 is numerically greater than -3, although its algebraical value is less.

NUMERICAL VALUE of an expression, in algebra, is the number obtained by attributing numerical values to all the quantities which enter the expression, and performing all the operations indicated. Thus, the numerical value of $a^2b - c^2d$, where $a = 2$, $b = 3$, $c = 1$ and $d = 2$, is 10.

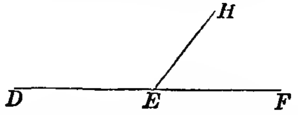
The numerical value of an expression generally varies with the values given to the quantities which enter it, but not always. Thus, $a - b$ is equal to 10, when $a = 12$ and $b = 2$: or, when $a = 16$ and $b = 6$, and so on, for an infinite number of sets of values of a and b.

O, the fifteenth letter of the English alphabet. As a numeral letter, it has been used to denote the number 11. With a dash over it thus, $\overline{\text{O}}$, it denoted 11,000.

OB'JECT-GLĂSS. The object-glass of a telescope is the lens which is directed towards the object viewed. It forms the image of the object, which is then viewed by the eye-glass or eye-lens, regarded as a simple microscope.

OB-LĀTE'. [L. *oblatus*; *ob*, on account of, and *fero*, to bear]. Flattened or depressed. If an ellipse be revolved about its conjugate axis, the volume generated is called an *oblate spheroid*. The earth, on which we dwell, is of the general form of an oblate spheroid. See *Figure of the Earth*.

OB-LĪQUE'. [L. *obliquus*, oblique]. Not direct, deviating from the perpendicular. A right line is oblique with respect to another, when it makes, on one side, an angle with it less than a right angle, and on the other side, an angle greater than a right angle.



The line HE is oblique to the line DF. One plane is oblique to another, when the diedral angles which they form with each other, are unequal. If two planes be passed through the lines DF and HE, respectively perpendicular to their plane, they will be oblique to each other.

An oblique angle is one either greater or less than a right angle; the angles HEF and HED are both *oblique* angles. Oblique-angled triangles are those in which all the angles are oblique. An *oblique circle*, in Spherical Projections, is one whose plane is oblique to the axis of the primitive plane. An oblique plane, in Dialing, is one which is oblique to the horizon. An oblique system of co-ordinates, in Analysis, is a system in which the co-ordinate axes are oblique to each other. Oblique projections are projections made by lines oblique to the plane of projection. An oblique cylinder or cone is one whose axis is oblique to the plane of its base.

OB'LONG. [L. *oblongus*; *ob*, for, and *longus*, long]. A name given to a rectangle whose adjacent sides are unequal. In common language, any figure approximating to to this form, is called an oblong; in fact, any body which is longer than it is wide, is often called an oblong. The prolate spheroid is often called an oblong spheroid.

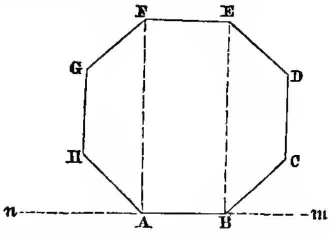
OB-TŪSE'. [L. *obtusus*, from *obtundo*, to beat against]. Blunt, opposed to sharp, or acute. An obtuse angle is an angle greater than a right angle: an obtuse polyhedral angle is one whose measure is greater than the tri-rectangular triangle. It is to be noted, that the measure of any polyhedral angle is the area of that portion of the surface of a sphere having its centre at the vertex of the angle and radius 1, which is intercepted by the faces of the polyhedral angle. An obtuse cone is a right cone, such that the angle

formed by two elements cut from the cone by a plane passed through the axis, is greater than a right angle. An obtuse hyperbola is an hyperbola in which the asymptotes make with each other an obtuse angle, or it is one in which the length of the conjugate axis is greater than that of the transverse axis. An obtuse hyperbola can only be cut from an obtuse cone. To cut an obtuse hyperbola from an obtuse cone, pass a plane through the vertex, cutting out two elements, which make an obtuse angle with each other, then will any parallel plane cut from the conic surface be an obtuse hyperbola. An obtuse ellipsoid is the same as a prolate spheroid. See *Prolate Spheroid.*

OC′TA-GON. [Gr. οκτω, eight, and γωνια, angle]. A polygon of eight angles or sides. A regular octagon is an octagon all of whose sides and angles are respectively equal to each other. The angle at the centre of a regular octagon is 45°, and the angle at the vertex of any angle is 135°. The area of a regular octagon, whose side is 1, is equal to 4.8284271, and for a regular octagon, whose side is equal to a, we have the formula,

$$A = 4.8284271\, a^2 \ldots$$

To construct a regular octagon on a given line as a side.



Let AB be the given line. Erect at A and B the perpendiculars AF and BE. Produce AB, in both directions, to m and n. Bisect the angles EBm and FAn by the lines BC and AH, and make BC and AH each equal to AB. Through C and H draw CD, and HG perpendicular to AB, and each equal to AB. With D and G as centres, and with a radius equal to AB, describe arcs of circles cutting BE and AF in the points E and F: join DE, EF, and FG; then will the polygon thus formed be a regular octagon.

To inscribe a regular octagon in a circle. Draw two diameters at right angles, and bisect both angles which they form with each other by diameters; join the adjacent points, two and two, in which these four diameters cut the circumference by straight lines, and the figure formed will be the required inscribed octagon.

To circumscribe a given circle by a regular octagon. Draw tangents to the circle at the points in which these four diameters cut the circumference, and they will, by their intersections, determine the required octagon.

OC-TAG′ON-AL. Appertaining to an octagon.

OC-TA-HĒ′DRON, OR OCTAEDRON. [Gr. οκτω, eight; and ἑδρα, base]. A polyhedron bounded by eight polygons. A regular octahedron is an octahedron bounded by eight equal and equilateral triangles. If the centre of each face of a cube be taken, and if planes be passed through these points, each plane passing through those lying in faces, which meet in the same angular point, these planes will, by their intersections, determine a regular octaedron; and, conversely, if the centre of each face of a regular octaedron be found, and planes be passed so that each plane shall pass through the centres of the faces, which meet at the same angular point, they will, by their intersections, determine a cube. If we denote the length of one edge of a regular octaedron by l, the area of the entire surface by A, the volume by V, the radius of the circumscribed sphere by R, and that of the inscribed sphere by r, there will exist the following relations between these quantities:

$$l = r\sqrt{6} = R\sqrt{2} = \sqrt{\tfrac{1}{6}A\sqrt{3}} = \sqrt[3]{\tfrac{3}{2}V\sqrt{2}} \ldots (1),$$

$$A = 12r^2\sqrt{3} = 4R^2\sqrt{3} = 2l^2\sqrt{3} \ldots\ldots\ldots (2),$$

$$V = 4r^3\sqrt{3} = \tfrac{4}{3}R^3 = \tfrac{1}{3}l^3\sqrt{2} \ldots\ldots\ldots (3),$$

$$R = r\sqrt{3} = \tfrac{1}{2}l\sqrt{2} = \tfrac{1}{2}\sqrt{A\sqrt{3}} = \sqrt[3]{\tfrac{3}{4}V} \ldots (4),$$

$$r = \tfrac{1}{2}R\sqrt{3} = \tfrac{1}{6}l\sqrt{6} = \tfrac{1}{6}\sqrt{A\sqrt{3}} \ldots\ldots\ldots (5).$$

OC-TAN″GU-LAR. Having eight angles.

OC′TANT. The eighth part of a circumference of a circle, or the half of a quadrant

OC-TŌ′BER. The tenth month of the year It contains 31 days.

ODD. An odd number is a whole number, which cannot be exactly divided by 2. The

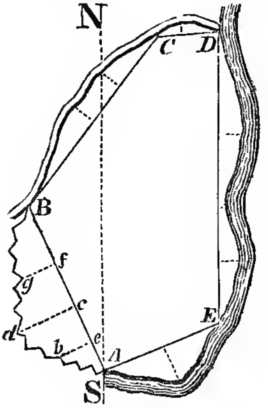
alternate numbers beginning at 1, as 1, 3, 5, 7, 9, &c., form the series of odd numbers. Every odd number, when divided by 2, leaves 1 for a remainder.

ODDLY ODD NUMBER. A number which, when divided by 4, leaves 3 for a remainder, or which is of the form $4n + 3$. Thus, 3, 7, 11, 15, &c., are oddly odd numbers. The alternate terms of the series of odd numbers, beginning at 3, constitute the series of oddly odd numbers.

O-DOM'E-TER. [Gr. *οδος*, way, and *μετρον*, measure]. An instrument employed for registering the number of revolutions of a carriage-wheel, to which it is attached. Knowing the length of the tire of the wheel, and the number of revolutions made between any two places, we can find the distance between the places. The principle of the odometer is entirely similar to that of the vernier. Two wheels of the same diameter, and turning freely on the same axis, are placed face to face; the edge of one is cut into 100 teeth, and that of the other into 99 teeth, and an endless screw works into the notches in each wheel. When the screw has turned 100 times around, the wheel having 99 teeth will have gained one notch on the other, which gain is shown by an index attached to one wheel, which passes over a graduated arc on the other. Every hundred turns are thus registered on the second wheel, and all turns less than 100, are shown by a separate index. Now, instead of the screw turning on its axis, it is found more convenient to have the screw fast, and to allow the weight of the machine to be suspended freely, so that as the carriage wheel turns, the effect is the same as turning the screw on its axis.

OFF'SET. In Surveying, a short course measured perpendicular to a longer one. The method of offsets is employed in surveying fields bounded by irregular lines.

Let ABCDE be a piece of ground to be surveyed. Assume stations at the principal points, A, B, C, D and E. Take with the compass the bearings from A to B, from B to C, from C to D, from D to E, and from E to A; measure with the chain, the distances AB, BC, CD, DE, and EA. At convenient points of the course AB, as *e*, *c*, and *f*, measure the offsets, *cb*, *cd*, and *fg*. Then, having measured these as well as the distances, A*e*, *ec*, *cf*, and *f*B, enough will be known to



determine the area which lies without the station line AB. The points *b*, *d* and *g*, of the fence which runs from A to B, are also determined. Erect, in a similar manner, offsets to the other courses, and determine the areas which lie without the station lines. These several areas, being added to the area within, will give the entire area of the ground. If the offsets fall within the station lines, the corresponding areas must be subtracted from the area which is bounded by the station lines.

The method of offsets is also employed with advantage in Surveying, for the purpose of making a map of a portion of country; as for example, in mapping a shore line. Principal stations are selected at the most prominent points, in the general direction of the line to be mapped, and the courses are run at suitable intervals, and opposite the points where the lines bend; offsets are measured, and their distances from the last station noted in the field-book. These measurements enable us to plot the principal points of the lines to be determined, from which the line can be sketched in by the eye.

The method of offsets is often employed in surveying large estates. Several prominent stations are selected, and the lines joining them surveyed with great care. These form a system of triangles, and may be plotted. Now, from these stations, let lines be run in suitable directions, noting where they cut

fences, streams, &c., and from proper points of them, let offsets be taken to angular points in fences, &c., then these lines and offsets being plotted, the entire area may be found from the plot. See *Surveying*.

OFFSET STAFF. A rod used in surveying for measuring offsets; it is usually 10 links in length, and is divided into ten equal parts, which are numbered from one end to the other.

O-PĀKE′. [L. *opacus*, shady]. In Shades and Shadows, a body which does not permit rays of light to pass through it. Opake bodies cast shadows and receive them.

OP-ER-Ā′-TION. [L. *operatio*, operation]. An operation in Mathematics is something to be done; generally some transformation to be made upon quantities, which transformation is indicated either by rules, or by symbols.

The operations indicated by rules are sufficiently explained in the rules themselves; those indicated by symbols cannot be fully understood without a complete knowledge of the nature and force of those symbols. There are four kinds of symbols employed in Mathematics. 1st. Those which stand for quantities; such as letters standing for numbers, time, space, or any of the geometrical magnitudes. 2d. Those of *relation*, as the signs, $=$, $>$, $: \ :: \ :$, &c., which indicate respectively, the relations of equality, inequality, proportion, &c. 3d. Those of abbreviation, as, $\therefore$ for *hence*, $\because$, for *because*; *exponents* and *co-efficients*, are likewise symbols of abbreviation, the symbol consisting in the manner of writing these numbers. 4th. *Symbols of operation*, or those employed to denote an operation to be performed, or a process to be followed; such are the symbols of algebra and the differential and integral Calculus, &c., which do not come under the preceding heads. Those of the 3d. class are generally regarded as symbols of operation.

SYMBOLS OF OPERATION are of two kinds. 1st. Those which indicate invariable processes and are, in all cases, susceptible of uniform interpretations. This kind includes most of what are usually called, the *signs* of algebra, as $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, &c. 2d. Those which indicate *general methods of proceeding* without reference to the nature of the quantity to be operated upon; for example, the symbol d in the expression $df(x)$, denotes that the function of x, whatever it may be, is to be differentiated; of this nature also, is the sign for integration and many others which have been employed in extending and generalizing the processes of the higher branches of Mathematics.

When a symbol is agreed upon, as denoting a certain process, it often happens that in the course of time, the meaning of the symbol becomes very much extended and generalized, so that its primitive meaning is either lost sight of, or at most, forms but an insignificant portion of its meaning. For example, an exponent was originally employed to denote the number of times that a factor was to be taken to produce a given quantity. But by extending this use, exponents have come to be *negative*, *fractional*, and even *imaginary*, and the operations which they denote, though perfectly comprehensible to the mathematician, would cease to be intelligible, if the meaning of the term exponent were restricted to its original signification.

In common Arithmetic, the symbol $-$, when written before two numbers, implies that the second is actually to be taken from the first, and so long as the symbol has this restricted signification, such expressions as $4 - 7$ are necessarily unintelligible; but in a more extended signification, it implies the operation of finding a number, or quantity, which being added to 7 will produce 4; this number is -3. In this point of view, the symbol can never give rise to any misunderstanding.

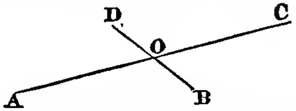
Again, the symbol $\sqrt{\ }$ implied, originally, that the square root of the quantity written under it is to be taken. But if we suppose that quantity to become negative, as $\sqrt{-1}$, the expression is unintelligible. But if we extend the signification of the symbol so that it shall imply the operation of finding an expression, which being taken twice as a factor, or being operated upon according to a certain process, shall produce -1, the symbol becomes as intelligible as any in the range of Mathematics.

We see, then, that the language of mathematical symbols is much more general than common language; in fact, as the science is extended, it is universally found that the symbols before used may be so interpreted as

to be amply sufficient to express every new operation that may arise from the process of generalization.

Of late years, a very extensive branch of analysis has been opened, called the *Calculus of operations;* this calculus is conducted on the principle of separating the symbols of operation from those of quantity, and then combining them, according to established principles, as though they represented quantities, and then deducing from the results, by interpretation, the truths required to be demonstrated. This branch of science is yet in its infancy, but already it has been the instrument of greatly extending the domains of science, and we may reasonably look to it for the next great step in the direction of mathematical progress. It is almost impossible to give any adequate account of the nature of this branch of mathematics, in the narrow limits to which we are restricted in this work.

OP′PO-SITE ANGLES. Angles lying on opposite sides of two lines which intersect



each other. The angles AOD and BOC are opposite; so are the angles AOB and DOC.

OR′DER [L. *ordo*, order, series]. Rank or class. In analysis, magnitudes are classed into orders, depending upon the degree of their equations. Algebraic magnitudes only are classed in this way. All algebraic lines whose equations are of the first degree are of the first order; those whose equations are of the second, third, &c., degrees, are respectively of the second, third, &c., orders. All lines of the first order are straight lines, and conversely all straight lines are of the first order. All lines of the second order are conic sections, that is, either ellipses, hyperbolas, parabolas, or some of their particular cases, and conversely all conic sections and their particular cases are of the second order.

Algebraic surfaces are classed into orders, according to the degrees of their equations. Those whose equations are of the first, second, third, &c., degrees, are surfaces of the first, second, third, &c., orders. All surfaces of the first order are planes, and conversely, all planes are surfaces of the first order. All surfaces of the second order are those whose plane sections are conic sections; that is, ellipsoids, hyperboloids, paraboloids, and their particular cases; conversely, all ellipsoids, hyperboloids, paraboloids, and their particular cases, are surfaces of the second order.

The following tables show the divisions and subdivisions of lines and surfaces of the second order:

I. LINES.

Order.	*Class.*	*Species.*	*Varieties.*
Second.	Ellipse,	Ellipse, Circle,	Point. Imaginary curve.
	Hyperbola,	Hyperbola, Equilateral Hyperbola,	Two straight lines intersecting.
	Parabola,	Parabola, Two parallel straight lines,	real, not coincident. real coincident. imaginary.

II. SURFACES.

Order.	*Class.*	*Species.*	*Varieties.*
Second.	Ellipsoids,	Ellipsoids, Ellipsoids of revolution, Sphere,	Point. Imaginary surface.
	Hyperboloids,	Hyperboloids of one nappe,	Hyperboloids of revolution of one nappe. Conic surfaces.
		Hyperboloids of two nappes,	Hyperboloids of revolution of two nappes. Conic surfaces.
	Paraboloids,	Elliptical Paraboloids,	Paraboloids of revolution. Cylinders with elliptical bases.
		Hyperbolic Paraboloids,	Cylinders with hyperbolic do. Two planes intersecting.
		Parabolic Paraboloids,	Cylinders with parabolic bases.

These are distinguished analytically by the relations which exist between the constants which enter their equations.

Lines and surfaces of other orders might be classified in a similar manner, but the discussion becomes more intricate as the order is higher.

OR′DI-NA-RY LIMITS of the roots of an equation: the limits ordinarily used in searching for the roots of a numerical equation. The ordinary limit is equal to 1 increased by that root of the numerical value of the greatest co-efficient, whose index is the number of terms that precede the first negative term. If any of the terms, preceding the first negative one, are 0, they must be counted in determining the index. Thus, the ordinary superior limit of the roots of the equation

$x^4 + 11x^2 - 25x - 67 = 0$, is $1 + \sqrt[3]{67}$ or 6.

It will be perceived that the cube root of 67 is between 4 and 5; and as it is desirable that the limit should be expressed in whole numbers, the highest number is taken. See *Limit.*

OR′DI-NĀTE. The ordinate of a point is one of the elements of reference, by means of which the position of a point is determined with respect to fixed straight lines, taken as co-ordinate axes. The ordinate of a point *to a diameter* of a conic section, is the distance of the point from that diameter, measured on a line parallel to a tangent drawn at the vertex of the diameter. The ordinate to a diameter is equal to half the chord through the point which is bisected by the diameter. See *Co-ordinate.*

OR′I-GIN OF CO-ORDINATE AXES. That point in the system in which the co-ordinate axes intersect. It is called the origin, because the co-ordinates of any point may be measured on the axes from this point. See *Co-ordinates.*

OR-THOG′ON-AL. [Gr. ορθος, right, and γωνια, angle]. Right angled. The orthogonal projection of a magnitude is that projection which is made by projecting lines drawn perpendicular to the plane of projection. See *Orthographic.*

OR-THO-GRAPH′IC PROJECTION. That projection in which points are projected by means of straight lines drawn through them, perpendicular to the plane of projection. All the projections of descriptive geometry are orthographic, also that particular kind of spherical projection called the *orthographic projection.* The name is almost exclusively applied in the latter case. The rules for making the orthographic projection of the circles of a sphere are few and simple. The projection of a circle is always an ellipse or some of its varieties. The rule for making the projection is this: project that diameter of the circle which is parallel to the plane of projection; this will be the transverse axis of the projection; project the diameter perpendicular to it, and this will be the conjugate axis of the projection. The length of the transverse axis is equal to the diameter of the circle projected; the length of the conjugate axis of the projection, is equal to the diameter of the circle multiplied by the cosine of the inclination of the plane of the circle to the plane of projection. When this inclination is 0, the cosine of it is 1, and the two axes of the projection are equal; that is, the projection of any circle parallel to the plane of projection, is an equal circle. If the inclination is 90°, the cosine is 0, and the conjugate axis is also 0; that is, the projection of a circle whose plane is perpendicular to the plane of projection is a limited straight line, equal in length to the diameter of the circle. These are the only particular cases.

The orthographic projection of the circles of the sphere may be regarded as the perspectives of the circles, the point of sight being at an infinite distance from the principal plane, or plane of projection, which is, in this case, the perspective plane.

OS-CU-LĀ′TION. [L. *osculatio,* a kissing]. A contact of one curve with another, at a given point, of the highest order possible. See *Osculatrix.*

OS′CU-LA-TO-RY. See *Osculatrix.*

Osculatory Circle. The osculatory circle is by far the most important of all the *osculatrices.* The most general equation of the circle is

$$(x - \alpha)^2 + (y - \beta)^2 = R^2.$$

By following the rule already given, the following equations of condition, for osculatory circles, may be deduced, viz:

$$(x'' - a)^2 + (y'' - \beta)^2 = R^2 \quad \cdot \quad (1),$$

$$\frac{dy''}{dx''} = -\frac{x'' - a}{y'' - \beta} \quad \cdot \quad \cdot \quad (2),$$

$$\frac{d^2y''}{dx''^2} = -\frac{1 + \frac{dy''^2}{dx''^2}}{y'' - \beta} \cdot \quad \cdot \quad (3);$$

from these the values of a, β and R may be deduced, which, being substituted in the assumed equation of the circle, will render it the equation of an osculatory circle to a given curve, at the point whose co-ordinates are x'' and y''. But in most cases the value of R alone is all that is needed; by combination, we find

$$R = \pm \frac{\left(1 + \frac{dy''^2}{dx''^2}\right)^{\frac{3}{2}}}{\frac{d^2y''}{dx''^2}}.$$

This value of R, when deduced for any particular curve, is called the *radius of curvature* of that curve, because at the point of osculation, the curvature of the given curve is the same as that of the osculatory circle at that point. If x'' and y'' vary so that the point of osculation shall coincide, in succession, with different points of the given curve, the locus of the other extremity of the radius of curvature is the evolute of the given curve. The radius of curvature of any given curve is always normal to that curve, and tangent to its evolute. See *Curvature*, *Radius of Curvature*, *Evolute*, *Involute*, &c.

Osculatory Circle in Space. A circle which passes through three consecutive points in space. It lies in the osculatory plane to the curve at the point of osculation. The normal plane to a curve in space, at any point, is a plane perpendicular to the tangent to the curve at the point. The centre of the osculatory circle is found at the point in which the osculatory plane at the point is pierced by the line of intersection of two consecutive normal planes. The general formula for the radius of the osculatory circle, or radius of curvature, at any point of a curve in space is

$$R = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2}}.$$

Osculatory Conic Sections. The most general form of the equation of the conic sections is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \cdot (1)$$

which represents

the parabola when $b^2 - 4ac = 0$,
the ellipse when $b^2 - 4ac < 0$, and
the hyperbola when $b^2 - 4ac > 0$.

Suppose that the equation of the given curve is

$$y = f(x) \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

and denote the co-ordinates of the given point of osculation by x'' and y''; then will

$$ay''^2 + bx''y'' + cx''^2 + dy'' + fx'' + f = 0 \, (3)$$

be the equation of condition that the conic section shall pass through the given point, and

$$y'' = f(x''),$$

the equation of condition that the given point shall lie upon the given curve. Now, since there are five arbitrary constants entering the most general form of the equation of the conic sections, it follows that at every point of the given curve there will always be an osculatory conic section, having, with the given curve, a contact of the fourth order. The nature of the conic section will depend upon the sign of $b^2 - 4ac$, in the equation of the osculatrix at the point in question.

To determine the circumstances of osculation. Since $b^2 - 4ac$ depends upon the co-ordinates of the point of osculation, it will vary as that point changes, and may, therefore, be regarded as a function of x'' and y'', supposed variables. If we follow the process indicated, the values of a, b and c, and consequently that of $b^2 - 4ac$, may be found in terms of x'', y'' and known quantities; and if this equation be combined with the equation $y'' = f(x'')$, and y'' eliminated, there will result an expression for the value of $b^2 - 4ac$ in terms of x'' alone and known quantities.

Place this expression equal to 0, and solve the resulting equation, and denote its roots by x''', x^{iv}, . . . &c. Then for every point of the curve corresponding to the real roots, the osculatory conic section is a parabola, and for all the points between each pair of these, taken in order, the osculatory conic section will be alternately an ellipse and a hyperbola, since $b^2 - 4ac$ can only change sign by passing through 0. If the roots are all imaginary the sign of $b^2 - 4ac$ will always remain the same, and the osculatory conic section will always be either an ellipse or an hyperbola. If the value of $b^2 - 4ac$ is equal to ϕ,

independently of all values of x'', the osculatory conic section will always be a parabola.

Osculatory Plane, to a curve of double curvature, is a plane which passes through three consecutive points of the curve. Or, if a plane be passed through three points of the curve whose abscissas differ from each other by the arbitrary quantity h, and then the value of h be diminished continually, till it is less than any assignable quantity, the plane will reach its limiting position and become osculatory. The angle between two consecutive osculatory planes is called the angle of *torsion*. The equation of an osculatory plane to any curve in space, is

$$\left.\begin{array}{l}(x - x'')(dy''d^2z'' - dz''d^2y'') \\ + (y - y'')(dz''d^2x'' - dx''d^2z'') \\ + (z - z'')(dx''d^2y'' - dy''d^2x'')\end{array}\right\} = 0;$$

in which x'', y'' and z'' are the co-ordinates of the point of osculation.

Osculatory Sphere, to a line of double curvature. A sphere passing through four consecutive points of the curve. If a circle be passed through three consecutive points of the curve, and a second circle pass through the second of these points and the next two consecutive ones, the two circles will have two consecutive points in common, and consequently will be tangent to each other; their planes will make with each other the angle of torsion, and a sphere passed through them both, will be the osculatory sphere to the curve. The general theory of osculatory surfaces is intricate, and of but little practical utility.

Osculatory Surfaces. If two surfaces have a point in common, and the partial successive differential co-efficients of the ordinates of the two surfaces of the first n orders, taken at the common point, respectively equal to each other, the surfaces have a contact of the n^{th} order. Since the most general equation of the sphere has but four arbitrary constants, it is impossible to assign to it a contact of the second order with any given surface. But a sphere may be made osculatory with any line drawn through the point of osculation on the surface.

Osculatrices to Curves in Space. If two curves in space have a point in common, and the partial differential co-efficients of z, of the first n orders of the two curves, taken at that point, are respectively equal to each other, the curves have a contact of the n^{th} order. This requires that the projections of the curves on the co-ordinate planes should have a contact of the n^{th} order; hence, if the projections of two curves in the three co-ordinate planes, respectively, have a contact of the n^{th} order, the curves themselves will also have a contact of the n^{th} order; and conversely, if one curve is given in kind, and the other completely, and such values be given to the constants as to make the projection of the curves osculatory, then will the curves in space be osculatory. We see, therefore, that the subject of osculation in space is reduced to a consideration of the matter of osculation in a plane. See *Osculatrix.*

OS-CU-LA'TRIX. If two plane curves have a point in common, and the first differential co-efficients of the ordinates taken at that point equal, the curves are said to have a contact of the *first order* at that point. If, in addition, the second differential co-efficients of the ordinates of the curves, taken at the point, are also equal, they have a contact of the *second order*. In general, if two plane curves have a point in common, and the first n successive differential co-efficients of the ordinates of the curves taken at the point respectively, equal, the curves are said to have a contact of the n^{th} order. A curve which has a higher order of contact with a given curve, at a given point, than any other curve of the same kind, is called an *osculatrix.*

The subject of the contact of curves may be viewed in another light. If two curves have a point in common, and if the consecutive ordinates of the two curves differ from each other by an infinitely small quantity of the second order, the curves have a contact of the first order. If this difference is of the third order, the contact is of the second order, and, in general, if it is of the $(n+1)^{\text{th}}$ order, the contact is of the n^{th} order. The definition of an osculatrix remains the same as before. These definitions indicate the method of solving the following two propositions:

First. *To find whether two given curves have any contact, and if so, to determine the order of contact.* Combine the equations of the curves, and find the values of the variables. For every pair of real values found for

x and y, there will be a point common to the two curves. Suppose that there is one common point. Differentiate the equations of the curves, and find the first differential co-efficients of the ordinates of the curves, and in them substitute for x and y the co-ordinates of the common points; if the results are equal, the curves have a contact of the first order, at least. Differentiate again, and find the second differential co-efficients of the ordinates of the two curves; substitute as before, and compare the results. If these are equal, the two curves have a contact of the second order, at least. Continue the operation of successive differentiation, substitution, and comparison, until two differential co-efficients of the ordinates, taken at the common point, are found, which are not equal; then will the number of successive differential co-efficients of the ordinates, taken at the common point, which were found equal, denote the order of contact of the two curves. If by the first combination, more than one common point is found, the same operation is to be gone through with for each common point. The equations resulting from the total operations, will indicate whether the curves have any contact, and the order of the contact.

SECOND. *To assign to a curve, given in kind, the highest order of contact that can be assigned to curves of that kind, with a given curve, at a given point.*

Assume the most general form of the equation of the kind of curve given. Substitute in it for x and y the co-ordinates of the given point of the given curve; there will result an equation of condition that the assumed curve shall pass through the given point. Differentiate the equations of both curves; find the first differential co-efficients of the ordinates; substitute in these for x and y the co-ordinates of the given point, and place the results equal; there will result a second equation of condition, which, with the preceding one, will cause the curves to have a contact of the first order. Continue the operation of successive differentiation, substitution, and formation of equations of condition, till as many such equations are formed as there are arbitrary constants in the equation of the curve given in kind. Combine these equations, and from them deduce the values of the constants required, and substitute these values in the assumed form of the equation of the curve given in kind, and the result will be the equation of the osculatrix of that kind. It will be seen that the highest order of contact that can be assigned to a curve, given in kind, with a given curve at a given point, is denoted by the number of arbitrary constants in the equation of that curve, diminished by 1. It may happen, however, that the conditions which make that number of successive differential co-efficients of the ordinate taken at the given point equal, will also make one or more of the successive ones equal, in which case there may be a higher order of contact at certain points, than that indicated. This, however, can only take place at certain points of any given curve. We have an instance of this in the case of the ellipse at the vertices of the axes. In general, it is impossible to assign a higher order of contact to a circle with the ellipse, at a given point, than the *second*, but at the vertices of the transverse and conjugate axes, the conditions which make the circle have a contact of the second order, will also make it have a contact of the third order. In general, at any point of a given curve, when the curve is symmetrical with the normal at that point, it happens, when the osculatrix is of an even order, that the imposed conditions give a contact of the next higher order. This is not to be regarded as an exception to the general rule.

The equation of the osculatrix of any given kind, with a given curve at a given point, may be arrived at by means of the following considerations:

If we denote the abscissa of the given point by x', and substitute for x in the equation of the curve the successive values,

$$x', \ x' + h, \ x' + 2h, \ x' + 3h, \ \&c.,$$

(h being arbitrary), we may deduce corresponding values for y, which, with the assumed values of x, will determine the position of a succession of points of the curve whose abscissas differ by the arbitrary quantity h. Now, the curve which is given in kind, may be made to pass through as many of these points as there are arbitrary constants in its most general equation. Substitute for x and y in the equation of the curve given in kind, the co-ordinates of these points, in succession, beginning with the first point, until as

many equations are found as there are constants to be determined. Combine these equations, and find the values of the constants in terms of the known quantities, and the arbitrary quantity h, and substitute them in the general form of the equation. This will be the equation of a curve of the given kind, which passes through as many points of the given curve as there are constants in the second equation. Now, if we suppose h to diminish, the several points will approach each other and the given point, in accordance with the law of the given curve, and when h becomes less than any assignable quantity, they will become consecutive, the curve will be the osculatrix, and the resulting equation will be the equation required. If the curve passes through two consecutive points, the contact is of the first order, if through three, it is of the second order; and, generally, if through $n+1$ points, the contact is of the n^{th} order.

OUNCE. [L. *uncia*, the twelfth part of a thing]. A unit of weight. In Troy weight, the ounce is the twelfth part of a pound, and is equivalent to 480 grains. In avoirdupois weight, the ounce is the sixteenth part of a pound, and is equivalent to $437\frac{1}{2}$ grains. See *Weight*.

OUT'LĪNE. The outline of a figure is a contour line which bounds the figure; thus, in perspective, the outline is the intersection of the enveloping visual cone of the body with the perspective plane. See *Perspective*.

ŌVAL. [L. *ovum*, an egg]. An egg-shaped figure, or a figure resembling an ellipse. An oval is sometimes used by carpenters instead of an ellipse, and may be formed from arcs of circles of different radii, and tangent to each other.

Oval of Descartes, or Cartesian. A curve such that the simultaneous increments of two lines drawn from the generating point of the curve to two fixed points, have always to each other a constant ratio. If the ratio is equal to -1, the oval becomes an ellipse; if it is equal to $+1$, it is an hyperbola. This kind of oval may be defined to be the locus of the vertex of a triangle having a given base, one of whose sides has a constant ratio to the other, increased or diminished by a given straight line. To find the equation of the Cartesian oval: denote the distance between the fixed points by $2c$; the distances from them to any point of the curve by r and r'. Then from the definition of the curve,

$$dr + mdr' = 0;$$

or, by integration,

$$r + mr' = 2a \ldots\ldots (1),$$

$2a$ being an arbitrary constant. Denoting the angle between r and $2c$ by ϕ, we have

$$\cos\phi = \frac{r^2 + 4c^2 - r'^2}{4cr} \ldots\ldots (2).$$

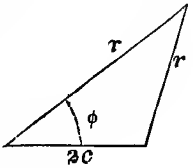
Combining equations (1) and (2), and eliminating r', we have

$$(m^2-1)r^2+4(a-m^2c\cos\phi)r+4(m^2c^2-a^2)=$$

which is the polar equation of a curve of the fourth order, except when

$$m = \pm 1,$$

in which case, after reduction, it becomes



$$r = \frac{a(1-e^2)}{1-e\cos\phi},$$

the polar equation of an ellipse or hyperbola. The Cartesian oval being revolved about its axis, generates a surface which must divide two media of different densities, so that rays of light emerging from a given point, shall be refracted accurately to another given point. If the radiant point is at an infinite distance, or if the rays are parallel, the surface becomes that of the ellipsoid.

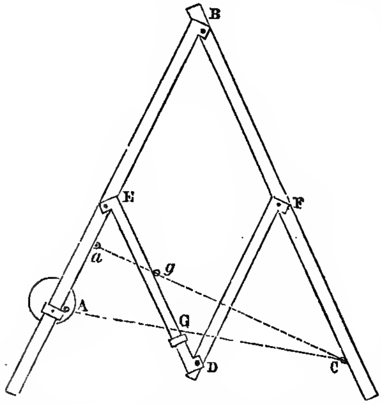
OX'Y-GON. [Gr. ὀξυς, sharp, and γωνια, angle]. A triangle having three acute angles.

P. The sixteenth letter of the English alphabet. As a numeral, it was formerly used to denote 100; with a dash over it, thus, $\overline{\mathrm{P}}$, it denoted 100,000.

PAIR OF VALUES. Two values so related that neither can exist without the other. Thus, in an equation between two variables, if any value be assumed for one, and the corresponding value of the other be deduced, the assumed and deduced values are called a pair of values. Conversely, if either of the deduced values be substituted, the assumed value will result.

PAN'TO-GRAPH. [Gr. παν, all, and γραφω, to write]. An instrument used in

copying plans, maps, and other drawings. The principal parts of the pantograph in most general use, are shown in the diagram.



It is essentially composed of four orass rulers or bars, jointed to each other at B,D,E,F. These joints should be executed with the greatest care, to insure smoothness and steadiness of motion, upon which the utility of the instrument principally depends. At the point C is fixed a small tube, which carries the tracing point or tracer, so fitted as to move freely within it, without shaking. The bar ED, and the lower part of the bar AB, are furnished each with a tube similar to that at C, but movable on the bar with a screw to fasten it down at any point. A pencil stem is arranged so as to fit either of the tubes in the same manner as the tracer; on the top of this stem is a cup to receive a weight to keep it down upon the paper, and the lower end carries a pencil or marking point. A silk cord is attached to the pencil stem, carried through eyes made for the purpose over the joints E,B,F, and fixed in a notch at the top of the tracer, so that the pressure of the thumb upon the cord lifts the pencil from the paper. There is also a flat leaden weight A, with a brass stem rising out of it, which fits in the tube in the same manner as the pencil and tracer; this is called the fulcrum, and is the point upon which the whole instrument turns; the weight has three or five short points on its under side, to keep it from shifting its place on the paper. The whole instrument is supported upon castors, which admit of free motion in all directions. The pin or fulcrum is placed near the edge of the weight, so as to allow room for the castor to work when the fulcrum is near the points A or D.

The length of the bars is so arranged that BF equals ED and BE equals FD, making the figure BEDF always a parallelogram.

Now, if the tracer at C is carried over the lines of the drawing to be copied, the fulcrum being fixed at A, and the pencil tube at G, the pencil will make an exact copy of the drawing half the size of the original; that is, each line will be half as long as in the drawing to be copied; for, the points A, G and C are capable of being brought close together, and when the instrument is open as in the figure, G is exactly half way between C and A; C, then, travels twice as fast as G, in the direction AGC, so that to whatever extent the pantograph may be opened, G and C being considered as points in a lever of which A is the fulcrum, it will be seen that C describes an arc of a circle of any radius; G at the same time describing a circle of half the radius, so that C moves in a direction perpendicular to AGC, twice as fast as G. Now it was shown above, that it moved twice as fast as G in the direction AGC, and as by the composition of these two motions all lines, whether rectilinear or curved, are produced, it follows that the pencil at G will produce a copy, all of whose lines are half the length of the corresponding ones in the original drawing, and which will have the same relative situation with respect to each other, that their homologous lines have to each other in the original. It will be apparent that the actual area of the drawing in the case considered, is only one-fourth that of the original, but it is customary to say that it is a drawing half the size of the original, because its lines are half of those to which they correspond.

To produce a copy whose lines shall be one-fourth their homologous lines in the original, we must shift the pencil to g, and the fulcrum to a, ag being one-fourth the length of aC, and so on for other proportions We may express the rule thus:

As the distance from the pencil to the fulcrum is to the distance from the tracer to the fulcrum, so is any line in the copy to its homologous line in the original.

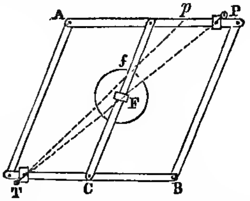
For the purpose of producing copies of any

fractional portion of the size of the originals, the arms bearing the tracing and pencil tubes are graduated and numbered so that these tubes can be set with great ease and accuracy. If it be required to produce a copy whose lines are more than half the length of their homologous lines in the original, the fulcrum must be placed on the arm ED, and the pencil on AE: so that for a copy of the full size of the original, the fulcrum must be at G, and the pencil at A. In the case last considered, that is, when the fulcrum is on the arm ED, the copy will be inverted.

The principle of the pantograph just described, is all that could be desired in the way of perfection, but it is found in practice, on account of the numerous joints and the necessary imperfection in its mechanical construction, that it is far from being an accurate instrument.

The pantograph is principally useful to the draughtsman, in enabling him to mark off the principal points in a reduced copy, through which the lines may afterwards be drawn by the usual methods of construction; for this purpose it is found to work successfully.

The annexed engraving shows another style of pantograph, which possesses some advantages over the one last described. In



the first place, the fulcrum being in the centre, it requires but one castor, which is placed at C, and makes it work much easier than the old instrument which has six, besides which, these six castors are a source of annoyance by getting off the edge of the drawing board and running over the drawing pins, or anything else that may happen to be in the way. Secondly, the shape of the instrument allows it to move as freely when nearly closed, as when it was wide open, which is not the case with the old one. The method of construction and of using the instrument is extremely simple. It is composed of five bars moving freely about each other at the points of junction, so arranged with regard to length, that AP and TB are always parallel to each other. F is the fulcrum furnished with a socket and a screw, through which the centre bar can be moved, and which can be fastened down at any of the divisions on the bar. This socket, with the bar, turns upon the pin rising out of the centre of the flat weight, shown in the diagram. The tracer, T, the fulcrum, F, and the pencil, P, must always be in a straight line. To produce a copy of the same size as the original, the fulcrum must be in the centre, and the pencil and tracer at equal distances from their respective arms, and consequently, from the fulcrum. For a half size copy, the pencil must be moved half way up the arm to *p*, and the fulcrum to *f*, in a straight line T*fp*, and so on for other proportions. The rule given for the other instrument is equally applicable to this.

PAN-TOM'E-TER. [Gr. *παν*, all, and *μετρον*, measure]. An instrument for measuring all sorts of angles and distances.

PAN-TOM'E-TRY. Universal measurement.

PÄR VALUE. [L. *par*, equal]. In Mercantile affairs, *par value* is the full value represented on the face of a note, bond, or other certificate of property. When any paper sells for less than its face, it is below par, if more than its face, it is above par. The term is used in buying and selling stocks, &c.

Par of Exchange is a term used in comparing the currency of different countries. Thus, the English sovereign is valued by law at $4.861, at our mints, and it is at this value that it must be reckoned in estimating the par of exchange.

Commercial Par of Exchange is a comparison of the coins of different countries, according to their commercial values. Thus, before the change of one standard of gold coin, the value of the English sovereign was $4,44$\frac{4}{9}$, and this is still the unit on which the exchange is calculated. The legal value of the English sovereign is fixed by Act of Congress, a little below its intrinsic value, viz.: at $4,86. Hence, the par exchange is found by adding such a per cent. to $4,44$\frac{4}{9}$, as will make the amount equal to $4,86, which is 9 per cent., very nearly. The par

of exchange can always be found when we know the unit in which the exchange is calculated, and the mercantile value of that unit.

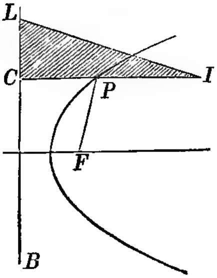
PA-RAB′O-LA. [L. *parabola;* Gr. παραβολη]. A curve having one or more infinite branches without rectilineal asymptotes.

The *conic*, or *common parabola*, is one of the conic sections. It is cut from the surface of a right cone with a circular base, by a secant line passed parallel to an element of the surface. This plane cuts all of the elements of the surface except one, and all in the same nappe. Hence, the curve has but one branch, and that extends to an infinite distance; the two parts of the branch approach parallelism as they recede from the vertex, and at a very short distance become sensibly parallel, so that, were the part towards the vertex removed, the remaining portion might be regarded as two parallel straight lines. If the cutting plane be moved parallel to its first position, towards the element to which it is parallel, the curve approaches to coincidence with a straight line, which is its axis, and finally, when it reaches the element, it is reduced to the axis or a single straight line extending indefinitely in both directions from the vertex of the cone. If the plane be moved still further in the same direction, the parabola passes to the other nappe of the cone, and has its concavity turned in the other direction. If the base of the cone remain the same, whilst the vertex is removed farther and farther from it, the two parts of the branch approximate to parallelism, and finally, when the vertex is at an infinite distance the cone becomes a cylinder, and the parabola reduces to two parallel straight lines. Now, if the secant plane be moved parallel to its first position, *from* the axis of the cylinder, the parallel lines will approach each other, and finally, when the plane becomes tangent to the cylinder, they will coincide. If the plane be moved still further, it will cease to cut the cylinder, and the parabola will then become two imaginary parallel straight lines. From this discussion we see that the parabola has for its extreme case, two parallel straight lines, which may be *real* and *separate*, *real* and *coincident*, or *imaginary*.

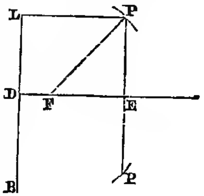
The parabola, like the ellipse and hyperbola, is a curve of the second order, and with them makes up all the lines of that order. It may be defined by means of any of its characteristic properties. The following definition is the one most commonly given.

The parabola is a plane curve, any point of which is equally distant from a fixed point and a fixed straight line. The fixed point is called the focus, and the fixed straight line the directrix. It is evident, that a straight line drawn through the focus and perpendicular to the directrix, will divide the curve symmetrically, for the conditions which determine a point above this line must also determine a second point at the same distance below the line. This line is therefore called an axis, and it is evident that it is the only axis of the curve.

From the preceding definition, the following constructions follow. 1. Let F be the focus, and BL the directrix of a parabola. Take a triangular ruler, LCI, and press one side, LC, against the directrix. At the point I, attach a string equal in length to IC, and fasten the other end at F; then press a pencil against the string, keeping the point of it against the ruler, and move the ruler along the directrix; then will the pencil-point trace an arc of the required parabola; for, in all positions of the pencil, we shall have PF = PC.



2. Let BL be the directrix, and F the focus of a parabola; assume any point on ED, perpendicular to BL through F, and at it, erect PP′ perpendicular to ED. With F as a centre, and a radius equal to ED, describe an arc of a circle cutting the perpendicular in the points P and P′; these will be points of the required parabola: for, by construction, we shall always have PL = PF. Having found a sufficient number of points, draw a curve through them, and it will be the curve required.

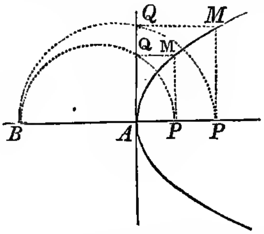


The point in which the axis cuts the curve is called the principal vertex, and if the origin of co-ordinates be taken at this point, the axis of X coinciding with the axis of the curve, its equation is

$$y^2 = 2px,$$

in which x and y are the co-ordinates of every point of the curve, and $2p$ is the *parameter*. The curve may be constructed, when $2p$ is known as follows.

Let AX and AY be the co-ordinate axes. Lay off a distance AB to the left of the origin, equal to $2p$, and assume any distance AP to the right, and through P draw an ordinate.

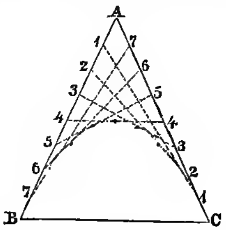


On BP as a diameter, describe a semicircle, cutting the axis of Y in the point Q; through Q draw a line QM, parallel to the axis of X, cutting the assumed ordinate in M; then is M a point of the curve; for from the construction, we have

$$PM^2 = 2p \times AP \text{ or } y^2 = 2px,$$

In this manner any number of points may be constructed; a curve drawn through them will be the required parabola.

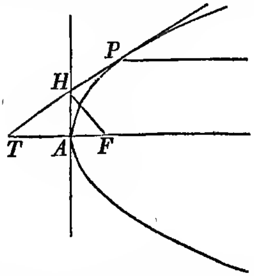
The following method of constructing an arc of a parabola is used in carpentry for laying out arches, &c. Construct an isosceles triangle, ABC, so that BC shall be equal to



the base of the required arc, and whose altitude shall be equal to twice the altitude of the required arc. Divide each of the equal sides of the triangle into any number of equal parts, say eight, and number them as in the figure. Join the corresponding numbers by straight lines, and draw a curve tangent to them all, and it will be the required arc.

If the curve is given, the elements may be found by the following construction:

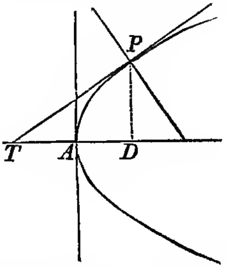
Let PA be the curve traced on a plane; draw any two parallel chords, and bisect them by a straight line; this will be a diameter. Suppose its vertex to be at P; draw any chord perpendicular to this diameter, and



bisect it by a straight line, parallel to the diameter already found; this line AA will be the axis of the curve, and A will be its vertex. Through the vertices P and A draw lines respectively tangent to the curve; they will intersect each other at H. From H draw the line HF perpendicular to PT, and find where it intersects the line TF; the point F is the focus. Lay off from A a distance to the left equal to AF, and draw a perpendicular to the axis; it will be the directrix of the curve.

The following are some of the properties of the curve, and the constructions to which they lead:

1. The subtangent on the axis is bisected at the vertex of the curve. This property

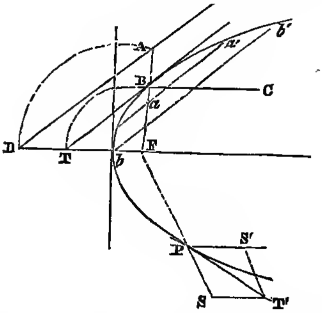


enables us to draw a tangent to the curve at a given point P of the curve.

Let AD be the axis, and A the vertex; draw PD perpendicular to AD, and lay off AT to the left, equal to AD; draw TP; it will be the tangent required.

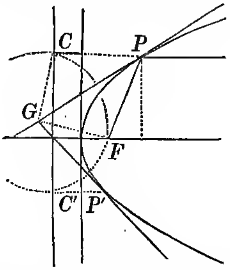
2. A diameter bisects all chords drawn in the curve, parallel to the tangent line at its vertex. This enables us to draw a tangent to the curve parallel to a given straight line. Draw two chords, aa', bb', parallel to the given line AD and bisect them by a straight line BC; at the point B, where this line cuts the curve, draw BT parallel to the given line, and it will be the tangent required.

3. If a tangent line be drawn to the curve at any point, and from the point of contact a straight line be drawn to the focus, the angle between this line and the tangent is equal to the angle between the axis and the tangent. This indicates methods of constructing a tangent at a given point, or parallel to a given line. Let F be the focus, and B a point on



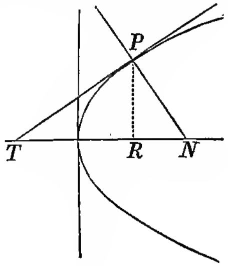
the curve. Draw BF, and with F as a centre, and FB as a radius, describe an arc of a circle, cutting the axis in T; draw TB, and it will be tangent to the curve at B. Again, let P be a given point. Draw FP, and prolong it to S; draw PS′ parallel to the axis, and make PS′ = PS; complete the parallelogram SS′, and draw the diagonal PT′; it will be tangent to the curve at the point P. Let AD be a given straight line; with F as a centre, and FD as a radius, describe an arc cutting AD in A; draw AF, and through B, the point in which AF cuts the curve, draw BT parallel to AD, and it will be tangent to the curve at B.

To draw a tangent to the curve through a point without. Let G be the point, F the focus, and CC′ the directrix. With G as a centre, and GF as a radius, describe the arc CFC′, cutting the directrix in C and C′;



through C and C′ draw CP and C′P′, parallel to the axis, cutting the curve in the points P and P′; draw GP and GP′, they will both be tangent to the curve.

4. The subnormal is constant, and equal to half of the perimeter, or to the focal ordinate. This enables us to construct a normal and tangent at a given point. Let P be a given

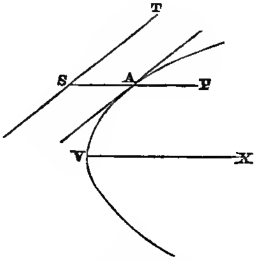


point; draw PR perpendicular to the axis, and lay off RN equal to half of the parameter; draw PN, and it will be normal to the curve at the point P; draw PT perpendicular to PN, and it will be tangent to the curve at P.

Similar constructions to the above may be made with respect to any diameter, and the tangent at its vertex, using oblique co-ordinates instead of rectangular ones.

5. A polar line with respect to a point taken anywhere in the plane of the curve, is a line such, that if from any point of it, two tangents be drawn to the curve, and a straight line be drawn through the points of contact, this will always pass through the given or assumed point, which is then called the pole of the polar line. Having given the pole to find the polar line: Let P be the pole; draw

PA parallel to the axis, cutting the curve at A; from A lay off a distance AS equal to AP; draw a tangent to the curve at the point



A, and through S draw ST parallel to the tangent; then will ST be the polar line of the point P. If P lies within the curve, ST will not cut the curve; if it lies on the curve, the polar line is the tangent at the point; and if it lies without the curve, the polar line cuts the curve, and only the points of it without the curve satisfy the definition of a polar line. To find the pole of any line taken as the polar line: Let ST be the assumed line; draw a tangent to the curve parallel to it, and through the point of contact A draw a line parallel to the axis. Lay off on this line AP equal to AS, and P will be the pole sought. If the polar line does not cut the curve, the pole falls within the curve; if it is tangent to the curve the pole is at the point of contact, and if the polar line intersects the curve, the pole lies without the curve. The polar line of the focus is the directrix of the curve.

6. The double ordinate through the focus, is equal to the parameter of the curve. The double ordinate to any diameter through the focus, is equal to the parameter of that diameter; in all cases, the parameter of any diameter is equal to four times the distance from the focus to the vertex of the diameter. If we denote the angle, which any diameter makes with the chords which it bisects, by a, then will the parameter of that diameter be equal to the parameter of the curve divided by $\sin^2 a$. If a straight line be drawn from the focus perpendicular to any tangent, the locus of its intersection with the tangent, is a straight line tangent to the parabola at its vertex. The area of any portion of the curve, bounded by the curve, axis, and an ordinate, is equal to two-thirds of the rectangle described upon the abscissa and ordinate of the extreme point.

The following equations seem to show the analytical relations existing between the elements of the curve:

1. The general equation,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

represents a parabola, when

$$b^2 - 4ac = 0;$$

this reduces to two parallel straight lines, when

$$bd - 3ae = 0,$$

which will be real and separate, when

$$d^2 - 4af > 0;$$

real and coincident, when

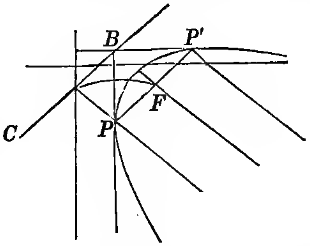
$$d^2 - 4af = 0;$$

and imaginary, when

$$d^2 - 4af < 0.$$

The parabola may be constructed from the general equation, as follows:

Solve the equation with respect to y, and place y equal to that part of its value, which is independent of the radical; the resulting equation is the equation of a diameter bisecting a system of chords parallel to the axis of Y, which construct. Place the radical



equal to 0, and find the value of x; lay off this value on the axis of X, and through its extremity draw a line parallel to the axis of Y; this will be the limit of the curve in the direction of the axis of X, and the point P, in which it intersects the diameter already constructed, is the point of contact of the limit. Solve the general equation with respect to x; place x equal to that part of its value independent of the radical, and this will be the equation of the diameter bisecting a system of chords parallel to the axis of X, which construct; place the radical equal to 0, and find, from the resulting equation, the

value of y; lay this off on the axis of Y, and through its extremity draw a straight line parallel to the axis of X: it will be the limit of the curve, in the direction of the axis of Y; and the point P′, in which it intersects the diameter last constructed, is the point of contact. Through the point B, in which the limits intersect, draw a line BC perpendicular to either diameter: it will be the directrix of the curve. With either P or P′, as a centre, and with the distance to the directrix, as a radius, describe an arc of a circle cutting the line PP′, joining the points of contact, in the point F. Then is F the focus, and a straight line drawn through it perpendicular to the directrix, is the axis of the curve. With these elements, the curve may be constructed with accuracy.

2. The equation of the curve referred to any diameter and the tangent to the curve at its vertex, is $y^2 = 2p'x$, in which $2p'$ is the parameter of the diameter, taken as the axis of X. If the axis of X coincides with the axis of the curve, the value of $2p'$ reduces to $2p$, the parameter of the curve.

3. The general polar equation of the parabola is

$$r^2 \sin^2 v + 2(b \sin v - p \cos v) r + b^2 - 2pa = 0,$$

in which r and v are the polar co-ordinates of every point of the curve; a and b, co-ordinates of the pole, and p the parameter of the curve.

If the pole is placed at the focus, which requires that $b = 0$, and $a = \frac{1}{2}p$, the equation may be reduced to the form

$$r = \frac{p}{1 - \cos v};$$

which is the form most used.

4. The equation of a tangent to the curve, referred to a diameter and tangent at its vertex, is

$$yy'' = p'(x + x''),$$

x'' and y'' being the co-ordinates of the point of contact; and when the diameter coincides with the axis, it becomes

$$yy'' = p(x + x''),$$

the co-ordinate axes being rectangular.

5. The equation of a normal to the curve, at a point $x''y''$, when referred to any diameter and the tangent at its vertex, is

$$y - y'' = -\frac{y''}{p'}(x - x'').$$

When the axis of X coincides with the axis of the curve, the equation reduces to

$$y - y'' = -\frac{y''}{p}(x - x'').$$

The curve whose properties have been discussed, is called the common parabola. There is a large class of curves, called parabolas. In general, any curve having an infinite branch, without having a rectilineal asymptote, is called a parabola. Curves having an infinite branch, or branches, to which rectilineal asymptotes may be drawn, are called hyperbolas. It sometimes happens that a curve has both parabolic and hyperbolic branches. The equation,

$$y^m = 2p'x^n,$$

embraces a large family of parabolas, of which the common, the cubic, and the semicubic parabolas are the most important. If $m = 2$, and $n = 1$, the equation becomes

$$y^2 = 2p'x,$$

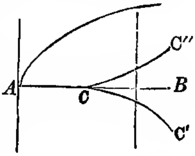
which represents the common parabola, a curve already considered.

If $m = 2$, and $n = 3$, the equation takes the form

$$y^2 = 2p'x^3, \quad \text{or} \quad y^2 = p^2x^3,$$

and the curve is called the *semicubical parabola*. It is the evolute of the common parabola PAP′.

The form of the curve is that represented in the figure. It has two infinite parabolic branches, CC′ and CC″, both convex towards the axis



of the curve AB, and both tangent to it at the same point C: C is therefore a cusp point of the first kind. If C be taken as the origin of a system of rectangular co-ordinate axes, the axis of X coinciding with the axis of the curve, then is the length of any arc of the curve, estimated from the origin, given by the formula,

$$z' = \frac{8}{27p^2}\left[\left(1 + \frac{9}{4}p^2x\right)^{\frac{3}{2}} - 1\right];$$

in which z' denotes the length of the arc, and x the abscissa of the extreme point. The area of any portion, included between the axis, the curve, and any ordinate, is expressed by the formula

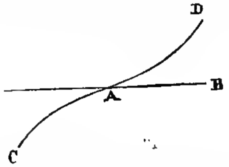
$$A = \tfrac{2}{5}xy,$$

in which x and y are the co-ordinates of the extreme point.

If $m = 1$, and $n = 3$, the equation takes the form

$$y = 2px^3, \quad \text{or} \quad y = ax^3,$$

and the curve is called the *cubic parabola.*



It has two infinite branches, AD and AC, extending in contrary directions, and both tangent to the axis at A. The branches are both convex towards the axis AB. The curve is not rectifiable; but the area, estimated from A to any ordinate, is expressed by the formula,

$$A = \tfrac{1}{4}xy;$$

in which x and y are the co-ordinates of the extreme point.

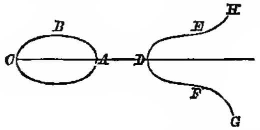
There is a remarkable parabola expressed by the general equation,

$$ay^2 - x^3 + (b - c)x^2 + bcx = 0 \ldots . (1).$$

By solving equation (1), we have

$$y = \pm\sqrt{\frac{x(x-b)(x+c)}{a}}.$$

The general equation gives a curve of two branches; the one, AC, an oval, and the other, EDF, bell-shaped; both being symmetrical with respect to an axis, CD.

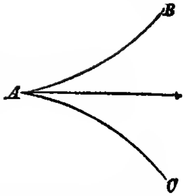


The bell-shaped branch is truly parabolic, and has two points of inflexion; one at F, and the other at E. If we suppose $c = 0$, the oval reduces to a point A, and the bell-shaped branch remains.

If we suppose $a = 0$, the curve takes a looped form, the points of inflexion reduce to the origin, and the oval joins the other branch at the origin, which then becomes a multiple point.

Finally, if we suppose both b and c to become 0, the curve reduces to the cubic parabola, and takes the form shown in the figure, the origin becoming a cusp point.

There is a multitude of other parabolas; but we have not the space to attempt even an account of their properties.



PAR-A-BOL'IC. Appertaining to the parabola.

Parabolic Conoid. The solid generated by revolving a parabola about its axis. The term *paraboloid* is preferable, and is most used. See *Paraboloid.*

Parabolic Spindle. A solid generated by revolving a portion of a parabola, limited by a straight line perpendicular to the axis of the curve, about that line as an axis. The volume of a parabolic spindle is equivalent to $\frac{8}{15}$ of its circumscribed cylinder.

Parabolic Spiral. A curve whose polar equation is

$$u^2 = 2pt,$$

in which u denotes the radius vector of any point, and t the corresponding angle. It is named from its analogy to the common parabola. It will be observed, that its equation is of the same form, and the curve can readily be constructed when the corresponding parabola is given.

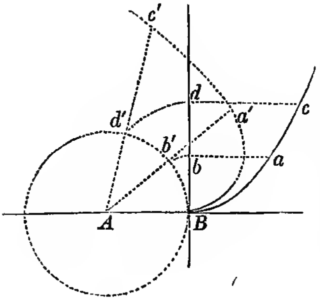
Describe from any point, as a pole, a circumference of a circle with the radius 1, and through the pole draw an initial line. From the point in which the directing circle cuts the initial line, lay off, on the circumference of the circle, the abscissa of any point of the parabola, and through the extremity of the distance draw a radius vector, making it equal to the corresponding ordinate of the parabola; then is the point, thus determined, a point of the spiral.

This is the curve, generally called the parabolic spiral. There is, however, another curve known by the same name, which may be conceived of from the following description:

If the axis BD, of a semi-parabola BCD, be wrapped around the circumference of a circle whose radius is r, any abscissa, as Bb, will coincide with an equal arc of the circle Bb', and the corresponding ordinate will take the direction of the normal A$b'a'$; the curve B$a'c'$, drawn through the extremities of the

ordinates in their new position, is called a parabolic spiral. Its equation is

$$(u - r)^2 = 2pt,$$



in which u and t are the same as before, and $2p$ the parameter of the parabola. If we conceive each ordinate, in its new position, to be moved towards the pole till its inner extremity reaches the pole, and at the same time consider the radius of the circle as 1, the corresponding curve will be that first considered.

PA-RAB'O-LOID. [Gr. παραβολη, and ειδος, form]. A volume bounded by a surface of the second order, such that sections made by planes passed in certain directions, are common parabolas. The name is in general, applied to the surface, and will be so used. It is a characteristic property of paraboloids, that they have no centres except in the extreme cases, when they have an infinite number of centres. There are three varieties of paraboloids, viz.: *elliptical*, *hyperbolic*, and *parabolic*.

Elliptical Paraboloids are those, in which all sections made by planes parallel to a straight line, called the axis of the surface, are parabolas, and all other sections are ellipses. When the sections made by planes perpendicular to the axis, are circles, the surface becomes the paraboloid of revolution. If the vertex is removed to an infinite distance, the parabolic sections become parallel straight lines, and the surface is an elliptical cylinder. Hence, the elliptical paraboloid of revolution, and the elliptical cylinder, are the particular cases of the elliptical paraboloid.

Hyperbolic Paraboloid is a warped surface, and may be generated by a straight line moving in such a manner, as to touch two given straight lines, and continue parallel to a given plane. It has also another generation; for, if we take any two elements of the first generation, and move a straight line so, that it shall constantly touch them, and continue parallel to a plane which is parallel to the directrices of the first generation, it will generate the same surface. Through every point of the surface two straight lines can always be drawn that lie entirely in the surface, which are, respectively, elements of the first and second generation, and the plane of these elements is tangent to the surface, at their point of intersection. The surface is named from the fact that any plane parallel to a tangent plane, cuts from the surface an hyperbola, whose asymptotes are parallel to the elements lying in the tangent plane, whilst all other planes cut from the surface, parabolas.

The hyperbolic cylinder is a particular case of the hyperbolic paraboloid. Every plane parallel to the axis cuts out two straight lines parallel to each other, which may be either *real*, *co-incident*, or *imaginary:* these are particular cases of the parabola; all other plane sections are hyperbolas. Another extreme case of the hyperbolic paraboloid, is determined by two planes which intersect.

All sections parallel to the axis or line of intersection of the planes, give parallel straight lines, a particular case of the parabola; all other plane sections give two straight lines which intersect, a particular case of the hyperbola. Hence, the particular cases of the hyperbolic paraboloid are the hyperbolic cylinder, and two planes which intersect. The parabolic paraboloid is a surface such that all plane sections of the surface are parabolas. The most general case of this surface is the cylinder with a parabolic base; and a particular case is two parallel planes, in which case every plane section is two parallel straight lines. When the term paraboloid is used alone, without the kind being specified, the paraboloid generated by revolving a parabola about its axis is meant. The volume of such a solid, limited by a plane perpendicular to the axis, is given by the formula,

$$V = \frac{\pi y^2 x}{2},$$

or one-half the volume of the cylinder, which has the same base and altitude.

If two planes be passed perpendicular to the axis of a paraboloid of revolution, the portion included between them is called a frustum of the paraboloid. The following formulas give the area of the surface and the volume of a frustum of a paraboloid:

$$A = \pi \cdot \frac{(2p + d^2)^{\frac{3}{2}} - (2p + d'^2)^{\frac{3}{2}}}{12p};$$

$$V = \cdot 3927\,(d^2 + d'^2)h\,;$$

in which $2p$ is the parameter, d the diameter of the lower base, d'' that of the upper base, and h the altitude of the frustum.

PAR-A-CEN′TRIC. [Gr. παρα, beyond, and κεντρον, centre]. A curve having the property that, when its plane is placed vertically, a heavy body descending along it, urged by the force of gravity, will approach to or recede from a fixed point, or centre, by equal distances in equal times.

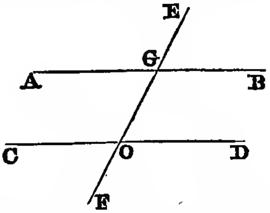
PAR′AL-LEL. [Gr. παραλληλος, from παρα, against; αλληλων, one another]. Having the same direction.

Parallel Circles, are those circles of the sphere whose planes are parallel to each other; every system of such circles has a common axis, and, consequently, their poles are also common. Circles lying in the same plane, and having a common centre, are sometimes, though improperly, said to be parallel; they are simply concentric.

Parallel Lines. Two straight lines are parallel to each other when they lie in the same direction. It follows from this definition, 1st., That they are contained in the same plane; and, 2d., That they cannot intersect how far soever both may be prolonged. Any number of straight lines are parallel to each other when they have the same direction, or when they are respectively parallel to a given straight line. Of a system of parallel lines, it follows that any two lie in the same plane; hence, if a plane be passed through any line of a system, and then be revolved about it as an axis, the plane may be made in succession to coincide with every line of the system.

If the straight lines AB and CD, lying in the same plane, be intersected by a third straight line EF in the points G and O, the angles formed about G and O have received particular names with reference to their relative positions, as follows:

1st. Those which lie between the first two and on the same side of the third, are called *interior angles on the same side;* as



BGO and DOG; also AGO and COG.

2d. Those which lie between the first two and on opposite sides of the third, but not adjacent, are called *alternate interior angles;* as

AGO and GOD; also COG and OGB.

3d. Those which lie without the first two, and on the same side of the third, are called *exterior angles on the same side;* as

EGB and FOD; also AGE and COF.

4th. Those which lie without the first two, and on opposite sides of the third, are called *alternate exterior angles;* as

AGE and FOD; also EGB and COF.

If the first two lines are parallel, the following relations between the angles will exist:

1st. The sum of the interior angles, on the same side, will be equal to two right angles.

2d. The alternate interior angles will be equal to each other.

3d. The sum of the exterior angles in the same side, will be equal to two right angles.

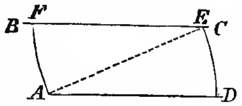
4th. The alternate exterior angles will be equal to each other.

Conversely, if any one of these relations exist, the first two lines will be parallel to each other.

If two straight lines in the same plane are perpendicular to a third straight line, the right angles formed will be severally equal to each other, the preceding conditions will be fulfilled, and the two lines will be parallel.

An important property of parallel lines is that their distance apart, at any point, is constantly the same, provided the distance is always measured in the same direction. The shortest distance is always found in the direction perpendicular to the parallels.

To draw a straight line through a given point parallel to a given straight line.



Let A be the given point, and BC the given line. From the point A, as a centre, with a radius greater than the shortest distance from A to BC, describe the indefinite arc ED; from the point E, as a centre, and with the same radius, describe the arc AF; make ED equal to AF, and draw the straight line AD; it will be the parallel required.

This problem may be solved more readily by the aid of a *parallel ruler*, or by means of the *rule and triangle*. See the articles *Parallel Ruler* and *Rule*.

In *Analytical Geometry* the condition that two straight lines, in the same plane, shall be parallel, is

$$a = a',$$

in which a and a' are the tangents of the angles which the lines make with one of the co-ordinate axes. Hence, to ascertain whether two straight lines are parallel, when the lines are given by their equations, *solve both equations with reference to the same variable; if the co-efficients of the other variable are equal, the lines are parallel.*

If one straight line is *completely given*, and the other *given in kind*, the second may be made parallel to the first by the following rule:

Solve both equations with reference to the same variable, then assign to the co-efficient of the other variable, in the second equation, a value equal to the co-efficient of that variable in the first equation, and the lines represented will be parallel.

If two straight lines in space are parallel, we have the analytical conditions,

$$a = a', \quad b = b',$$

in which a and a' are the tangents of the angles which the projections of the lines on the plane XZ make with the axis of Z; b and b' are the tangents of the angles which the projections of the lines on the plane YZ make with the axis of Z. Hence, to ascertain whether two lines, in space, are parallel, *solve the equations of their projections on the plane* XZ, *with reference to* x, *and see if the co-efficients of* z *are equal; if so, solve the equations of their projections on the plane* YZ, *with reference to* y, *and see if the co-efficients of* z *are equal; if these are also equal, the lines themselves are parallel.*

If one line is completely given, and the other given in kind, they may be rendered parallel as follows: *Make their projections on the planes* XZ *and* YZ *respectively parallel by the rule already given for making two lines in the same plane parallel, and the lines themselves will be parallel.*

Parallel Planes. Two planes are parallel when they lie in the same direction. From this definition, it follows that they can never intersect, or meet each other, how far soever both may be extended.

Two planes are parallel when they are both perpendicular to the same straight line: they are also parallel when two lines of the one which intersect are respectively parallel to two lines of the other.

If two parallel planes be intersected by a third plane, the lines of intersection will be parallel to each other, and each will be parallel to the other plane.

A straight line is parallel to a plane when all its points are equally distant from it.

In Analytical Geometry, two planes are parallel when

$$c = c' \quad \text{and} \quad d = d',$$

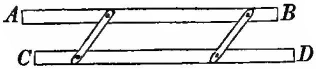
in which c, c', d and d', are respectively the co-efficients of x and y in the equations of the planes, when both have been solved with respect to z.

To ascertain whether two planes given by their equations are parallel, *solve both with reference to* z, *and see if the co-efficients of* x *and* y *are respectively equal, if so, the planes are parallel.*

If one plane is completely given, and the other given in kind, the latter may be made parallel to the former by *solving both equations with respect to* z, *and then giving such values to the arbitrary constants as shall make the co-efficients of the other variables respectively equal.*

If two planes are parallel, their traces on the co-ordinate planes are respectively parallel; and conversely, unless both the planes are parallel to one of the co-ordinate axes.

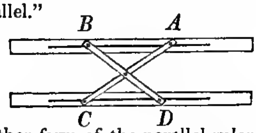
PARALLEL RULER. A mathematical instrument for drawing parallel lines. It is constructed as follows:



Two rectangular rules of wood or metal are connected by cross pieces, usually of brass, which are of equal length, and so attached by means of a hinge joint, that the two rulers may be made to recede from or approach towards each other at pleasure, so that if one remains fast the other will constantly be parallel to it. Its use is obvious.

If it is required to draw a straight line parallel to another straight line, and passing through a given point, the instrument is laid down so that the lower edge of the part CD shall coincide with the given line; the instrument is then opened till the upper edge of the part AB passes through the given point; a line drawn along the extreme edge will then pass through the given point, and be parallel to the given line.

The mathematical principle on which the construction of this instrument depends, is simply this; "if the opposite sides of a quadrilateral are equal to each other, they will also be parallel."



Another form of the parallel ruler consists of two rulers as before, connected by pieces which cross each other and turn upon a common point O, at their intersection. The ends A, B, C and D, of the cross pieces, are fitted so as to slide freely in narrow grooves, cut longitudinally in the rulers.

Another ruler for drawing parallel lines, consists of a heavy rectangular piece of wood, which has two longitudinal rollers on its under side, and which are sunk nearly flush with the lower surface. Its accuracy depends upon the nicety with which the cylindrical rollers are constructed, and upon their exact parallelism.

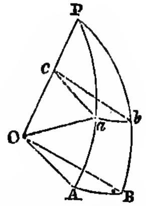
PARALLEL SAILING, in Navigation, is sailing on a parallel of latitude.

In the solution of problems, in parallel sailing, three cases may arise.

1st. *Having given the latitude of the parallel, and the difference of longitude of the two points, to find the distance sailed*;

2d. *Having given the latitude of the parallel and the distance sailed, to find the difference of longitude*; and

3d. *Having given the difference of longitude and the distance sailed, to find the latitude of the parallel.*



Let a and b represent the two places on the surface of the sphere at any arc of latitude, PA and PB meridians through a and b, and AB an arc of the equator. Since the radius ac of the parallel of latitude is equal to the radius of the sphere AO into the cosine of the latitude, and because the circumferences of the two circles are to each other as their radii, it follows that the arc ab is equal to the arc AB multiplied by the cosine of the latitude of the parallel; or designating ab, the distance sailed, by d, AB the difference of longitude by L, and Aa, the latitude of the parallel by l, we shall have the formula

$$d = L \cos l \quad . \quad . \quad . \quad (1);$$

from which we readily deduce

$$L = \frac{\cos l}{d} \quad . \quad . \quad . \quad . \quad (2),$$

and $$\cos l = \frac{L}{d} \quad . \quad . \quad . \quad (3).$$

The interpretation of these formulas gives rules for solving the three oases mentioned, as follows:

1st. *The distance sailed is equal to the difference of longitude multiplied by the cosine of the latitude;*

2d. *The difference of longitude is equal to the cosine of the latitude divided by the distance sailed;* and

3d. *The cosine of the latitude of the parallel is equal to the difference of longitude divided by the distance sailed:*

The distance sailed is expressed in nautical miles, the difference of longitude in minutes, and the cosine used is the natural cosine.

PARALLEL SPHERE, in spherical projections, is that position of the sphere in which the circles of latitude are all parallel to the horizon. This position evidently requires

the axis to be perpendicular to the horizon, which must then coincide with the equator.

Parallels of Latitude, in Navigation, are those circles of the sphere which have their planes parallel to that of the equator. There are an infinite number of such circles, four of which have received particular names; the two whose planes pass through the poles of the ecliptic are called polar circles, the one north of the equator being the Arctic, and the southern one the Antarctic circle; the two whose planes pass through the solstitial points are called tropics, the one north of the equator being the tropic of Cancer, and the one south of the equator the tropic of Capricorn.

These circles relate principally to the projection of the celestial sphere, but are usually delineated on terrestrial projections.

In Astronomy the term parallel of latitude is applied to those circles of the celestial sphere whose planes are parallel to the Ecliptic.

PAR-AL-LEL'O-GRAM. [Gr. παραλληλος, parallel, and γραμμα, a diagram]. A quadrilateral whose opposite sides are parallel to each other, taken two and two. The opposite sides are equal to each other, taken in pairs, as are also the opposite angles. If one angle of a parallelogram is a right angle, all the other angles are also right angles, and the parallelogram is a rectangle. If two adjacent sides of a parallelogram are equal, the remaining sides are also equal to each other, and the figure is a rhombus: If, in addition, the included angles between the equal sides are right angles, the figure is a square.

The diagonals of a parallelogram mutually bisect each other; and conversely, if the diagonals of a quadrilateral mutually bisect each other the quadrilateral is a parallelogram. The diagonals of an equilateral parallelogram, or rhombus, are at right angles to each other: and conversely, if two straight lines mutually bisect each other at right angles, the figure formed by joining their extremities, two and two, is an equilateral parallelogram or rhombus. The diagonals of a rectangle are equal to each other. The area of a parallelogram is equal to the product of its base by its altitude. Any two parallelograms having the same or equal bases are to each other as their altitudes; if they have equal altitudes they are to each other as their bases; generally, any two parallelograms are to each other as the product of their bases and altitudes. The sum of the squares described upon the two diagonals of a parallelogram, is equivalent to the sum of the squares described upon the four sides.

PAR-AL-LEL-O-PIP'ED-ON. A polyhedron bounded by six parallelograms. If the parallelograms are rectangles, the solid is a rectanglar parallelopipedon. If they are squares, the solid is a cube. The opposite faces are equal to each other, as are also the diagonally opposite polyhedral angles. If straight lines be drawn through the centres of the opposite parallel faces, they will all intersect at the same point. If a plane be passed through any two diagonally opposite edges, it will divide the solid into two equivalent triangular prisms. The volume of any parallelopipedon is equal to the product of its base and altitude. Two parallelopipedons having equivalent bases, are to each other as their altitudes, or having equal altitudes, are to each other as their bases. Generally, any two parallelopipedons are to each other as the product of their bases and altitudes.

PA-RAM'E-TER. [Gr. παραμετρεω, to measure with another thing]. A name given to a constant quantity entering the equation of a curve. The term is principally used in discussing the conic sections. In the parabola the parameter of any diameter is a third proportional to the abscissa and ordinate of any point of the curve, the abscissa and ordinate being referred to that diameter and the tangent at its vertex. In all cases the parameter of any diameter is equal to four times the distance from the focus to the vertex of the diameter. The parameter of the axis is the least possible, and is called the parameter of the curve.

In the ellipse and hyperbola, the parameter of any diameter is a third proportional to the diameter and its conjugate. The parameter of the transverse axis is the least possible, and is called the parameter of the curve. In all of the conic sections, the parameter of the curve is equal to the chord of the curve drawn through the focus, perpendicular to the axis. The parameter of a conic section and the foci, are sufficient data for constructing the curve.

PÄRT. [*pars*, *partis*, a part]. A portion

of a thing, regarded as a whole. Thus, an arc of a circle is a part of a circumference.

The term, part, is used technically to signify some particular element of a figure. Thus, in a right-angled spherical triangle, the sides adjacent to the right angle, the complement of the other two angles and the hypothenuse, are called *circular parts.*

PÄR'TIAL DIFFERENTIAL. A differential of a function of two or more variables obtained by differentiating with respect to one of the variables only. A partial differential may be of the first, or of a higher order. There are as many partial differentials of the first order, of a function, as there are independent variables, and the number increases by one for each successive order. There are two kinds of partial differentials of a higher order than the first, viz.: those obtained by differentiating successively with respect to the same variable, and those obtained by differentiating successively with reference to different variables. If

$$u = f(x, y),$$

the partial differentials of the first order are denoted by the symbols,

$$\frac{du}{dx}dx, \text{ and } \frac{du}{dy}dy.$$

Those of the second order, by

$$\frac{d^2u}{dx^2}dx^2, \ \frac{d^2u}{dxdy}dxdy, \text{ and } \frac{d^2u}{dy^2}dy^2$$

and so on, for the higher orders. The system of notation adopted, indicates the variables with respect to which the differentiations has been performed.

PÄR-TIC'U-LAR CASE. In Analysis, an extreme case, or one resulting from making some extreme supposition upon the value or relation of the arbitrary constants, which enter the equation of a magnitude. Thus, if

$$b^2 - 4ac = 0,$$

in the general equation of the second degree, between two variables, we have the *general* case of the parabola. If, in addition, we suppose

$$bd - 2ae = 0.$$

the equation represents two parallel straight lines, which is called a *particular* case of the parabola, and is to be regarded as the extreme case of the curve, or the case towards which the parabola approaches, as

$$bd - 2ae$$

grows smaller and smaller. Extreme cases are nearly the same as limits; they may be regarded as limits of species, whereas, what we term limits are generally limits, or extreme cases of individual magnitudes. In the case of cones, regarded as a species of surface, if we suppose the vertex to recede from the base, regarded as a fixed line, the cone approximates to the cylinder, and if we make the extreme supposition that the vertex is placed at an infinite distance, the cone becomes a cylinder, which is then regarded as the limit, or particular case of a cone. Almost every general case admits of a particular case or limit.

PÄR-TIC'U-LAR INTEGRAL. The integral of a differential, in which a particular value has been assigned to the arbitrary constant. In every integral, as obtained by integrating, one arbitrary condition may always be assigned; this is done by giving a particlar value to the arbitrary constant; after the particular condition has been assigned, the integral is said to fulfill the *particular* condition, and is therefore called a *particular* integral.

To illustrate: let it be required to find an expression for the area of a portion of the common parabola. We have the general formula,

$$A = \int yax,$$

which for the parabola becomes

$$A = \int \sqrt{2p}\, x^{\frac{1}{2}} dx$$

$$= \frac{2\sqrt{2p}\, x^{\frac{3}{2}}}{3} + C = \tfrac{2}{3}\, xy + C.$$

This is the indefinite integral, and expresses the area between the curve, the axis, and any two ordinates. Let it now be required to estimate the area from the principal vertex, when

$$x = 0, \quad \text{and} \quad y = 0;$$

we have,

$$A = 0$$

for that ordinate, whence,

$$0 = 0 + C, \quad \text{or,} \quad C = 0;$$

making this substitution, we have,

$$A' = \tfrac{2}{3}\, xy,$$

in which $\frac{2}{3}xy$ is the particular integral; it expresses the area between the curve and the axis of x, estimated from the *particular* ordinate drawn through the vertex up to any other ordinate.

PĂTH. The path of a point is the curve described by a point when it moves continuously according to some definite mathematical law. It is the locus of the point when moving in accordance with the same law. If a point moves, subject to the condition of remaining on a double curved surface, the path is necessarily a curved line, and is generally a curve of double curvature.

The problem of finding the shortest path between two points on a curved surface, is one belonging to a calculus of variations. Let it be required to find the shortest path between two points on the surface of a sphere. The equation of a sphere is

$$x^2 + y^2 + z^2 = R^2,$$

from which we deduce the equation of variation,

$$x\delta x + y\delta y + z\delta z = 0 \cdots\cdots (1),$$

which expresses the general relation between the variations of the co-ordinates of the point x, y, z, on the surface.

If we denote any arc of the path between two points on the surface of the sphere by s, we have the relation

$$d\left(\frac{dx}{ds}\right)\delta x + d\left(\frac{dy}{ds}\right)\delta y + d\left(\frac{dz}{ds}\right)\delta z = 0 \cdot (2),$$

which must be satisfied, in order that the path be a minimum. Combining equation (2) and (1), and eliminating δz, and substituting for dx, dy, and dz, their values, $2xdx$, $2ydy$, $2zdz$, we have

$$\left\{xd\left(\frac{dz}{ds}\right) - zd\left(\frac{dx}{ds}\right)\right\}\delta x$$
$$+ \left\{yd\left(\frac{dz}{ds}\right) - zd\left(\frac{dy}{ds}\right)\right\}\delta y = 0 \cdot (3).$$

Since δx and δy are entirely independent of each other, equation (3) requires that we should have

$$xd^2z - zd^2x = 0, \text{ and}$$
$$yd^2z - zd^2y = 0, \therefore yd^2x - xd^2y = 0.$$

Integrating these equations, we find

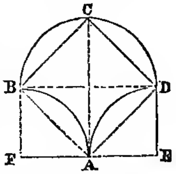
$$xdz - zdx = ads;$$
$$ydz - zdy = bds, \quad\text{and}\quad ydx - xdy = cds.$$

Multiplying both members of these equations respectively, by y, x and z, and dividing by the common factor ds, we deduce

$$ay + bx + cz = 0,$$

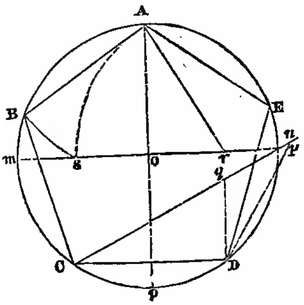
which is the equation of a plane passing through the centre, taken as the origin of co-ordinates. Hence, the shortest path is upon the arc of a great circle joining the two points. It must be the arc of a great circle, since it lies at the same time upon the surface of the sphere, and in the plane passing through the centre of the sphere. In like manner, the shortest path between any two points lying on any double curved surface may be determined.

PEL'I-COID. In Geometry, a figure of a hatchet-shaped form. The figure ABCD, included between the semi-circle BCD, and the two quadrants BA and DA, is a pelicoid. The area of the pelicoid is equivalent to the square ABCD, which is in turn equivalent to the rectangle FBDE. The perimeter of the pelicoid is equal to the circumference of the circle whose diameter is BD.



PEN'CIL OF RAYS. In Shades and Shadows, a system of rays diverging from a point. If the point is taken at an infinite distance, the rays may be regarded as parallel, and the pencil becomes a beam of rays.

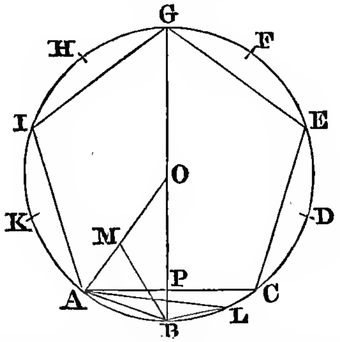
PEN'TA-GON. [Gr. πεντε, five, and γωνια, angle]. A polygon of five angles or five sides. If the sides and angles are all equal, each to each, the pentagon is regular, and



may be inscribed in a circle. To inscribe a regular pentagon in a given circle:

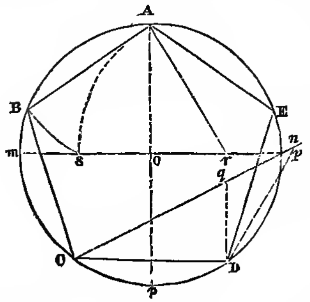
1st. Draw two diameters, Ap and nm, at right angles to each other, and bisect the radius on in r; from the point r as a centre, and with rA as a radius, describe the arc As, and from the point A as a centre, and with the distance As as a radius, describe the arc sB; join AB and it will be one side of the regular pentagon; apply AB as a chord five times from any point, and the polygon formed will be a regular inscribed pentagon.

2d. Divide the radius AO in extreme and mean ratio at the point M; take OM, the greater segment, and lay it off from A to B,



the side AB will be one side of a regular inscribed decagon, and if the alternate vertices of this polygon be joined by straight lines they will form a regular inscribed pentagon.

To construct a regular pentagon on a given line as a side: Let CD be the given side;



draw Dq perpendicular to CD at D, and equal to one half of it. Draw Cq, and produce it till gp is equal to Dg; from C and D as centres, with the radius Dp, describe arcs cutting each other at o; then with o as a centre, and a radius oC describe a circle, and apply the chord CD five times, and the figure formed will be the regular pentagon required. A pentagon may be circumscribed about a circle by drawing tangents to the circle at the vertices of a regular inscribed pentagon; these will, by their intersections, form a regular circumscribed pentagon.

PEN-TAG′ON-AL. Having five angles.

PEN′NY-WEIGHT. A unit of weight equivalent to the twentieth part of an ounce Troy.

PERCH. [L. *pertica*, a perch]. A unit of measure for surfaces, employed chiefly in measurement of land. The perch is a square rod, and is equivalent to $30\frac{1}{4}$ square yards, or $272\frac{1}{4}$ square feet. There are 160 perches in an acre.

PER′FECT NUMBER. [L. *perfectus*, complete]. A number which is equal to the sum of all its different divisors. Thus 6 is a perfect number, since

$$6 = 1 + 2 + 3;$$

and so is 28, for

$$28 = 1 + 2 + 4 + 7 + 14.$$

If the geometrical progression

$$1, 2, 4, 8, 16, \&c.,$$

be continued until the sum of the terms is a prime number, then will the product of this sum by the last term be a perfect number.

Or, the rule may be given thus: since the sum of n terms of the progression is $2^n - 1$, and the last term is 2^{n-1}, we shall have the product of the sum by the last term

$$2^{n-1}(2^n - 1),$$

and this will be perfect whenever $2^n - 1$ is prime.

If we make $n = 1$, $n = 2$, $n = 3$, $n = 5$, and $n = 7$, we find the corresponding perfect numbers 1, 6, 28, 496, and 8128. The next perfect numbers after these, in their order, are 33550336, 8589869056. 137438691328, and 2305843008139952128.

PE-RIM′E-TER. [Gr. περι, about, and μετρον, measure]. The bounding line of a plane figure. In a polygon the length of the perimeter is equal to the sum of all of the sides of the polygon. If the polygon is regular, and inscribed in a given circle, the length of the perimeter increases as the number of sides is increased, having for its limit

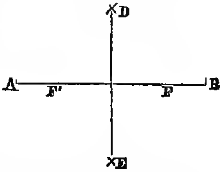
the length of the circumference of the circle. The perimeter of a plane curve is a curved line, and the length is the same as the length of the circumference.

PĒ'RI-OD. [L. *periordus*, a period]. In Extraction of Roots, a number of figures considered together. In extracting the n^{th} root of a number, we separate its digits into groups of n, each beginning at the right hand; these groups are called periods.

PE-RI-OD'IC FUNCTIONS. A function in which equal values recur in the same order, when the value of the variable is uniformly increased or diminished. Thus, the sine of x is a periodic function of x, varying from 0 to 1, as x varies from 0 to 90°; from 1 to 0, as x varies from 90° to 180°; from 0 to -1 as x varies from 180° to 270°; and from -1 to 0 as the arc varies from 270° to 360°. From 360° to 360° + 90°, the function goes through the same variations as from 0 to 90°; and in general, the function goes through the same variations from $n \times 360°$ to $n \times 360° + 90°$, as it does from 1° to 90°, and so on for the other quadrants. All the direct trigonometrical functions are periodic. The ordinate of the cycloid is a periodic function of the abscissa. There are many other periodic functions.

PER-MU-TĀ'TION. [L. *permutatio*, from *per* and *muto*, to change]. The results obtained by writing any number of factors, or letters one after another, in every possible order, so that each shall enter every result and enter it but once. To find the number of permutations of n letters, let us denote the number of permutations of $n - 1$ letters by Q; then, by introducing an n^{th} letter, it is plain that in each of the Q permutations this n^{th} letter may have n places; that is, it may be written before the first letter, between each two letters, and after the last letter, in succession, giving for each permutation n new permutations. Hence, the whole number of permutations is equal to Qn. Now, if $n = 2$, $Q = 1$; hence, the number of permutations of 2 letters is equal to $1 \cdot 2$. If $n = 3$, then from what has just been shown, $Q = 1 \cdot 2$, and the whole number of permutations is $1 \cdot 2 \cdot 3$, and so on; hence, in general, the number of permutations of n letters is equal to $1 \cdot 2 \cdot 3 \cdot 4 \cdots n$; that is, to the continued product of the natural numbers from 1 to n inclusively. If the actual product indicated in each permutation be found, it will be the same in each case. The theory of permutations is of use in deducing formulas for the number of combinations of m letters or factors, taken in sets of n; formulas which are of extensive application in deducing other formulas, and in the expression and summation of series. One of the most elementary demonstrations of the Binomial Theorem depends upon the principles of combinations.

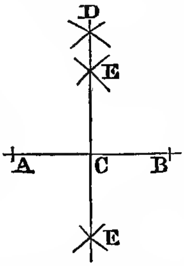
PER-PEN-DIC'U-LAR. [L. *per* and *pendeo*, to hang]. When one straight line meets another straight line, so as to make the two angles formed equal to each other, the lines are said to be perpendicular to each other. A



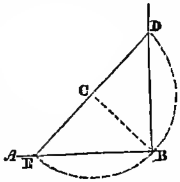
straight line is perpendicular to a plane curve when it lies in the plane of the curve, and is perpendicular to a tangent to the curve at the point of contact. Such a line is generally called a normal. A straight line is perpendicular to a plane when it is perpendicular to every straight line drawn through its foot in that plane. A straight line is perpendicular to a curved surface when it is perpendicular to a tangent plane to the surface at the point of contact. Such a line is generally called a normal line to the surface. Conversely, these magnitudes are perpendicular to the straight line under the same circumstances. A plane is perpendicular to a plane or other surface when it passes through a straight line which is perpendicular to the surface. Two curves in the same plane are perpendicular to each other at a common point of intersection of the curves, when the tangent lines to the curves at this point are perpendicular to each other. Two surfaces are perpendicular to each other at a common point of intersection, when the tangent planes to the surfaces at this point are perpendicular to each other. Two straight lines in space which do not intersect, are perpendicular to each other when a straight line drawn through any point of either one, parallel to the other, is

perpendicular to the first; or when two straight lines drawn through any point whatever, respectively parallel to the two lines, are perpendicular to each other.

To erect a line perpendicular to a given line AB, at a given point C. Lay off on each side of C convenient and equal distances CA and CB; with A and B as centres, and with a radius greater than CB, describe arcs intersecting each other at E; join E and CE by a straight line, and it will be perpendicular to AB at C. Either point E, is sufficient to determine the perpendicular, but by determining both points the correctness of the construction may be tested. A third point may be found by using the same centres as before and a different radius, which ought also to fall upon the same line, if the perpendicular is correctly determined.



If it is required to erect a perpendicular to a straight line, AB, at one extremity, B; take any point, C, as a centre, and with CB as a radius, describe a circumference of a circle, cutting the line AB in E. Draw EC, and produce it till it cuts the circumference in D; draw DB; it will be the required perpendicular. Perpendiculars to a straight line through a point, either upon, or without a given straight line, are usually drawn by the aid of a triangular ruler. See *Ruler*.



PERPENDICULAR IN PERSPECTIVE. A straight line perpendicular to the perspective plane. A perpendicular may be drawn through any point, and every such perpendicular vanishes at the centre of the picture. See *Perspective*.

PER-PE-TU'I-TY. [L. *perpetuitas*, everlasting]. In Annuities, the sum of money which will buy an annuity to last for ever. It is equal to the product of the annual value of the annuity, multiplied by the number of years it will take any sum to double at simple interest. Thus, any sum at 5 per cent. simple interest will double in 20 years, hence the value of a perpetuity of $100 per annum is $2000. In all cases, the perpetuity is equal to the annual payment multiplied by the reciprocal of the rate per cent. at which the perpetuity is computed.

PER-SPEC'TIVE. [L. *per* and *specio*, to see]. The object of perspective is to make such a representation of an object upon a surface as shall present to the eye, situated at a particular point, the same appearance that the object itself would present, were the surface removed.

Perspective consists of two parts: *First*, the accurate delineation of the principal lines of the picture: and *Second*, the shading and coloring of the picture so as to produce the desired effect of distance, &c. The first part, called *linear perspective*, is purely mathematical, and this part only will be considered. Perspective drawings may be made upon any surface, but we shall only consider them made upon a plane. The plane upon which the representation is made is called the *perspective plane*, and is generally supposed to be *vertical*. The point at which the eye is supposed to be situated is called the *point of sight*; all that part of space situated on the same side of the perspective plane with the eye, is said to be *in front* of the perspective plane, all on the other side is said to be *behind* the perspective plane.

A *visual ray* is any straight line passing through the point of sight. A *visual plane* is a plane passing through the point of sight. A *visual cone* is a cone whose vertex is at the point of sight.

The *perspective of a point*, is the point in which a visual ray through the point pierces the perspective plane; if the given point and its perspective are on the same side of the eye, the perspective is said to be *real*, if on opposite sides it is *virtual*.

The *perspective of a straight line* is the intersection of the perspective plane with the visual plane passing through the line.

The perspective of a curved line is the intersection of the perspective plane with the visual cone passing through the line.

The outline of the perspective of any body, is the intersection of the perspective plane with the enveloping visual cone; the line of

contact of this enveloping cone with the body, is called the *apparent contour* of the body. The term *cone* is here used in its most enlarged sense. It may sometimes happen that the enveloping visual surface may be pyramidal, as is the case in finding the perspective of a cube, or other polyhedron, or it may be composed of both conical and pyramidal surfaces; all of these surfaces come under the general denomination of *conical surfaces*.

The perspective of a body is generally obtained by finding the perspective of the principal lines of the body, embracing all those included within the apparent contour. The perspective of any point of a body may be found by drawing a visual ray through it and determining the point in which it pierces the perspective plane. This operation is tedious, and to shorten the process other methods have been devised, the best of which is that of diagonals and perpendiculars. The following definitions of terms are given as necessary to a complete understanding of this method:

A *perpendicular* is a straight line perpendicular to the perspective plane. A *diagonal* is a horizontal line, making an angle of 45° with the perspective plane. Through any point in space one perpendicular and two diagonals can always be drawn.

The *centre of the picture* is the point in which the perpendicular, through the point of sight, pierces the perspective plane. The *horizon* is the intersection of the perspective plane with a horizontal visual plane. It passes through the centre of the picture, and is horizontal.

The vanishing point of a line is the point in which a line drawn parallel to it, through the point of sight, pierces the perspective plane. Every system of parallel lines has the same vanishing point, which is a point common to the perspectives of all the lines of the system. The centre of the picture is the vanishing point of all perpendiculars. If a line is parallel to the perspective plane, its vanishing point is at an infinite distance. The *vanishing points of diagonals* are the points in which the diagonals, through the point of sight, pierce the perspective plane. They are in the horizon of the picture, and at distances from the centre of the picture equal to the distance from the point of sight to the perspective plane.

Magnitudes, to be put in perspective, are given by their projections, or by their distances above a horizontal visual plane, and from the perspective plane. To find the perspective of any point, draw any two lines through the point, and find their perspectives; their point of intersection is the perspective required. The most convenient auxiliary lines are the perpendicular and a diagonal through the point. To find the perspective of the perpendicular, find the point where it pierces the perspective plane, and join it by a straight line with the centre of the picture: this will be the perspective. To find the perspective of the diagonal, find the point where the diagonal pierces the perspective plane, and join it by a straight line with the proper vanishing point of diagonals; this will be the perspective of the diagonal. To ascertain the proper vanishing point of any diagonal, conceive it produced till a part of the diagonal comes in front of the perspective plane, then if this line inclines to the right, it vanishes at the right hand vanishing point of diagonals, otherwise it vanishes at the left hand one.

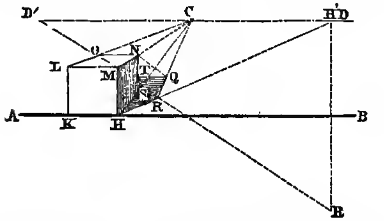
The vanishing point of rays is the point in which a ray of light through the point of sight pierces the perspective plane; the vanishing point of horizontal projections is the point in which the projection of the same ray on the horizontal plane through the point of sight intersects the horizon of the picture. These two points are in the same straight line, perpendicular to the horizon. When the former is assumed or given, the latter can be found by drawing through it a straight line perpendicular to the horizon and finding the point in which it intersects the horizon.

The shadow which any point casts upon any surface, lies upon the ray of light, and upon the projection of that ray upon the surface. Hence, to find the perspective of the shadow cast by any point upon a horizontal plane, find the perspective of the projection of the point upon the plane, and join it by a straight line with the vanishing point of horizontal projections of rays. Join the perspective of the point with the vanishing point of rays: the point in which these two lines intersect, is the perspective required. These principles are enough to find the perspective of all bodies, and the perspectives of their shadows; but, certain constructions, in par-

ticular cases, serve to facilitate the operations of finding the perspectives of bodies and of their shadows.

To illustrate the rules above given, let it be required to find the perspective of a cube and its shadow in the horizontal plane.

For convenience, take the perspective plane through the front face of the cube, and let the horizontal plane of the base be taken as the plane on which the shadow is cast, and suppose AB to be the line of intersection of

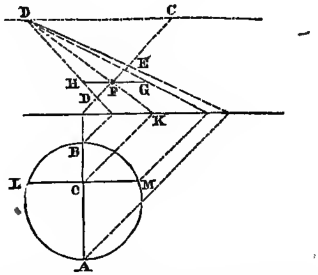


these planes. Assume DD′ parallel to AB, as the horizon of the picture. Assume C as the centre of the picture, D and D′ equally distant from C, as the vanishing points of diagonals, R as the vanishing point of rays of light, and H′ as the vanishing point of projections of rays. Construct the square HL, to represent the front face of the cube, and it will be its own perspective. The four edges that pierce the perspective plane at H, K, L and M, are perpendiculars, and their indefinite perspectives are found by drawing through these points straight lines to C. The diagonal through the back left hand upper vertex pierces the perspective plane at M, and MD′ is its perspective. The point O, in which this line intersects LC, is the perspective of the back left hand upper vertex. Through O draw ON parallel to AB, and it will be the perspective of the back upper edge of the cube; draw NS perpendicular to AB, and this will be the perspective of one of the back vertical edges of the cube: the figure HONSLK is the perspective of the cube.

To find the perspective of its shadow on the horizontal plane. Draw MR: it is the perspective of the ray through M; draw HH′; it is the perspective of the horizontal projection of the same ray, and R, their point of intersection, is the perspective of the shadow of M on the plane. Draw NR and RC, intersecting in Q; then is Q the perspective of the shadow cast by the point whose perspective is N. Draw through Q a line parallel to AB, and limited by OR. The figure HRQTSH is the perspective of the outline of the shadow cast on the horizontal plane.

The following rules for finding the perspectives of circles, are of much use in practical operations:

First. To find the perspective of a horizontal circle.



Draw a diameter AB of the circle perpendicular to the perspective plane, find its perspective DE, and bisect it in F; draw the perspective of a diagonal through F, and find the diagonal KC, of which it is the perspective; draw the chord LM, through C, parallel to AB, and find its perspective HG; then are DE and HG conjugate diameters of the ellipse of perspective, which may, therefore, be constructed by known rules.

In the figure, we have supposed the horizontal plane of projection to have been revolved so that the part behind the ground-line falls below the ground-line. The perspective will always be an ellipse, or some of its particular cases when the perspective of all its points are real; that is, when a plane, parallel to the perspective plane, through the point of sight passes entirely in front of the circle.

Second. To find the perspective of any circle whatever. Draw two tangents to it parallel to the perspective plane; then draw the diameter through their points of contact, and find its perspective: draw the perspective of a diagonal through its middle point, and find the diagonal corresponding to it; through the point in which it intersects the diameter taken, draw a chord of the circle parallel to the tangent, and find its perspec-

tive; this, with the perspective of the diameter already found, are conjugate diameters of the ellipse of perspective.

These methods, with suitable modifications, serve to find the perspectives of circles, however situated.

The principles already explained, serve to find the perspective of any body, whatever may be its form, and also the perspective of its shadow.

The principles of mathematical perspective are intimately connected with the arts of design, and a knowledge of their application is indispensable to the architect, the engraver, and the skillful mechanic. The practice of perspective is particularly necessary to the painter and the sculptor. Perspective alone enables us to represent fore-shortenings with accuracy, and its aid is required in the accurate delineation of even the simplest of natural objects.

OBLIQUE PERSPECTIVE. The perspective is said to be oblique when the perspective plane is taken obliquely to the principal face of the object delineated.

PARALLEL PERSPECTIVE. The perspective is said to be parallel when the perspective plane is taken parallel to the principal face of the object represented.

ISOMETRICAL PERSPECTIVE. See *Isometrical Projection.*

PIĒRCE. [Fr. *percer*, to penetrate]. A line is said to pierce a surface, when of three consecutive points of the line, the middle one lies in the surface, and each of the remaining two lies on opposite sides of the surface.

PILING SHOT AND SHELLS. Shot and shells are generally piled at arsenals, navy yards, &c., in regular piles of a pyramidal or wedge-shaped form. The piles are named from the form of their bases, *triangular*, *square* and *rectangular*.

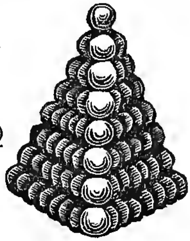
The triangular pile is made up of a succession of triangular layers, equilateral, and diminishing from bottom to top, so that the number of shot in a side of any layer shall be one less than in the layer directly below to the top layer, which consists of a single shot.

The number of balls in a complete triangular pile is equal to the sum of the series,

$1, 1+2, 1+2+3,$ &c. $\cdots$ to $1+2+3+\cdots+n$,

or,

$$1+3+6+\cdots+\frac{n(n+1)}{2}$$



The formula *for summing a series by the method of differences*, is

$$S = na + \frac{n(n-1)}{1\,.\,2}d_1 + \frac{n(n-1)(-2)}{1\,.\,2\,.\,3}d_2 + \frac{n(n-1)(n-2)n-3}{1\,.\,2\,.\,3\,.\,4}d_3 + \text{\&c.} \ldots (1).$$

Series,	1	3	6	10	15	21,	&c.
1st order of diff.,	2	3	4	5	6		&c.
2d " "	1	1	1	1			&c.
3d " "	0	0	0				&c.

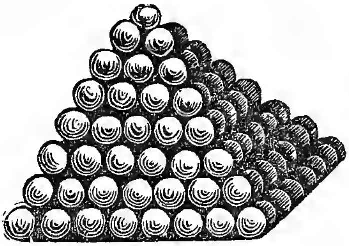
Hence,

$a=1,\ d_1=2,\ d_2=1,\ d_3=0,\ d_4=0,$ &c. . .

Substituting these in formula (1), and reducing, we have

$$S = \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3} \ldots\ldots (2).$$

The square pile is formed, as in the annexed figure.



The number of balls in the top layer is 1^2, in the next layer 2^2, in the next, 3^2, and so on. To find the number of balls in a pile of n layers, we have the series,

	1	4	9	16	25	36,	&c.
1st order of diff.,	3	5	7	9	11		&c.
2d " "	2	2	2	2			&c.
3d " "	0	0	0				&c.

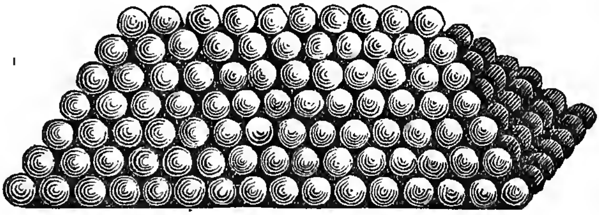
Hence,

$a=1,\ d_1=3,\ d_2=2,\ d_3=0,\ d_4=0,$ &c.

Substituting these in formula (1) and reducing, we have,

$$S = \frac{n(n+1)(2n+1)}{1.2.3} \cdots\cdots (3).$$

The rectangular pile is formed as in the annexed figure. The top layer contains $(m+1)$ balls, the second layer contains $2(m+2)$, the third, $3(m+3)$, and so on. To find a formula for the number of balls in a complete rectangular pile, we have the series



	$1.(m+1)$,	$2(m+2)$, $3(m+3)$,	$4(m+4)$	&c.
1st ord. diff.,	$m+3$,	$m+5$,	$m+7$,	&c.
2d " "		2	2	&c.
3d " "		0		&c.

Hence,

$a = m+1$, $d_1 = (m+3)$, $d_2 = 2$, $d_3 = 0$, $d_4 = 0$, &c.

Substituting in formula (1) and reducing, we have,

$$S = \frac{n(n+1)(1+2n+3m)}{1.2.3} \cdots (4).$$

In order to find the number of balls in an incomplete pile, compute the number that the pile would contain if complete, and the number required to complete it; the difference of these two numbers is the number of balls in the pile.

Formulas (2), (3) and (4), may be written thus,

Triangular, $S = \frac{1}{3} \cdot \frac{n(n+1)}{2}(n+1+1) \cdots (2)$,

Square, $S = \frac{1}{3} \cdot \frac{n(n+1)}{2}(n+n+1) \cdots (3)$,

Rectangu'r, $S = \frac{1}{3} \cdot \frac{n(n+1)}{2}\big((n+m)+(n+m)+(m+1)\big) \cdots (4)$.

Now, since

$$\frac{n(n+1)}{2}$$

is the number of balls in the triangular face of each pile, and the next factor in each case denotes the number of balls in the longest side of the base, *plus* the number in the sides of the base opposite, *plus* the number in the top parallel row, we have the following practical rule for finding the number of balls in any pile.

Add to the number of balls in the longest side of the base the number in the parallel side opposite, and also the number in the parallel top row; multiply this sum by one-third of the number of balls in the triangular face of the pile, and the result will be the number of balls in the pile.

This rule is easy to remember, and is equally applicable to each of the three forms; it is often called the *workman's rule.*

Where space is an object, the rectangular pile is preferable to either of the others, and the longer the pile, the greater the number of balls that can be piled upon a given area, having a given breadth. One long pile is more economical of space than two or more short ones. The square pile occupies most space for the number of balls contained in it.

PĪNT. A unit of measure of capacity, equivalent to one-eighth of a gallon, or about $39\frac{5}{8}$ cubic inches. See *Measures.*

PLAN. In Descriptive Geometry and Surveying, a representation of the horizontal projection of a body. The plan of an object is the same as its horizontal projection. The term is particularly applied to architectural drawings.

PLĀNE. [L. *planus*, even, flat]. A surface such that, if any two points be taken at pleasure and joined by a straight line, that line will lie wholly in the surface. A plane is supposed to extend indefinitely in all directions. A plane may be generated by a straight line moving in such a manner as to touch a given straight line, and continue parallel to its first position. A plane may

also be generated by revolving one straight line about another straight line, perpendicular to it, as an axis of revolution. The rectilineal equation of a plane may be reduced to the form,

$$z = cx + dy + g,$$

in which x, y and z, are the co-ordinates of every point of the surface, and c, d and g, constants. The plane is given when c, d and g are known, and may be constructed by points, as follows: Assume any two values for x and y, and substitute them in the equation; there will result a corresponding value for z, which with the assumed value of x and y will be the co-ordinates of a point that may be constructed by known principles. In like manner, any number of points may be found and constructed; a surface passed through them will be the surface required; three points are sufficient to fix the position of a plane provided they are not in the same straight line.

Planes are generally constructed by finding the points in which they cut the co-ordinate axes, and then passing a plane through these three points. Planes are given in Descriptive Geometry by their traces, that is, by their intersections with the planes of projections; they may, in like manner, be determined analytically. To find the equation of the trace of a plane upon the planes XY, XZ and YZ respectively: make in the equation of the plane, z, y and x, respectively equal to 0 in the equation; the resulting equations will be the equations of the required traces. Two traces will be sufficient to fix the position of a plane.

If we take two planes, whose equations are

$$z=cx+dy+g, \quad \text{and} \quad z=c'x+d'y+g',$$

they will be parallel when

$$c = c' \quad \text{and} \quad d = d'.$$

They will be perpendicular to each other, when

$$1 + cc' + dd' = 0.$$

In general, the angle which they make with each other, may be determined by means of the formula,

$$\cos V = \frac{1 + cc' + dd'}{\sqrt{1 + c^2 + d^2}\sqrt{1 + c'^2 + d'^2}}.$$

To find the line of intersection of two planes, combine their equations and eliminate one variable; the resulting equation will be the equation of the projection of their intersection on the plane of the other two. Combine the equations again, eliminating a second variable, and the resulting equation will be that of the projection of the line of intersection upon a second plane, and these will be sufficient to determine the line of intersection.

If the equation of a plane is

$$z = cx + dy + g,$$

and the equations of a straight line are

$$x = az + a, \quad \text{and} \quad y = bz + \beta,$$

the line and plane will be parallel when

$$1 - ac - bd = 0;$$

they will be at right angles when

$$a = -c \quad \text{and} \quad b = -d;$$

and in general, the angle which they make with each other is given by the formula

$$\sin A = \frac{1 - ac - bd}{\sqrt{1 + a^2 + b^2}\sqrt{1 + c^2 + d^2}}.$$

To find the point in which a line pierces a plane, combine their equations and find the corresponding values of the *variables*; these will be the co-ordinates of the required point.

PLANE ANGLE. A portion of a plane lying between two straight lines, meeting at a point. The lines are called *sides* of the angle, and their common point is the *vertex*. See *Angle*.

PLANE CHART. A chart constructed so that the parallels of latitude and longitude are represented by straight lines parallel to each other, and at the same distance from each other, in every latitude.

PLANE CURVE. A curve all of whose points lie in the same plane.

PLANE DIRECTOR. A plane parallel to every element of a warped surface of the first class. See *Warped Surface*.

PLANE FIGURE. A portion of a plane limited by lines either straight or curved. When the bounding lines are straight, the figure is *rectilinear*, and is called a *polygon*. When they are curved, the figure is *curvilinear*.

PLANE GEOMETRY. That part of Geometry which treats of the relations and properties of plane figures.

HORIZONTAL PLANE. A plane parallel to the surface of still water, or, parallel to a

tangent plane to the earth's surface at the place. In Perspective the term implies a horizontal plane passing through the point of sight. See *Descriptive Geometry.*

Oblique Plane. A plane making an oblique angle with a horizontal plane.

Objective Plane. In Surveying, the horizontal plane upon which the object to be delineated is supposed to stand. It is usually taken as the horizontal plane of projection.

Plane of a Dial. The plane upon which the hour lines of the dial are constructed. See *Dial.*

Plane of Projection. One of the planes to which points are referred in descriptive geometry for the purpose of determining their relative position in space. See *Descriptive Geometry.*

Plane of Rays. In Shades and Shadows, a plane parallel to a ray of light.

Perspective Plane. The plane upon which the perspective of an object is drawn. See *Perspective.*

Principal Plane. In Spherical Projections, the plane upon which the projection of the different circles of the sphere are projected. It is generally taken through the centre of the sphere to be projected.

Plane Problem. A problem which can be solved geometrically, by the aid of the right line and circle only.

Plane Sailing. The method of computing the position of a ship and her path, under the supposition that the surface of the earth is a plane. See *Navigation.*

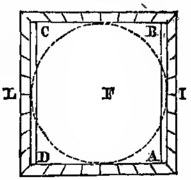
Plane Scale. A scale upon which are graduated chords, sines, tangents, secants, rhumbs, geographical miles, &c. The scale is principally used by navigators in their computations, in plotting their courses, &c.

Plane Surveying. That branch of Surveying in which the curvature of the earth's surface is not taken into consideration. In this branch the surface of the earth is regarded as a plane. Such is the ordinary field and topographical surveying, where only very limited portions of the earth's surface are considered. See *Surveying.*

Plane Table. An instrument used in surveying for plotting in the field without the necessity of taking field notes. It is particularly employed in the filling in of a trigonometrical survey; it is also of some use in land surveying. Where exactness is required, the plane table is of little value, but in making approximate sketches, it commends itself on account of the rapidity with which operations can be carried on.

The plane table consists of a square board or limb, mounted upon a tripod. Two leveling plates are attached, one to the tripod and the other to the limb, and are connected by a ball and socket joint. Four leveling screws, working through one leveling plate and against the other, serve to regulate the lateral motions of the table with respect to the axis of the instrument. The limb may be moved in azimuth around the axis of the instrument, which motion may be checked by a clamp screw; small motions in azimuth are then communicated by a tangent screw. The limb of the instrument is made horizontal by the aid of a small detached spirit level, by laying it over two of the leveling screws, and bringing the bubble to the centre, then placing it over the other two, and again bringing the bubble to the centre.

The upper face of the limb is bordered by a brass plate about an inch in width, and its centre is marked by a steel pin, F. The perimeter of the limb is graduated to degrees and fractions of a degree, as follows; suppose a circle to be described with F as a centre, and tangent to the



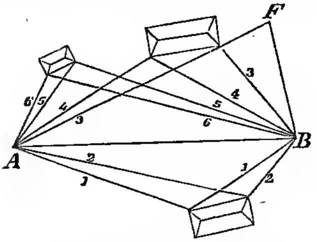
sides of the brass plate. Let the circle be graduated to the required unit, and then suppose straight lines to be drawn from the centre, F, through these points of division; the intersection of these lines with the brass plate are marked and numbered from the point I through 180° around to L, and from L through 180° around to I again. In some plane tables the numbering is from 0° to 360°. There are generally diagonal scales of equal parts, DC and AB cut in the plates. *Used in plotting.* Near the outer edges of the limb, two small grooves are made to receive two plates of brass, DC and AB, which are drawn to their places by means of milled-headed screws, which pass through the table from the

under side and screw firmly into the plates. The object of these plates is to secure the paper on which the drawing is to be made. By loosening the screws, and pushing up the plates, the paper may be introduced; then, by turning the screws back again, the plates are drawn down, and the paper is held tightly. The paper might be slightly moistened, which would secure a smoother surface when it is dried. A ruler accompanies the table, with two sights like compass sights, or sometimes with a telescope, in their stead. One edge of the ruler is beveled, and this edge is so placed that it is in the plane of the openings through the sights. The sights are constructed so as to fold down for convenience in carriage. A compass is sometimes attached for determining the bearings of lines.

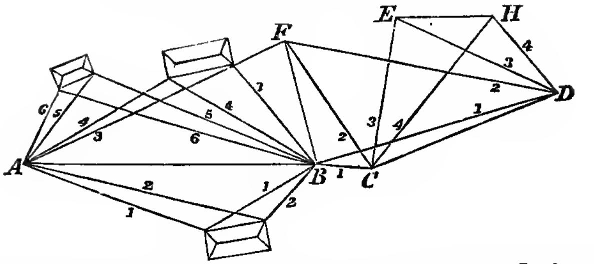
The plane table, as described, is used for two distinct purposes; 1st. To measure horizontal angles; and 2d, For determining the shorter lines of a survey, both in extent and position.

To measure a horizontal angle. Place, by means of a plumb line, the centre of the table exactly over the angular point; then level the table and clamp the limb; after which place the ruler with the sights raised so that the beveled edge shall rest against the steel pin at the centre; direct the sights to the left hand object, and read the reading at each end of the ruler, and take a mean of the results for the first reading. Then direct the sights o the right hand object, and take the second reading in like manner. If the ruler has not passed over the 0° point of the limb, the excess of the second reading over the first is the value of the required angle. If the ruler has passed the 0° point, the first reading must be subtracted from 180° and the difference added to the second reading; the sum will be the value of the angle required.

To determine lines in extent and position. Having fastened a sheet of paper on the table, examine the lines and objects which are to be determined in position and select for a base a convenient line which is connected with some point of the triangulation, taking care that as many prominent objects as possible may be seen from its two extremities. Then place the plane table over one extremity of the base, so that the point on the paper corresponding may be exactly over the extreme point of the base. Clamp the limb and make it truly horizontal. Mark the point corresponding to the end of the base by a needle, and pressing the ruler against the needle, direct the sights to the other extremity of the base. With a fine-pointed pencil, draw a straight line along the beveled edge of the ruler, and lay off on it, from a scale of equal parts, the length of the base, and mark the second point; then direct the sights, in succession, to all the principal objects that are visible from the first station, and draw pencil lines along the edge of the ruler. Next plant the plane table so that the second end of the plotted base line shall be over the second end of the base line in the field; level the instrument, and having placed the needle at the second end of the base line, bring the beveled edge of the ruler to coincide with the plotted base, and then turn the limb of the instrument till the sights are directed to the first end of the base; clamp the limb and direct the sights, in succession, to every object sighted from the first station, marking the points of intersection of these lines with the corresponding ones from the first station. To illustrate: let it be required to determine the relative position of several



houses. From station A, and on one of the line of the triangulation, as AB, measure the base line AB, which we will suppose equal to 300 yards. Place the plane table over A, and sight to the corners of the houses, and mark the lines 1, 2, 3, 4, &c. Then move the table to B, place the plotted line AB in the direction from B to A, and sight to the same corners as before. and draw the lines as in the figure; the points at which they intersect the corresponding lines before drawn, determine the plot of the corners of the houses; the front lines of the houses may then be drawn on the paper, and upon these the plots of the houses themselves may be con-



structed. In like manner the plot in the next figure is constructed.

When one sheet of paper is filled, and there is yet more work to be done, let the paper be removed and another sheet substituted in its stead, after which the table may be used as before. In order that the two sheets may be put together, and form one entire plan, it is necessary that two points that were determined on the first sheet should also be determined on the second sheet; then by placing the line joining these points in the two sheets one upon the other, all the lines in the two sheets will have the same relative position as the corresponding lines in the field, and so on for as many sheets as it may be desirable to use. In the example, the two points F and B have been determined on each sheet, and the two parts, when placed as shown in the figure, make up a continuous plan.

The plane table is much used in surveying for plotting coast line, the prominent points being determined by it, and the minor details sketched in by the eye.

Plane Triangle. A triangle lying entirely in the same plane.

Plane Trigonometry. That part of trigonometry which treats of the relations and properties of the sides and angles of plane triangles.

Vertical Plane. A plane perpendicular to the horizon, or to a horizontal plane. In perspective it is the vertical plane passing through the point of sight and perpendicular to the perspective plane.

PLA-NIM'E-TRY. [L. *planus*, plane, and Gr. μετρον, measure]. That branch of applied geometry which treats of the measurement of plane areas. The term is used in contradistinction to stereotomy, which treats of the measurement of volumes. See *Mensuration*, *Surveying*, &c.

PLAN'I-SPHERE. [L. *planus*, plane and sphere]. A projection of the various circles of the sphere upon a plane. See *Spherical Projections*.

Platonic Bodies. So called from Plato, who treated of them. They are the five regular polyhedrons, viz.: the *tetrahedron*, *hexahedron*, *octahedron*, *dodecahedron*, and the *icoshedron*. See *Polyhedron regular*.

PLOT. A drawing of the projections of the prominent objects of a portion of the earth's surface upon a plane, in which the lines are proportional to the corresponding lines in nature.

PLOT'TING. The operation of representing the lines of a survey upon paper drawn to a scale, and bearing to each other the proper relative positions and lengths. There are various methods of plotting; some of the principal ones only will be considered.

To plot a field survey from the field notes.

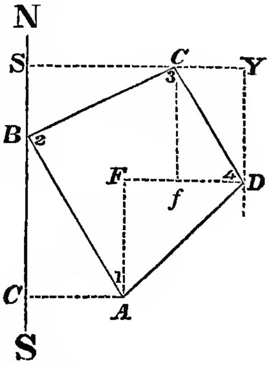
1*st method:*

Field-notes of a Survey.

Sta.	Bearings.	Dist.	Lat.	Dep.
1	$N 31\frac{1}{2}° W$	10.	+ 8.71	− 5.24
*2	$N 62\frac{3}{4}\ E$	9.25	+ 4.40	+ 8.21
3	$S\ 36\ \ E$	7.60	− 6.01	+ 4.46
4	$S\ 45\frac{1}{2}\ W$	10.40	− 7.10	− 7.43

Select some station, as the principal station (in this instance, the second), and mark it with a star, in the field-notes. Draw any line, as NS, to represent the meridian passing through the principal station, and on it take any point B, to represent that station. From a scale of equal parts, lay off, on NS towards N, a distance BS equal to 4.40, and at S erect a perpendicular to NS, and from the same scale lay off SC equal to 8.21; draw BC, and it will be the plot of the first course from B. Through C draw Cf parallel to NS,

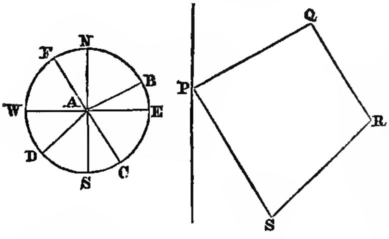
and lay off Cf equal to -6.01, and at f erect the perpendicular fD, making it equal to 4.46; draw CD: it is the plot of the second course. Plot, in the same manner, the remaining courses.



2d method:

With a protractor lay off the angle NBC, equal to $62\frac{3}{4}°$, and on the line BC lay off, from the scale of equal parts, BC equal to 9.25. Through C draw the line Cf parallel to NS, and at C construct the angle fCD equal to 36°, and make CD equal to 7.60. In the same manner, plot the remaining courses. Or, the angle at each point, as C, between two adjacent sides, may be computed and laid off, and the plot continued, as before.

Both of these methods of plotting are liable to error, particularly the latter; and an error, once committed, is carried on through the work, to be swelled by the continual errors made at each step of the operation. To avoid, as much as possible, this accumulation of error, a third method is adopted.



3d method:

With any radius, AE, describe a circle, and draw two diameters, NS and EW, at right angles to each other. Then construct the angles

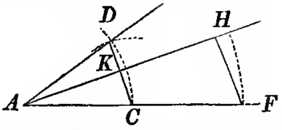
NAB $= 62\frac{3}{4}°$, SAC $= 36$, SAD $= 45\frac{1}{2}°$,
and NAF $= 31\frac{1}{2}°$.

Through P, any assumed point, draw PQ parallel to AB, and equal to 9.25; through Q draw QR parallel to AC, and equal to 7.60; through R draw RS parallel to AD, and equal to 10.40; and through S draw SP parallel to AE, and equal to 10: the work, if right, ought to close; the last end of the last line falling upon the first end of the first line.

This method of laying off angles admits of great accuracy. The angles may be laid off with a circular protractor; or, what is still better, they may be determined by the

Method of Chords.

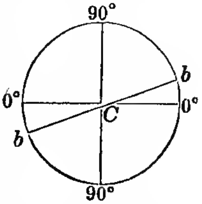
The principle of this method is, that the chord of a given arc is equal to the sine of half the arc with double the radius.



Let DAF be any given angle, and AH a line bisecting it. Let DC be the chord of the arc CD described with a given radius, and HF, parallel to CD, the sine of half the given angle to a radius AF equal to 2AC. Then, from the similar triangles of the figure, since AF = 2AC, we have

$$HF = 2KC = CD.$$

To apply this principle: let a circle be described upon some convenient portion of the paper with a radius of five units of the scale, say 5 inches, and draw two diameters at right angles to each other, taking care to make one one of them parallel to the meridian, base line, or other prominent line of the survey. Now, if we regard the radius of the table of natural sines as 1 ten, say ten inches, the number



of inches in the length of the sine of any arc will be found by removing the decimal point one place to the right. To lay off any angle, say 14° 29′: half the angle is 7° 14′ 30″, and the natural sine of this angle, as found in the tables, is 0.126005; or,

to the radius 10 inches, it is $1^{in}.26$. Take $1^{in}.26$ in the dividers, and with 0, as a centre, describe an arc cutting the circle in b; from the opposite end of the same diameter 0, describe a second arc cutting the circle in b; draw bb: it ought to pass through the centre of the circle, and if it does so, it will make with 00 an angle equal to 14° 29′. In the same manner, lines may be constructed, making any given angle with a given line. After one circle has become defaced by continued constructions, a new circle may be taken on some other portion of the plot; but care must be taken to make the new 00 line parallel to the primitive one.

It is plain that we might employ the points 90° as centres, by using the complements of angles instead of the angles themselves.

4th method:

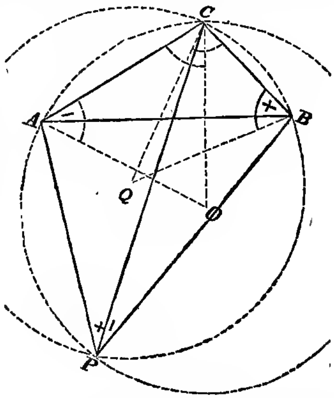
The fourth method is that of rectangular co-ordinates. Draw, in the plane of the plot, two straight lines at right angles to each other, taking one through some prominent point, and parallel to the meridian or some principal line of the survey. Then, from the field-notes, compute the distance of each point to be plotted from these right lines; then, any point may be constructed by laying off, on each axis, a distance equal to the distance of the point from the other axis, and drawing through the extremities of these distances lines parallel to the axes, respectively: the point in which these intersect, is the plot of the required point. This method of plotting is applicable to the projection of maps of a portion of the surface of the earth, sufficiently great to take into account the curvature. See *Projection of Maps*.

To plot a survey made by means of offsets:

Plot the main lines by one of the preceding methods, and lay off on them distances corresponding to the offsets. At each point, thus determined, erect a perpendicular to the plotted course, and on it lay off a distance equal to the offset: the points, thus determined, are points of the required plot. The plotting-scale is a useful instrument in plotting by offsets. See *Plotting-Scale*.

To plot the problem of the three points:

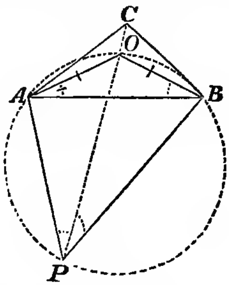
Let A, B, and C, be three points whose positions are absolutely determined and plotted upon paper, and suppose that the angles APC $= a$, CPB $= b$, and APB $= a + b$, have been measured from a fourth point P, to find the position of the point P upon the plot



of the survey. Construct the angles ACO and CAO, each equal to $90° - a$, and with the point O, as a centre, and radius OA, describe a circumference of a circle. Then construct the angles CBO′ and BCO′, each equal to $90° - b$ and with O′ as a centre, and O′B as a radius, describe a second circumference intersecting the first in P. Then is P the plot of the required point.

This method is used for locating buoys, and sounding points upon a harbor; P, representing the position of a boat, at the required point, and A, B, C, fixed points on shore, as light-houses, spires, &c.

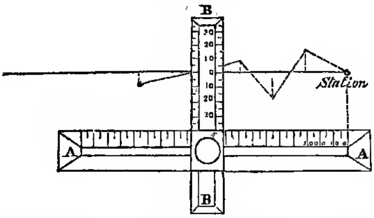
A second method of plotting the problem of the three points, is the following. The data being the same as before, construct the



angle ABO equal to APC, or a, and the angle BAO equal to BPC, or b. Pass a circle through the three points, O, A and B. Draw the line CO, cutting the circumference

in P: then will P be the plot of the point required. When the point O falls near C, the first construction will be the most accurate. If the auxiliary circle passes through C, the problem is indeterminate. The plot may also be made by means of the station pointer. See *Station Pointer*.

PLOT′TING SCALE. The plotting scale consists of two ivory scales at right angles to each other, as shown in the figure. The



longer scale, AA, contains a slit, or dovetail groove, nearly its whole length, in which slides a button carrying the cross scale, BB. The scale AA is graduated into equal parts, numbered from 0 to the limit of the scale, beginning at one end of the groove. The portion of the cross scale, projecting beyond AA, is divided into two equal parts, by a line parallel to AA, and these points are graduated into parts equal to those on the main scale, AA, and numbered from the middle division each way to the limits of the scale.

To use the scale in plotting, lay it upon the paper so that the edge of the scale, AA, shall be parallel to the plot of the line from which the offsets are taken, and so that the 0 point of the scale shall fall in a line through the beginning of this line, perpendicular to it, and at such a distance from it, that the 0 point of the cross scale shall fall upon the station line. The scale is held fast in this position by small points of needles, projecting from the lower side of the scale.

Then slide the button and cross scale along till the edge of the scale, BB, indicates upon AA, the distance of the first offset from the beginning of the line. Then mark with a point the extremity of the offset as shown by the cross scale. This extremity will sometimes fall above and sometimes below the station line. Having found, in this manner, a sufficient number of points, draw a line through them, and it will be the plot of the line required. This method of plotting is of use in plotting the course of a crooked stream, or a winding line of coast. The plotting scale has other applications which will readily suggest themselves to the experienced draughtsman.

PLUMB′-LĪNE. [L. *plumbum*, lead]. A line, or string, having a heavy piece of lead attached to one extremity, used for the purpose of fixing the axis of an instrument exactly over a given point. In the axis of the instrument, as the theodolite, for example. and at the lower part, a hook is fastened, over which the thread of the plumb-line is passed, and the piece of lead, generally of a conical shape, is suffered to swing freely. When the bob of the plumb-line settles, the direction of the string indicates the vertical through the hook, and by moving the instrument backwards and forwards, the point of the bob may be brought exactly to coincide with the given point of the surface of the earth. The plumb-line is also used in the carpenter's and stone-layer's level. See *Level*.

POINT. [L. *punctun*, from *pungo*, to prick]. A point is that which has position only, but neither length, breadth, or thickness. The extremities of a limited line are points. That which separates two adjacent parts of a line is a point.

Conjugate Point of a Curve. A point of a curve which has no consecutive points, that is, a point whose co-ordinates satisfy the equation of the curve, but those of its consecutive points do not. Conjugate points arise from oval branches reducing to points. The analytical test of a conjugate point, is that the first differential co-efficient of the ordinate of the curve, taken at the point, is imaginary. Hence, to find the conjugate points of a curve, differentiate its equation and find an expression for the differential co-efficient of the ordinate, and seek for those values of the variables that will, at the same time, satisfy the equation of the curve and render the differential co-efficient imaginary: each pair of values that will satisfy these conditions will correspond to a conjugate point. See *Singular Points*.

Point of Concurrence. A point common to two lines, but not a point of tangency, or a point of intersection. Such, for instance, is

the point in which a cycloid meets its base. In this case, the point of concurrence is also a cusp point.

POINT OF CONTACT. The point of a given line at which tangency takes place. The union of the curve and tangent at that point is so intimate, that they may be regarded as coinciding, for an infinitely short distance, in the immediate neighborhood of this point.

The point of contact is such a point, that if any secant be drawn through it, and then be revolved about this point till the second point of secancy unites with it, the secant will become a tangent.

The point of contact of two curved lines, is a point common to the two lines, at which, if a straight line be drawn tangent to one of the lines, it will be tangent to the other also.

The point of contact of two surfaces, is a point such that if any number of secant planes be passed through this point, the sections cut out of one surface will be respectively tangent to the sections cut out of the other; or, if a plane be passed tangent to one surface at the point, it will also be tangent to the other surface at the same point.

POINT OF CONTRARY FLEXURE, or, POINT OF INFLEXION. A point at which a curve, from being convex towards a line not passing through it, becomes concave towards the same line, or the reverse. See *Inflexion: Singular Points.*

POINT OF INTERSECTION. A point in which two lines cross each other. See *Intersection.*

POINT IN PERSPECTIVE. Is used technically to designate some of the principal positions connected with the perspective of an object: for which, see *Perspective.*

POINT OF SIGHT. The point at which, if the eye be placed, the picture will present the same appearance as the object itself would, were the picture removed. This is sometimes called the point of view. See *Perspective.*

The point of sight, sometimes means the point from which the object is actually viewed, to have the appearance of the picture.

POINTS OF THE COMPASS. In Navigation, the 32 points of division of the compass-card in the Mariner's Compass. The angular space between two consecutive points is 11° 15′, and each space is sub-divided into half and quarter points. The 4 principal points are called cardinal points, and are lettered on the card, N., S., E., W., the initial letters of North, South, East, and West. See *Navigation.*

PŌ'LAR CIRCLES. Two circles of latitude, whose planes pass through the poles of the ecliptic. They are about 23° 28′ from the poles, and limit the regions on the surface of the earth, in which the sun never sets for a portion of the year, and never rises for an equal portion: the length of duration of this phenomenon increases as the place is nearer to the pole, where there is six months of light, and 6 months of darkness. The circle nearest the north pole, is called the *Arctic Circle;* that nearest the south pole, the *Antarctic Circle.*

POLAR CO-ORDINATES. Elements of reference, by means of which points are referred to a system of polar co-ordinates. In a plane system, these elements consist of a variable angle and a variable distance called the radius vector. In space, they consist of two variable angles and a variable right line, still called the radius vector. See *Polar System.*

POLAR DIAL. A dial whose plane is parallel to a great circle passing through the poles of the earth. It is generally perpendicular to the meridian of the place. The style is then parallel to the plane of the dial, and the hour lines will therefore be parallel to each other. At the equator it is a horizontal dial.

POLAR DISTANCE. The distance of the circle of a sphere from its pole, estimated on the arc of a great circle of the sphere passing through the pole of the circle. This distance is generally expressed in degrees.

POLAR DISTANCE of a point on the surface of a sphere, is the distance of the point from the pole of the sphere, measured on the arc of a great circle passing through the point and the pole. Every circle, and every point on the surface of a sphere, has two polar distances, and unless the contrary is expressed, the lesser one is understood.

POLAR EQUATION of a line, or surface; an equation which expresses the relation between the polar co-ordinates of every point of the line or surface. To find the polar equation of any magnitude, substitute in the rectangular equation of the magnitude, for

the variables, their values taken from the formulas for passing from a rectangular to a polar system; the resulting equation will be the polar equation required.

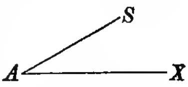
Polar Line of a Point, in the plane of a conic section, a line such that if from any point of it two straight lines be drawn tangent to the conic section, the straight line joining the points of contact, will pass through the given point, which is called a *pole*. The terms polar line and pole are correlative, and neither has any signification except with regard to the other. Every straight line in the plane of a conic section, is a polar line with respect to some point. The polar line of a point is always parallel to a tangent to the curve at the vertex of the diameter drawn through that point. The polar lines of all points on the same diameter, are parallel to each other. The point in which any polar line intersects the diameter through the pole, is called the polar point. The polar point and pole lie on opposite sides of the curve, but on the same side of the centre. If the pole approach the curve, the polar point also approaches it; and, conversely, if the polar point approaches the curve, the pole also approaches, and the two meet and pass each other on the curve. When the pole is at the centre, the polar point is at an infinite distance, and when the pole is at an infinite distance, the polar point is at the centre. These two points are conjugate to each other.

To construct the polar line of any point in the plane of a conic section. Draw a diameter through the point, and draw a tangent to the curve at its vertex; then, if the given point lies within the curve, draw a chord parallel to the tangent through the given point, and at the point in which it intersects the curve, draw a tangent to the curve; the point in which it cuts the diameter, before drawn, is the *polar* point; through this point, draw a straight line parallel to the tangent at the vertex of the diameter; it will be the polar line required: if the given point lies without the curve, draw a tangent to the curve through the point, and through the point of contact draw a chord parallel to the tangent at the vertex of the diameter, and produce it indefinitely; this is the polar line required.

In the case of the parabola, the polar point and the pole are always at the same distance from the vertex of the diameter drawn through them. In all the conic sections the directrix is the polar line of the focus. From the preceding discussion, it would seem that there is no polar line to points when the diameter passing through them does not intersect the curve; but the exception is only apparent, for, in that case, the polar line is parallel to the tangent to the conjugate of the given hyperbola at the vertex of the diameter through the point of contact. Since there are an infinite number of diameters passing through the centre of a conic section, it follows that the centre has an infinite number of polar lines, all of which must necessarily be at an infinite distance.

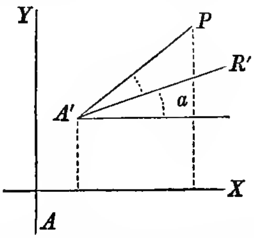
Polar Projection of the Sphere. A projection of the circles of the sphere on the plane of one of the polar circles. This projection is employed in connection with Mercator's, to represent the region of the earth about the polar circles. Charts of the polar regions are generally constructed in accordance with the principles of this projection for nautical purposes. See *Projection*.

Polar System of Co-ordinates in a Plane. A system of co-ordinates in which points are referred to a fixed straight line, and a fixed point of that line by means of a variable angle and a variable distance. Let AX be a fixed straight line, called the *initial* line, A a fixed point of that line, called the *pole*, and S any point whatever in the plane SAX; then will the position of the point S be completely given with respect to the system when the angle SAX is given, and the distance AS is known. The first element of reference is denoted by v, and is estimated from the initial line AX around in a direction contrary to the motion of the hands of a watch through any number of degrees. The second element of reference is denoted by r, and is estimated from A outwards to any limit whatever, but it can never be negative.



Let AX and AY be two axes of co-ordinates at right angles to each other. Let A′R′ be the initial line, A′ the pole of any polar system in the same plane, and P any point in that plane. Denote the co-ordinates

of the point referred to the rectangular system by x and y, the polar co-ordinates of the



same point by v and r, the co-ordinates of the pole by a' and b', and the angle, which the initial line makes with the axis of X, by a; then from the figure we shall have

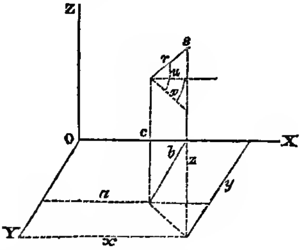
$$x=a'+r\cos(v+a),\quad y=b+r\sin(v+a).$$

These are the formulas for passing from a rectangular system to a polar system in the same plane. It is customary to take the initial line parallel to the axis of X, in which case, the formulas become

$$x = a' + r\cos v,\quad y = b' + r\sin v.$$

A polar system, in space, is a system in which points are referred to a fixed plane, a fixed line of that plane, and a fixed point of that line, by means of two variable angles, and a variable distance. The fixed plane is called the *initial plane*, the fixed line is the *initial line*. and the fixed point is the *pole*.

In passing from a rectangular system to a polar system in space, the initial plane is taken parallel to the plane XY, and the initial line parallel to the axis of X. If we denote the rectangular co-ordinates of any point



in space by x, y and z, the co-ordinates of the pole by a, b, and c, the radius vector of any point S by r, the angle which it makes with the initial plane by u, and the angle which its projection on the initial plane makes with the initial line by v, we shall have, from the figure,

$$x = a + r\cos u\cos v,\quad y = b + r\cos u\sin v,$$
$$z = c + r\sin u.$$

These are formulas for passing from a rectangular system in space to a polar system in space.

PŌLE. A unit of measure, equivalent to $16\frac{1}{2}$ feet, or $5\frac{1}{2}$ yards. It is called a rod, and is principally used in land surveying.

In a polar system of co-ordinates, the point from which the radius vector of any point is estimated.

Pole of a Polar Line. A point in the plane of a conic section, such that if any straight line be drawn through it, cutting the curve in two points, and tangents be drawn to the curve at these points, they will intersect each other on the given line. See *Polar Line.*

Poles of a Circle of a Sphere. The points in which a diameter of the sphere, perpendicular to the plane of the circle, pierces the surface of the sphere.

Poles of a Sphere. The two points in which the axis of the sphere pierces the surface. See *Axis.*

POL'Y-GON. [Gr. πολυς, many, and γωνια, angle]. A portion of a plane bounded on all sides by limited straight lines. These lines are called sides of the polygon, and the points in which they meet are called vertices of the polygon.

Polygons are classified according to the number of their sides or angles. Those of three sides are called *triangles;* those of four sides, *quadrilaterals:* those of five sides, *pentagons;* those of six sides, *hexagons;* those of seven sides, *heptagons;* those of eight sides, *octagons;* those of nine sides, *nonagons;* those of ten sides, *decagons;* those of eleven sides, *undecagons;* those of twelve sides, *dodecagons;* and so on. Polygons having all their sides equal to each other are called *equilateral;* those having all their angles equal to each other, are called *equiangular.* Polygons which are both *equilateral* and *equiangular* are called *regular* polygons. Regular polygons may have any number of sides. When the number of sides is infinite, a regular polygon becomes a circle; hence, the circle is the *superior* limit of

regular polygons. The regular triangle is the inferior limit of regular polygons. Of regular polygons inscribed in the same circle, the greater the number of sides the greater the perimeter. Of regular polygons circumscribed about the same circles, the less the number of sides the greater the perimeter. The perimeter of a regular circumscribed triangle is equal to

$$r \times 10.39236\,;$$

that of a regular inscribed triangle is

$$r \times 5.19618\,;$$

and the perimeters of all regular polygons, whether inscribed or circumscribed, lie between these limits.

The following properties are common to all convex polygons:

1. Every polygon may be divided into as many triangles as it has sides, less two.

2. The sum of the interior angles of every polygon is equal to two right angles, taken as many times as the polygon has sides, less two.

3. If all the sides be prolonged in the same direction, the sum of the exterior angles thus formed, is equal to four right angles.

The following are some of the properties of regular polygons:

1. Every regular polygon may have a circle inscribed within it, and another circumscribed about it.

2. If at the vertices of a regular inscribed polygon, tangents be drawn to the circle, they will, by their intersection, determine a regular polygon of the same number of sides circumscribed about the circle. If the points of contact of a regular circumscribed polygon be joined, two and two, in their order, by straight lines, these chords will form a regular inscribed polygon, having the same number of sides.

It has been stated that any regular polygon may be inscribed in a circle, but geometrical constructions have only been discovered for a limited number. The only regular polygons having less than 100 sides, which can, by known methods, be inscribed in a circle by geometrical construction, are those having the following number of sides; viz: 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, or not quite one-fourth of the entire number.

In general, since a regular triangle, square, and pentagon may be inscribed in a circle, and since, also, having any inscribed polygon, we may always inscribe one having double the number of sides, it follows that whenever the number of sides of a regular polygon is equal to 2^n, 3×2^n, or 5×2^n, it can always be inscribed. To these may be added those in which the number of sides is equal to $2^n + 1$, whenever this number is prime.

Spherical Polygon. A portion of the surface of a sphere bounded by arcs of more than two great circles, which are called sides. Spherical polygons are classed like plane polygons, according to the number of sides which bound the figure, or according to the number of their angles.

The sum of all the sides of a convex spherical polygon is always less than the circumference of a great circle.

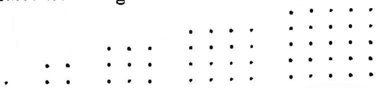
The area of a spherical polygon is equal to that of the trirectangular triangle of the same sphere, multiplied by the quotient obtained by dividing the sum of all its angles, diminished by two right angles taken as many times as the polygon has sides less two, by 90°. Or, denoting the right angle by 1, and the area of the trirectangular triangle by T,

$$A = (s - 2n + 4)\, T.$$

POL-YG′ON-AL. Appertaining to a polygon.

Polygonal Numbers. Series of numbers, each term of which is formed from the preceding by adding to it the corresponding term of an arithmetical progression. They are called polygonal numbers because the number of points in each series can be arranged in the form of a polygon, which gives the name to the series. Thus the numbers 1, 3, 6, 10, 15, &c., are triangular numbers, because they indicate the proper number of points necessary to form triangles, as

The numbers 1, 4, 9, 16, 25, &c., are square numbers, since the corresponding number of points may be arranged in squares, as indicated in the figure



The numbers 1, 5, 12, 22, &c., are pentag-

onal numbers. The rule for arranging the points in figures is this: draw a small regular polygon of the required number of sides, and from one vertex draw lines to each of the other vertices and prolong them indefinitely. On one of the sides, through the assumed vertex, lay off distances equal to two, three, four, &c., times the length of one side of the first polygon and on these lines construct polygons similar to the first polygon and having for a common angle that whose vertex was assumed. Then distribute the points at distances on the perimeters of these several polygons equal to the side of the first polygon, and the polygonal figures will be formed.

The particular progression employed in deducing a series of polygonal numbers is called the directing progression. The common difference of the directing progression is always equal to the number indicating the order of the polygonal number, less two.

The following table will indicate the method of forming series of polygonal numbers:

1st order.	Direct. Progress'n,	1, 2, 3, 4, 5, 6, 7, 8, 9, &c.
	Triangular Num.	1, 3, 6, 10, 15, 21, 28, 36, 45, &c
2d order.	Direct. Progress'n,	1, 3, 5, 7, 9, 11, 13, 15, 17, &c.
	Square Numbers,	1, 4, 9, 16, 25, 36, 49, 64, 81, &c.
3d. order.	Directing Progression,	1, 4, 7, 10, 13, 16, 19, &c.
	Pentagonal Numbers,	1, 5, 12, 22, 35, 51, 70, &c.
4th order.	Directing Progression,	1, 5, 9, 13, 17, 21, 25, &c.
	Hexagonal Numbers,	1, 6, 15, 28, 45, 66, 91, &c.
5th. order.	Directing Progression,	1, 6, 11, 16, 21, 26, 31, &c.
	Heptagonal Numbers,	1, 7, 18, 34, 55, 81, 112, &c.
	&c.,	&c., &c., &c.
nth order.	Directing Progression,	1, $n-1$, $2n-3$, $3n-5$, &c.
	$\mathcal{N}$gonal Numbers,	1, n $3n-3$, $6n-8$, &c.

To find the general term of any series of polygonal numbers. The m^{th} term of a series of polygonal numbers of the n^{th} order is evidently equal to the sum of m terms of an arithmetical progression, whose first term is 1, and common difference $n-2$: hence, denoting this term by t, we have the formula,

$$t = \tfrac{1}{2}[(n-2)m^2 - (n-4)m],$$

which becomes for the series of

Triangular numbers, $t = \frac{1}{2}(m^2 + m)$.
Square " $t = m^2$.
Pentagonal numbers, $t = \frac{1}{2}(3m^2 - m)$.
Hexagonal " $t = \frac{1}{2}(4m^2 - 2m)$.
Heptagonal " $t = \frac{1}{2}(5m^2 - 3m)$
&c. &c. &c.

Multagonal numbers, $t = \frac{1}{2}[(n-2)m^2 - (n-4)m]$.

To find the sum of m terms of a series of polygonal numbers of the n^{th} order.

The first term d_1 of the first order of differences is equal to $n+1$; the first term d_2 of the second order of differences n; and the first terms of the remaining orders of differences, are all 0. Substituting these in the formula,

$$S = ma + \frac{m(m-1)}{1.2}d_1 + \frac{m(m-1)(m-2)}{1.2.3}d_2 + \&c$$

and we have,

$$S = m + \frac{m(m-1)}{1.2}(n+1) + \frac{m(m-1)(m-2)}{1.2.3}n.$$

If we make n, in succession, equal to 1, 2, 3, &c., we have for the sum of m terms, in the

Triangular series, $S = \frac{1}{6}m(m^2+3m+2)$,
Quadrangular " $S = \frac{1}{6}m(2m^2+3m+1)$,
Pentangular " $S = \frac{1}{6}m(3m^2+3m+0)$,
Hexangular " $S = \frac{1}{6}m(4m^2+3m-1)$,
Heptangular " $S = \frac{1}{6}m(5m^2+3m-2)$,
&c., &c., &c., &c.

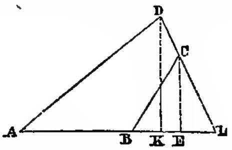
It is a property of polygonal numbers that every number is the sum of one, two, or three triangular numbers: the sum of one, two, three, or four square numbers; the sum of one, two, three, four, or five pentagonal numbers; and in general, the sum of one, two, three, ... &c., ... n, multangular numbers.

POL-Y-GON-OM'E-TRY. [Gr. πολυς, many, γωνια, angle, μετρον, measure]. This is an extension of some of the principles of Trigonometry to the case of polygons. The following enunciation of some of the leading principles of polygonometry, will show the analogy between this branch of Mathematics and Trigonometry.

1. In any polygon, any one side is equal to the algebraic sum of the products, obtained by multiplying each of the other sides into the cosines of the angles which they severally make with the required side. Thus, in the polygon, ABCD, we have

$$AB = BC\cos CBE + DC\cos DEL + DA\cos DAB.$$

The angles are estimated from the prolonga-



tion of AB around to the sides, or sides produced.

2. The perpendicular let fall from any vertex of the polygon, upon any side taken as a base, is equal to the algebraic sum of the products of the sides from this point, around to the base in either direction, multiplied respectively by the sines of the angles which they make with the base. Thus,

$$DK = DA \sin DAB,$$

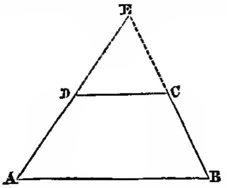
or,

$$DK = DC \sin DEL + CB \sin CBE \ldots$$

and so on, for a polygon having any number of sides.

3. The square of any side of a polygon, is equal to the sum of the squares of all the other sides, *minus* twice the algebraic sum of the products of the sides, taken two and two by the cosines of their included angles. Thus, in the quadrilateral ABCD, we have,

$$AB^2 = BC^2 + DC^2$$
$$+ AD^2 - 2(BC \times CD \cos BCD$$
$$+ CD \times DA \cos CDA + BC \times AD \cos AEB).$$



This proposition is equally true whatever may be the number of sides of a polygon.

POL'Y-GRAM. [Gr. πολυς, many, γραμμα, writing]. In Geometry, a figure composed of many lines.

POL-Y-HĒ'DRAL ANGLE. An angle bounded by three or more plane angles, having a common vertex, which is called the *vertex* of the polyhedral angle. The plane angles are the *faces* of the polyhedral angle. The lines in which the plane angles meet, are the *edges* of the polyhedral angle. If we describe a sphere having its centre at the vertex of the polyhedral angle, and a radius equal to 1, the portion of the surface intercepted between the faces of the polyhedral angle may be taken as the measure of the angle, the area of the trirectangular triangle of the same sphere being regarded as 1. A polyhedral angle is *acute* when its measure is less than 1, it is *right* when its measure is 1, and it is *obtuse* when its measure is greater than 1.

If two planes meet, they form an angle which is called a *diedral* angle, and this may be considered as the limiting case of a polyhedral angle, being bounded by two angles, each equal to 180°; any point of the edge may be taken as the vertex, and the measure of the angle will be a lune, having the same angle as that included within the planes. It is more usual, however, to take as the measure the plane angle formed by two straight lines, one in each plane, both drawn perpendicular to the common intersection at the same point. Both these measures amount to the same thing, so far as comparison of diedral angles with each other is concerned.

When an angle is bounded by three plane angles, it is called a *triedral* angle. When the plane angles which bound a polyhedral angle are equal. and equally inclined to each other, the polyhedral angle is said to be regular.

POL-Y-HĒ'DRON. [Gr. πολυς, many, and εδρα, sides, or faces]. A solid. bounded by polygons. The bounding polygons are called *faces*; the lines in which they meet are called *edges*, and the vertices of the polyhedral angles are called *vertices* of the polyhedron. A straight line joining two vertices not in the same face, is called a *diagonal*, and a plane passing through three vertices not in the same face, is called a *diagonal plane.*

When the faces are regular polygons, the polyhedron is said to be *regular*; there are but five such polyhedrons, viz.. the regular *tetrahedron, hexahedron, octahedron, dodecahedron.* and *icosahedron.* See *Regular Polyhedrons.*

The principal irregular polyhedrons considered in Geometry, are the *parallelopipedon*, the *pyramid*, and the *prism*. For an account of these several solids, see the corresponding articles.

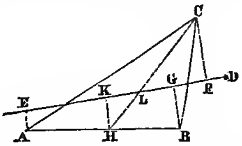
Polyhedrons are classed, according to the number of faces which bound them, into *tetrahedrons*, *pentahedrons*, *hexahedrons*, *heptahedrons*, *octahedrons*, *nonahedrons*, *decahedrons*, &c.

POL-Y-Nō'MI-AL. [Gr. πολυς, many, ονομα, name]. In Algebra, an expression composed of two or more terms connected by the signs, plus or minus. A polynomial of two terms is called a *binomial;* one of three terms, a *trinomial*, &c.

POLYNOMIAL FORMULA. See *Multinomial Formula.*

PO'RISM. [Gr. πορισμος, acquisition; from πορος, a passage]. A name given by the ancient geometers to a class of propositions having for their object to find the conditions that will render certain problems indeterminate. The determinate solution of a problem requires that there should be given as many *independent* conditions as there are required parts. Now if any supposition made upon the data of the problem causes one of the given conditions to become dependent upon the others, it is evident that the solution will be no longer determinate. The discovery of the conditions necessary to make the given conditions dependent upon each other, is the object of the porism.

The nature of porisms, and the difference between them and ordinary problems, will be best illustrated by an example.



Having giving a triangle ABC, and a point D, in the plane of the triangle, to find a straight line through D, such that the sum of the perpendiculars let fall upon the line, from the two vertices of the triangle, on one side of the line, shall be equal to the perpendicular let fall upon it from the vertex on the other side of the line. Suppose the problem solved and that DE is the required line, and that the sum of the perpendiculars AE and BG is equal to the perpendicular CF. Bisect AB in H, and draw CH cutting DE in L; draw also HK perpendicular to DF. Then from the figure we shall have

$$HK = \tfrac{1}{2}(AE + BG)$$

whence $2HK = CF$. But from the similar triangles, LHK and LCF, we have

$$HK : CF :: HL : CL$$

whence $2HL = CL$: that is, the line DE cuts the line CH at a point, one-third of the distance from H to C. We have therefore the following construction for the required line. Bisect the base by a straight line drawn from the vertex; take a point in this line one-third of the distance from the base to the vertex, and through this and the given point draw a straight line, and it will be the right line required. This is a determinate problem, and evidently admits of but one solution in the general case. It is plain that for the same triangle, whatever may be the position of the point D, the point L will remain the same, and as long as D and L do not coincide, there will be but one solution. Now if we suppose the point D to coincide with L, it is evident that there will be an infinite number of solutions, for every straight line drawn through L will fulfill the required conditions. We may enunciate this new and indeterminate problem as follows:

To find in the plane of a triangle a point such that any straight line being drawn through it, and perpendiculars let fall upon it, from the vertices of the three angles, the sum of the two perpendiculars on one side will be equal to the perpendicular on the other side; or the algebraic sum of the perpendiculars will be equal to 0.

This proposition is a porism, and the method of deducing it from that of a common problem indicates the distinctive properties of the porism.

The preceding porism is only a particular case of a more general one, which may be enunciated as follows:

To find a point in the plane of a polygon such that any straight line being drawn through it, and perpendiculars being let fall upon it from the vertices of the polygon, the sum of the perpendiculars on one side will be equal to the sum of the perpendiculars on the other side, or the algebraic sum of all the perpendiculars will be equal to 0.

If we regard the perpendiculars on one side as positive, and those on the other side as negative, the algebraic sum of all the perpen-

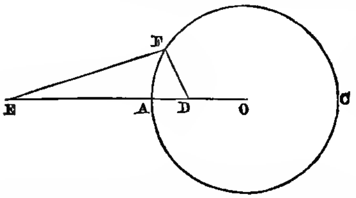
diculars must be equal to 0. The point to be determined is evidently the centre of gravity of the polygon.

As an illustration of the difference between a theorem and porism, let us take the following:

Having given a circle AFC, and a point E, in its plane, let a point D be assumed on EO, so that

$$EO \times OD = AO^2;$$

then, if from a given point F on the circumference, the straight lines FE and FD be



drawn, it is required to show that the ratio of EF to DF is equal to the ratio of EA to DA.

This proposition is a theorem. If now the proposition be enunciated as follows, it becomes a porism, viz:

Having given a circle, and a point E in the plane of the circle, to find a second point D, on EO, such that two lines EF and FD, drawn from any point F of the circumference shall bear to each other a given ratio.

Playfair's definition of a porism is the following:

"A porism is a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions."

PO-SI'TION. [L. *positio*; from *positus*, a placing or setting]. A rule in arithmetic for solving certain problems, which would otherwise require the aid of algebra. It is sometimes called *false position*, or false supposition, because in it, untrue numbers are assumed, and by their means the true answer to a problem is determined. It is also sometimes called the rule of *trial* and *error*, because it proceeds by the *trial of false numbers*, and thence discovers true ones by a comparison of the *errors* committed.

The exact solution of problems can only be made by the rule of position, when they give rise to equations of the first degree. When the problems are of a higher degree than the first, results may be found which are approximately correct, and by continued application a high degree of approximation may be attained. In this way the rule has been applied to find the roots of equations of the higher degrees. It is also useful in solving exponential equations; and in general, all kinds of transcendental equations. It may be applied with advantage in extracting the higher roots of numbers.

It is divided into two parts, *single* and *double position*. Single position explains the method of solving problems in which the results are proportional to the assumed numbers. That is, when the required number is to be multiplied and divided by certain numbers, or when it is to be increased or diminished by any aliquot part of the number, &c.

The rule is as follows:

Assume any number, and perform upon it the successive operations indicated in the enunciation of the problem; then will the result obtained be to the true result, as the number assumed is to the number required.

Example: What number is that, which being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum will be 125?

Assume the number 72. Then, from the conditions of the question,

$$72 + 36 + 24 + 18 = 150;$$

now, by the rule,

$$150 : 125 :: 72 : x, \quad \therefore x = 60,$$

the required number.

Double Position explains the method of solving problems in which the results are not proportional to the assumed numbers; that is, when the numbers sought, or their parts, or their multiples, are increased or diminished by some absolute number, or the like.

RULE. Take any two convenient numbers, and proceed with them separately, according to the conditions of the problem, noting the results. Then, as the difference of these results, is to the difference of the assumed numbers, so is the difference between the true result and either of the deduced results, to the correction to be applied to the number corresponding to that result.

When the deduced result is too small, the

correction is to be added; when too large, it is to be subtracted.

Example: What number is that, which, being multiplied by 6, the product increased by 18, and the sum divided by 9, the quotient will be equal to 20?

FIRST POSITION.		SECOND POSITION.	
18		30	
6		6	
108		180	
18		18	
9)126		9)198	
14	1st result.	22	2d result.

Then, by the rule,

$$8 : 12 :: 20 - 14 : x \quad \therefore x = 9;$$

hence, $18 + 9 = 27$, true result. Or,

$$8 : 12 :: 22 - 20 : x \quad \therefore x = 3;$$

hence, $30 - 3 = 27$, true result.

POSITION of a point or magnitude, in Geometry, is its place with respect to certain other objects, regarded as fixed. Thus, in the system of rectilinear co-ordinates, a point is given in position, when its distances from three co-ordinate planes are known. In analysis, the constants which enter the equation of a magnitude of any kind, make known its position with respect to the co-ordinate axes, or the system in which the magnitude is taken. Analytical Geometry is sometimes called *Geometry of Position.*

POS'I-TIVE QUANTITIES. [L. *positivus*, placed]. Those affected with the sign +. The term positive is used in contradistinction to negative; the two indicating quantities taken in a diametrically opposite sense. The sense in which a positive quantity is to be taken is purely conventional, but when once assumed, the negative quantities must be regarded in a contrary sense.

For instance, if it is agreed to represent distances estimated from a point along a straight line in either direction, by a positive symbol, then will a negative symbol indicate distances estimated from the same point in the contrary direction. If it is agreed to estimate time, from a particular epoch, forward, by a positive symbol, then will time backward from the fixed epoch be represented by a negative symbol, and so on. It is in accordance with this principle that positive and negative results are to be interpreted in analysis.

POS'TU-LATE. [L. *postulatum*; from *postulo*, to demand]. The enunciation, in Geometry, of a self-evident problem. It differs from an axiom, which is the enunciation of a self-evident proposition. The axiom is more general than the postulate. The following are some of the postulates of Geometry:

1. A straight line may be drawn from one point to another.
2. A limited straight line may be prolonged to any length.
3. A limited straight line may be bisected, that is, divided into two equal parts.
4. If two limited straight lines are unequal in length, the length of the shorter one may be laid off upon the longer one.
5. A straight line may always be drawn, bisecting a given angle.
6. A perpendicular may always be drawn to a given straight line through any point, either upon or without the line.
7. An angle can always be constructed equal to a given angle.
8. A straight line may always be drawn through a given point parallel to a given straight line.
9. A circle can always be described, having its centre at a given point, and with any given radius.

POUND. [L. *pondus*, weight, *pendo, to* weigh]. A unit of weight. Pounds are of different kinds, as pounds Troy, pounds Avoirdupois, &c. A cubic inch of distilled water. at 62° Fahr., the barometer being 30 inches, weighs 252.458 Troy grains, and the Troy pound is equal to 5760 of these grains. The Avoirdupois pound is equal to 7000 Troy grains, so that the Troy pound is to the Avoirdupois pound as 144 is to 175. See *Weights.*

POUND. A unit of currency in the British system, also in several other foreign systems. The British pound sterling is equivalent to $4,84 of our currency, though its commercial value varies from $4,83 to $4,86.

POW'ER. The power of a quantity in Algebra, is the result obtained by taking that quantity a certain number of times, as a factor. We may regard the unit 1 as the base of the powers of all quantities, the base itself being called the 0 power. Now, denote any quantity whatever by a, and multiply 1 by it,

the result may be written a and is called the *first power* of a; multiply this by a, the result may be written a^2, and is called the *second power* of a, and so on; after n successive multiplications by the same quantity a, the result may be written a^n, and is called the n^{th} power of a.

Commencing with 1, or the 0 *power* of a, we have the series

$$1,\ a,\ a^2,\ a^3,\ a^4,\ a^5,\ \ldots\ a^n,$$

which is called the series of *ascending* powers. Were we to commence with a^n, and divide successively by a, or what is the same thing, multiply by $\frac{1}{a}$, we should obtain the same series in an inverse order; if on reaching 1 we continue the successive division by a, or multiplication by $\frac{1}{a}$, we shall have

$$\frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \frac{1}{a^4},\ \&c.,\ \frac{1}{a^n}.$$

which, from analogy with the terms of the ascending series, are denoted by the symbols

$$a^{-1},\ a^{-2},\ a^{-3},\ a^{-4},\ \ldots\ a^{-n};$$

this is called the series of *descending* powers. The two series may be written together, thus:

$$a^{-n}\ldots a^{-4},\ a^{-3},\ a^{-2},\ a^{-1},\ a^0,\ a^1,\ a^2,\ a^3,$$
$$a^4,\ a^5,\ a^6,\ a^7,\ a^8,\ a^9,\ a^{10},\ a^{11},\ a^{12}\ldots a^n,$$

in which any term may be derived from the preceding one by multiplying it by a, or from the succeeding one by dividing it by a, or multiplying it by $\frac{1}{a}$. The numbers written at the right and above the quantity a, are called exponents, and denote the degree of the power, or the number of times that a has been taken as a factor, or the number of successive multiplications that have been made, beginning at the *base* 1. In accordance with the well established rules for the interpretation of negative results, it follows that a negative exponent indicates the number of successive divisions by the quantity a, beginning at the base 1. We have seen above that any quantity affected with a negative exponent is equal to that power of the reciprocal of the quantity which would be denoted by the same exponent, with its sign changed. Thus,

$$a^{-n} = \left(\frac{1}{a}\right)^n.$$

The number a is called a *root* of the different powers; thus, a is the *square root* of a^2, the *cube root* of a^3, the n^{th} *root* of a^n. The terms power and root are correlative, and are thus used in mathematics.

Fractional powers are those indicated by fractional exponents, as, $a^{\frac{m}{n}}$. By the rules for the multiplication of quantities we have

$$a^{\frac{m}{n}} = a^{\frac{1}{n}} \times a^{\frac{1}{n}} \ldots = \left(a^{\frac{1}{n}}\right)^m.$$

These principles enable us to explain the nature of all quantities affected with negative and fractional exponents. By a combination of these principles we have the following table of analytical equivalents:

(1). $a^{\frac{m}{n}} = a^m \times a^{\frac{1}{n}} = a^m \times \sqrt[n]{a}$

(2). $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$

(3). $a^{-\frac{m}{n}} = \dfrac{1}{a^{\frac{m}{n}}} = \left(\dfrac{1}{a}\right)^{\frac{m}{n}}$

The following are some of the properties of powers:

The difference of the like powers of two quantities is always divisible by the difference of the quantities; that is, when m is a whole number,

$$\frac{x^m - y^m}{x - y} = x^{m-1} + x^{m-2}y + x^{m-3}y^2 + \cdots$$
$$+ x^2y^{m-3} + xy^{m-2} + y^{m-1}\ldots$$

If $x = y$ the quotient reduces to mx^{m-1}, whatever may be the value of m.

2. The expression $x^m - y^m$ is divisible by $x - y$, and also by $x + y$, when m is a positive even number.

3. The expression $x^n + y^n$ is divisible by $x + y$ when n is an odd whole number, and positive.

4. The expression $x^m - x^n$ is divisible by $x - 1$, and it is also divisible by $x + 1$ when $m - n$ is an even whole number.

5 The expression $x^m + x^n$ is divisible by $x + 1$ when $m - n$ is an odd whole number.

6. Neither the sum nor the difference of any two powers of a degree superior to the second, is equal to a perfect power of the same degree.

7. If m is a prime number, and x any number not divisible by m, then will the remainder arising from the division of x by m, be the

same as that from the division of x^m by m, and consequently $x^{m-1} - 1$ will be exactly divisible by m.

By means of this principle we readily deduce the following table of the forms of powers of numbers with regard to certain fixed numbers taken as moduli:

2[d] powers are of the form		$5n$ or	$5n \pm 1$
3[d] powers	" "	$7n$ or	$7n \pm 1$
4[th] powers	" "	$5n$ or	$5n + 1$
5[th] powers	" "	$11n$ or	$11n \pm 1$
6[th] powers	" "	$13n$ or	$13n \pm 1$
7[th] powers	" "	$17n$ or	$17n \pm 1$

and generally, when $m + 1$ is prime,

m^{th} powers are of the form $(m + 1)n$, or $(m + 1)n + 1$; and when $2m + 1$ is prime,

m^{th} powers are of the form $(m + 1)n$, or $(m + 1)n + 1$.

8. All the terms of the $(n + 1)^{th}$ order of differences of a series of like powers of the natural numbers, are equal to 0.

The same principle holds in regard to the differences of any series of like powers of the terms of any arithmetical progression.

COMMENSURABLE IN POWER. Two quantities that are not commensurable, but which have any like powers commensurable, are said to be commensurable in power. Thus, the side and diagonal of a square are incommensurable, but their squares are commensurable, being to each other as 1 to 2: they are thus commensurable in the second power.

POWER OF AN HYPERBOLA. The rhombus described upon the abscissa and ordinate of the vertex of the curve when referred to its asymptotes. It is equivalent to one-eighth of the rectangle of the axes, or to one-eighth of the parallelogram described upon any pair of conjugate diameters.

PRAC'TI-CAL. An application of what is *theoretical* or *scientific*. That which may be accomplished or effected.

PRACTICAL ARITHMETIC, GEOMETRY, &c. The application of the principles and truths of the science of arithmetic, geometry, &c., to the wants of life.

Most of the ordinary operations of business are only so many practical applications of rules or principles which have been deduced from a consideration of the truths of science.

PRAC'TICE. [Gr. πρακτικη, from πρασσα, πραττω, to do, to act]. An easy and concise method of applying the rules of arithmetic to questions which occur in trade and business. It is only a particular case of the Rule of Three, in which the first term is 1. For example, If 1 yard of cloth cost half a dollar, what will 60 yards cost? This is an example of the nature of the questions that are solved by the rule of Practice.

The general rule for Practice is: take the sum of such aliquot parts of the given number of things, as the given price is of the unit of currency of the next higher order, and the result will be the price of the thing in terms of that unit. Thus: What will be the cost of 5320 bushels of wheat be at $3^s\,6^d$ per bushel?

....	6)5320
$3^s\,4^d$ is $\frac{1}{6}$ of a £ · $\frac{1}{8}$ of 5320 is	20)886.6666
2^d is $\frac{1}{120}$ or $\frac{1}{20}$ of $\frac{1}{6}$ of a £ · $\frac{1}{120}$ of 5320 is	44.3333
	931

Hence, the cost is £931.

It is only experience that can give facility in the application of the rules of Practice.

PRE-FIX'. [L. *præfigo*, to fix before]. To write before, as, to prefix a co-efficient, to prefix 0's, &c.

PREM'I-SES. [L. *præmissa*, dispatched before]. In logic, the first two propositions of a syllogism, from which the inference is drawn. Thus:

All tyrants are detestable.
Cæsar was a tyrant,

are premises, and if their truth be admitted, the conclusion, that Cæsar was detestable, follows as a matter of irresistible inference. The entire syllogism reads as follows:

All tyrants are detestable;
Cæsar was a tyrant;
Therefore, Cæsar was detestable.

Of the two terms of the conclusion, the predicate (*detestable*) is called the *major* term, and the subject (*Cæsar*) the *minor* term; and these, with the middle term (*tyrant*) make up the three terms of the syllogism, each being used twice. The premiss into which the major term enters is called the *major premiss*, the one into which the minor term enters is called the *minor premiss*. In the example given,

All tyrants are detestable,

is the major premiss, and

Cæsar was a tyrant

is the minor premiss. In the reasoning of

mathematics the premises are axioms, definitions, and propositions already established, whether expressed in mathematical or in common language. See *Demonstration*.

PRĪ′MA-RY. [L. *primarius*, chief, principal]. First or lowest in order; that which stands highest in rank, as opposed to secondary. Thus, we say that the unit 1 is the primary base of all numbers, because to it all numbers are ultimately referred in order to show the relations which they bear to each other. Beginning with the first or primary order of units, they are collected upon the base 1 till ten are collected; then, beginning with tens, we collect upon the secondary base 10 or 1 ten, till ten of these are collected; then, we collect upon 100 or 1 hundred, as a base of the third order, and so on; and before we can acquire a distinct idea of a number, we are obliged to refer it back through the different units to the primary base 1. The base 1 is the measure of the relation of equality; that is, it is the result obtained by dividing one thing by an equal thing of the same kind. See *Number*.

PRIME. [L. *primus*, first]. A number or quantity, is prime, when it cannot be exactly divided by any other number or quantity, except 1. Two numbers or quantities are *prime* with respect to each other, when they do not admit of any common divisor except 1. The numbers 2, 3, 5, 7, &c., are prime numbers, and the numbers 7, 12, and 25, are prime with respect to each other.

There has not been any rule discovered by means of which prime numbers can be found by a direct process. A method of finding prime numbers by sifting out those which are not prime, was discovered by Erastosthenes, and called by him, for that reason, a sieve. The method is as follows: Since every even number is divisible by 2, we may confine our attention to the odd numbers. For this purpose, write down the series of odd numbers from 1 to any desired limit—suppose to 99, for example.

1 3 5 7 9 11 13 15 17 19 21 23 25 27

29 31 33 35 37 39 41 43 45 47 49 51 53

55 57 59 61 63 65 67 69 71 73 75 77 79

81 83 85 87 89 91 93 95 97 99.

We begin with the first prime number after 2, which is 3, and over every third number, from that place, we put a point, because those numbers are divisible by 3; as, 9, 15, 21, &c.

Then, from 5 a point is placed over every fifth number, they all being divisible by 5; as, 15, 25, 35, &c.

Again, every 7th number from 7 is pointed as before; as, 21, 35, 49, &c.

Now, all that remain, viz.: 1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97, are prime numbers; for there is no prime number between 7 and $\sqrt{100}$ that will divide either of them; and if a number cannot be divided by a prime number less than the square root of itself, it is a prime number. If, therefore, we add to the numbers thus found, the only even prime number, 2, we shall have all the prime numbers in the first hundred. This method becomes exceedingly tedious, beyond a certain limit, and, in order to find prime numbers beyond this limit, analysts have sought to find a formula; but, thus far, none of general application has been found.

I.—Table of Prime Forms.

No.	Prime forms.	Equivalent forms.
1	$4n + 1$	$y^2 + z^2$
2	$6n + 1$	$y^2 + yz + z^2$
3	$8n + 1,\ 7$	$y^2 - 2z^2$
4	$8n + 1,\ 3$	$y^2 + 2z^2$
5	$12n + 1$	$y^2 - 3z^2$
6	$12n + 11$	$3y^2 - z^2$
7	$14n + 1,\ 9,\ 11$	$y^2 + 7z^2$
8	$20n + 1,\ 9,\ 11,\ 19$	$y^2 - 5z^2$
9	$20n + 1,\ 9$	$y^2 + 5z^2$
10	$20n + 3,\ 7$	$2y^2 + 2yz + 3z^2$
11	$24n + 1,\ 19$	$y^2 - 6z^2$
12	$24n + 5,\ 25$	$6y^2 - z^2$
13	$24n + 5,\ 11$	$2y^2 + 3z^2$
14	$24n + 1,\ 7$	$y^2 + 6z^2$
15	$28n + 1,\ 9,\ 25$	$y^2 - 7z^2$
16	$28n + 3,\ 19,\ 27$	$7y^2 - z^2$
17	$30n + 1,\ 19$	$y^2 + 15z^2$
18	$30n + 17,\ 23$	$3y^2 + 5z^2$
19	$40n + 1,\ 9,\ 31,\ 39$	$y^2 - 10z^2$
20	$40n + 3,\ 13,\ 27,\ 37$	$2y^2 - 5z^2$
21	$40n + 1,\ 9,\ 11,\ 19$	$y^2 + 10z^2$
22	$40n + 7,\ 13,\ 23,\ 37$	$2y^2 + 5z^2$
23	$120n + 11,\ 29,\ 59,\ 101$	$5y^2 + 6z^2$
24	$120n + 13,\ 37,\ 43,\ 67$	$10y^2 + 3z^2$
25	$120n + 1,\ 31,\ 49,\ 79$	$y^2 + 30z^2$
26	$120n + 17,\ 23,\ 47,\ 113$	$2y^2 + 15z^2$

II.—TABLE OF PRIME NUMBERS.

0

0	1	2	3	4	5	6	7	8	9
1	01	11	07	01	03	01	01	09	07
2	03	23	11	09	09	07	09	11	11
3	07	27	13	19	21	13	19	21	19
5	09	29	17	21	23	17	27	23	29
7	13	33	31	31	41	19	33	27	37
11	27	39	37	33	47	31	39	29	41
13	31	41	47	39	57	41	43	39	47
17	37	51	49	43	63	43	51	53	53
19	39	57	53	49	69	47	57	57	67
23	49	63	59	57	71	53	61	59	71
29	51	69	67	61	77	59	69	63	77
31	57	71	73	63	87	61	73	77	83
37	63	77	79	67	93	73	87	81	91
41	67	81	83	79	99	77	97	83	97
43	73	83	89	87		83		87	
47	79	93	97	91		91			
53	81			99					
59	91								
61	93								
67	97								
71	99								
73									
79									
83									
89									
97									

1

0	1	2	3	4	5	6	7	8	9
09	03	01	01	09	11	01	09	01	01
13	09	13	03	23	23	07	21	11	07
19	17	17	07	27	31	09	23	23	13
21	23	23	19	29	43	13	33	31	31
31	29	29	21	33	49	19	41	47	33
33	51	31	27	39	53	21	47	61	49
39	53	37	61	47	59	27	53	67	51
49	63	49	67	51	67	37	59	71	73
51	71	59	73	53	71	57	77	73	79
61	81	77	81	59	79	63	83	77	87
63	87	79	99	71	83	67	87	79	93
69	93	83		81	97	69	89	89	97
87		89		83		93			99
91		91		87		97			
93		97		89		99			
97				93					
				99					

2

0	1	2	3	4	5	6	7	8	9
03	11	03	09	11	03	09	07	01	03
11	13	07	11	17	21	17	11	03	09
17	29	13	33	23	31	21	13	19	17
27	31	21	39	37	39	33	19	33	27
29	37	37	41	41	43	47	29	37	39
39	41	39	47	47	49	57	31	43	53
53	43	43	51	59	51	59	41	51	57
63	53	51	57	67	57	63	49	57	63
69	61	67	71	73	79	71	53	61	69
81	79	69	77	77	91	77	67	79	71
83		73	81		93	83	77	87	99
87		81	83			87	89	97	
89		87	89			89	91		
99		93	93			93	97		
		97	99			99			

3

0	1	2	3	4	5	6	7	8	9
01	09	03	01	07	11	07	01	03	07
11	19	09	07	13	17	13	09	21	11
19	21	17	13	33	27	17	19	23	17
23	37	21	19	49	29	23	27	33	19
37	63	29	23	57	33	31	33	47	23
41	67	51	29	61	39	37	39	51	29
49	69	53	31	63	41	43	61	53	31
61	81	57	43	67	47	59	67	63	43
67	87	59	47	69	57	71	69	77	47
79	91	71	59	91	59	73	79	81	67
83		99	61	99	71	77	93	89	89
89			71		81	91	97		
			73		83	97			
			89		93				
			91						

4

0	1	2	3	4	5	6	7	8	9
01	11	01	27	09	07	03	03	01	03
03	27	11	37	21	13	21	21	13	09
07	29	17	39	23	17	37	23	17	19
13	33	19	49	41	19	39	29	31	31
19	39	29	57	47	23	43	33	61	33
21	53	31	63	51	47	49	51	71	37
27	57	41	73	57	49	51	59	77	43
49	59	43	91	63	61	57	83	89	51
51	77	53	97	81	67	63	87		57
57		59		83	83	73	89		67
73		61		93	91	79	93		69
79		71			97	91	99		73
91		73							87
93		83							93
99		89							99
		97							

5

0	1	2	3	4	5	6	7	8	9
03	01	09	03	07	01	23	01	01	03
09	07	27	09	13	03	39	11	07	23
11	13	31	23	17	07	41	17	13	27
21	19	33	33	19	19	47	37	21	39
23	47	37	47	31	21	51	41	27	53
39	53	61	51	37	27	53	43	39	81
51	67	73	81	41	31	57	49	43	87
59	71	79	87	43	57	59	79	49	
77	79	81	93	49	63	69	83	51	
81	89	97	99	71	69	83	91	57	
87	97			77	73	89		61	
99				79	81	93		67	
				83	91			69	
								79	
								81	
								97	

This table contains a list of all the prime numbers up to 6000. To use it, look for the figure denoting thousands over one of the sub-tables, then under it, at the head of the table, look for the figure denoting the hundreds, follow the line down, and if the remaining figures of the numbers occur it is prime; if not, it is not. For example, 5381 is prime—4755 not.

The following are some of the properties of prime numbers, shown in Table I.

1. If a number cannot be divided by a number less than the square root of itself, it is a prime number.

2. All prime numbers are of the form

$$4n \pm 1;$$

that is, if a prime number be divided by 4, the remainder will be ± 1. All prime numbers are also of the form

$$6n \pm 1.$$

Many other forms might be given, some of which are written on p. 447, in a tabular form. The converse of these propositions is not true; for every number of one of the forms is not necessarily a prime number.

3. There cannot be three prime numbers in arithmetical progression unless the common difference of the progression is divisible by 6, or unless the first of these prime numbers is 3; in which case there may be three prime numbers in such progression, but in no case can there be more than three.

4 If n is a prime number,

$$1 + 1.2.3.4 \ldots n$$

is divisible by n.

5. If n is a prime number, and r any number whatever, not divisible by n, then will r^n, when divided by n, leave the same remainder as r when divided by n.

6. Under the same supposition as before, $r^{n-1} - 1$ is divisible by n.

7. The square of every prime number of the form $4n + 1$, is itself of the form

$$y^2 + 25z^2.$$

Table I. gives, in a condensed form, most of the properties of prime numbers.

The distribution of prime numbers does not follow any known law; but from the preceding table and from other tables continued much further, it is evident that for a given interval, the number of primes is generally less, the higher the beginning of the interval is taken. The whole number of primes, up to 10,000, is 1230; between 10,000 and 20,000, it is 1033; between 20,000 and 30,000, it is 983, and so on; between 90,000 and 100,000, it is 879.

The following formula has been given for determining the number of primes, up to the number x, when x is a very great number, viz.:

$$N = \frac{x}{A \log x - B};$$

in which N denotes the number of primes, and A and B constants to be determined by trial.

When x is a very great number, and the Naperian system of logarithms is used, A is nearly equal to 1, and B is nearly equal to 1.08366, giving the formula

$$N = \frac{x}{lx - 1.08366}.$$

This formula is deduced empirically; that is, it is found to satisfy the results given in the tables; but no demonstration of it can be given. It has been found, that of all the numbers less than a million million of millions, only one out of 40 is a prime, whilst the number of primes under the square of the same number is but one out of 82. We infer from this, that we might name a series of numbers beginning with one so high, that a million, or any other number, however great, of numbers should succeed without containing one prime number. Nevertheless, there cannot be an end of prime numbers; for if so, let p be the last prime number, and let N denote the product of all the prime numbers, $2, 3, 5, \ldots p$. Now, every number is either prime or divisible by a prime; but $N + 1$ is not divisible by $2, 3, 5, \ldots$, or p, since it leaves a remainder 1 in every case. Hence, $N + 1$ is prime, which is necessarily greater than p, the greatest prime number, which is absurd; hence, there can be no limit to the number of prime numbers.

Prime Factors. The prime numbers that will exactly divide the number.

If we denote the prime factors of any number by a, b, c, d, &c., and the number of times that they enter, respectively, by m, n, p q, &c., the number itself will be denoted by $a^m \cdot b^n \cdot c^p \cdot d^q \ldots$ &c.; and the whole number of its divisors (including 1 and the number itself,) will be

$$(m + 1)(n + 1)(p + 1)(q + 1) \ldots \text{\&c.};$$

call this number N; then the number of numbers less than N, and prime with respect to N, is denoted by

$$N \cdot \frac{m-1}{m} \cdot \frac{n-1}{n} \cdot \frac{p-1}{p} \cdot \frac{q-1}{q} \ldots \text{\&c.}$$

To resolve any number into its prime factors, we commence by dividing it successively by 2, as often as possible; after which we divide it successively by 3, as often as possible; then by 5, and so on until we see, from the table, that the quotient obtained is prime; then we gather the divisors and the last quotient together: these are all of the prime factors. For example, let it be required to resolve 504 into its prime factors:

Operation.
$$\left.\begin{array}{r} 2)504 \\ \hline 2)252 \\ \hline 2)126 \\ \hline 3)\ 63 \\ \hline 3)\ 21 \\ \hline 7 \end{array}\right\} \text{hence, } 504 = 2^3 \cdot 3^2 \cdot 7.$$

To find all the different divisors of 504, we form all the different products of the prime factors taken in sets of 1 and 1, 2 and 2, 3 and 3, &c.; these will be the required divisors. The highest prime number that has hitherto been shown to be prime, is

2147483647.

Prime and Ultimate Ratios. A method of analysis, devised and first successfully employed by Newton in his Principia. It is an extension and simplification of the method

known amongst the ancients as the method of exhaustions. To conceive the idea of this method, let us suppose two variable quantities constantly approaching each other in value, so that their ratio continually approaches 1, and at last differs from 1 by less than any assignable quantity; then is the *ultimate ratio* of the two quantities equal to 1.

In general, when two variable quantities simultaneously approach two other quantities, which, under the same circumstances, remain fixed in value, the ultimate ratio of the variable quantities is the same as the ratio of the quantities whose values remain fixed. They are called *prime*, or *ultimate* ratios, according as the ratio of the variable quantities is receding from or approaching to the ratio of the limits. This method of analysis is generally called the method of limits.

Prime Vertical. In Navigation and Surveying, a vertical plane which is perpendicular to a meridian plane at any place.

Prime Vertical Dial. In dialing, a dial drawn upon the plane of the prime vertical of the place, or a plane parallel to it.

PRIM'I-TĪVE. Original, not derived.

Primitive Axes of Co-ordinates, or Primitive System. That system to which the points of a magnitude are first referred with reference to a second set or second system, to which they are afterwards referred, and which is called the new set of axes, or the new system. See *Transformation of Coordinates.*

Primitive Circle. In Spherical Projections, the circle cut from the sphere to be projected, by the primitive plane. It is generally a great circle.

Primitive Plane. In Spherical Projections, the plane upon which the projections are made. This plane is generally taken through the centre of the sphere, and in most cases is made to coincide with some principal circle of the sphere, as the equator, or one of the meridians.

PRIN'CI-PAL. [L. *principalis*, from *princeps*]. In Arithmetic, the name given to a sum of money put out at interest.

Principal Axis of a conic section, that axis which passes through the foci. In the case of the parabola, it is the diameter through the focus.

In the case of the circle, the foci coincide at the centre, and every straight line through that point is a principal axis.

Principal Plane. In surfaces of the second order, a plane that bisects a system of parallel chords of the surface perpendicular to it. In every surface of the second order there is always one principal plane. A principal plane of a surface of the second order, is analogous in its properties to an axis of a line of the second order.

Whenever the surface has one centre only, and that at a finite distance, it always has three principal planes, which, by their intersection, determine the position of the axes of the surface.

If the surface is one of revolution, any plane passed through the axis is a principal plane, and consequently, there are an infinite number of them.

In the case of the sphere, any plane passed through the centre is a principal plane.

The principal plane of a surface always divides the surface, as well as the volume bounded by the surface, into two equivalent and symmetrical parts.

Principal Point. In Perspective, the projection of the point of sight upon the perspective plane; it is the same as the centre of the picture.

Principal Ray. In Perspective, the ray drawn through the point of sight, perpendicular to the perspective plane.

PRIN'CI-PLE. [L. *principium*]. A truth which has been proved, or which is evident.

Principle of 9's. See *Nines.*

PRISM. [Gr. πριςμα; from πριω, to cut with a saw]. In Geometry, a polyhedron in which two of the faces are equal polygons of any kind, having their homologous sides parallel; all of the remaining faces are parallelograms. The equal and parallel polygons are called *bases* of the prism, and the parallelograms taken together make up the *lateral*, or *convex surface*. The distance between the bases is called the *altitude*. The lines in which the lateral faces meet, are called *lateral edges*. When the lateral edges are perpendicular to the planes of the bases, the prism is *right*, when they are oblique to them, the prism is *oblique*. Prisms take their names from the polygons which form their bases

A triangular prism is one whose bases are triangles. A quadrangular prism is one whose bases are quadrilaterals, and so on. When the base of a prism is a regular polygon, and the lateral edges perpendicular to the plane of the base, the prism is called a regular prism. A regular prism, with an infinite number of faces, differs insensibly from a cylinder, and we therefore regard the cylinder as the limit of a regular prism.

The following are some of the properties of prisms :

1. The convex surface of any right prism is equal to the perimeter of either base, multiplied by the altitude.

2. The volume of any prism is equal to the area of the base, multiplied by its altitude.

3. The convex surfaces of any two right prisms, are to each other as the products of perimeters of their bases and altitudes.

The volumes of any two prisms are to each other as the products of their bases and altitudes.

4. The sections made in the same prism by secant parallel planes are equal polygons.

5. Every prism is equivalent to the sum of three pyramids, having an equal base and an equal altitude.

PRIS'MOID. [Gr. πριςμα and ειδος]. A volume somewhat resembling a prism. The right prismoid is the frustum of a wedge made by a plane parallel to the back of the wedge. Its volume may be found by adding together the areas of the two bases, and four times the area of a section midway between the bases, then multiplying the sum by one-sixth of the altitude.

The prismoidal solids used in railroad cutting and embankment, are bounded by six quadrilaterals, the end ones parallel to each other, the base horizontal, or slightly inclined, the sloping sides making equal angles with the base, and the superior surface making any angle with the horizon. The volume of such a solid is equal to one-sixth of the sum of the end sections, plus four times the mean sections multiplied by the length of the section.

PROB-A-BIL'I-TY. [L. *probabilitas*, likelihood]. Likelihood of the occurrence of an event, in the doctrine of chances. The quotient obtained by dividing the number of favorable chances by the whole number of chances, both favorable and unfavorable. The word chance is here used to signify the occurrence of an event in a particular way, when there are two or more ways in which it may occur and when there is no reason why it should happen in one way rather that in another.

Thus, if a die is thrown into the air, it will necessarily fall upon one of its six faces, but no reason can be assigned why it should fall upon one rather than upon another, and we therefore say that the chance of its falling on one face is equal to the chance of its falling on another. Now the whole number of chances in this case is six, and since it can fall upon but one face, we say that the probability of its falling on any one face is $\frac{1}{6}$. The probability that a given face will not turn up is $\frac{5}{6}$, because in this case the whole number of chances is 6, and the number of those unfavorable to the occurrence of the event is 5.

Again, suppose that there are five balls, three black ones and two white ones, placed in an urn, and one of them drawn out. In this case there are five chances in all, three in favor of drawing a black ball, and two in favor of drawing a white one ; hence, the measure of the probability of drawing a white ball is $\frac{2}{5}$, and the measure of the probability of drawing a black one is $\frac{3}{5}$. The sum of these two probabilities is 1, which indicates a certainty of drawing either a black ball or a white one.

Every contingent event gives rise to two complementary probabilities, one in favor of the occurrence of the event, and the other against its occurrence, and as it must either occur or not occur, the sum of these probabilities is always equal to 1, which indicates a certainty.

In general, denote the number of chances in favor of the occurrence of an event by m, and the number of chances opposed to its occurrence by n; the whole number of chances will be $m + n$. Now the probability of the occurrence of the event is measured by the fraction $\dfrac{m}{m+n}$; and the probability of the non occurrence of the event is measured by $\dfrac{m}{m+n}$; and we have

$$\frac{m}{m+n} + \frac{n}{m+n} = 1.$$

Now, if we denote the probability of the occurrence of any event of p, the probability that it will not occur is $1 - p$.

Probability of the simultaneous occurrence of two or more events. In order to investigate this subject, let us consider the case of two dice, having each 6 faces, and let it be required to find the measure of the probability that on being thrown into the air both will turn up aces. The probability that the first die will turn up ace is $\frac{1}{6}$, but the second die may turn up any one of its six faces in connection with this ace; hence, the probability that the aces will turn up together is $\frac{1}{6}$th of $\frac{1}{6}$, or $\frac{1}{36}$. The same reasoning may be applied to any number of independent events; hence, we conclude, in general, that the probability of the simultaneous occurrence of any number of independent events, is measured by the continued product of the probabilities of the occurrence of the events taken separately. Thus, the probabilities of throwing three aces with three dice, is

$$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}.$$

Probability of successive events occurring in any given order. The probability of the same event occurring twice, successively, is determined in the same manner. The probability that a single die will turn up ace twice in succession is $\frac{1}{6} \times \frac{1}{6}$ or $\frac{1}{36}$, and the probability that it will turn up ace three times in succession is

$$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216},$$

and so on. The probability that an ace will not turn up the first throw is $\frac{5}{6}$, and the probability that it will not turn up twice in succession is $\frac{25}{36}$.

This result is entirely independent of the probability that an ace will not turn up at either of the two throws. To investigate the nature of this probability, it may be remarked that four cases may occur. *First*, an ace may turn up at both throws; *second*, an ace may not turn up at either throw; *third*, an ace may turn up at the first throw, and not at the second; and *fourth*, an ace may turn up at the second throw and not at the first. The measure of the first probability is, as we have seen, equal to $\frac{1}{36}$; the measure of the second probability is $\frac{25}{36}$; the measure of the third probability is

$$\frac{1}{6} \times \frac{5}{6} \text{ or } \frac{5}{36};$$

and the measure of the fourth probability is

$$\frac{5}{6} \times \frac{1}{6}, \text{ or } \frac{5}{36},$$

and the sum of these is

$$\frac{1}{36} + \frac{25}{36} + \frac{5}{36} + \frac{5}{36} = 1.$$

To generalize this case, suppose m to be the number of white balls in an urn, and n the number of black balls, and that when a ball has been drawn it is immediately replaced, so that at each trial the number of chances is $m + n$. Let p denote the probability of drawing a white ball, on any trial, and q the probability of drawing a black ball; whence,

$$p = \frac{m}{m+n} \quad \text{and} \quad q = \frac{n}{m+n}.$$

First, let us consider the probabilities of the occurrence of the different possible events on two trials. There are but four ways in which the results can occur, viz: *First*, a white ball may be drawn at both trials; *second*, a black ball may be drawn at both; *third*, a white ball may be drawn at the first, and a black ball at the second; and *fourth*, a black ball may be drawn at the first and a white one at the second trial.

The probability of the first occurrence is

$$p \times p = p^2.$$

The probability of the second occurrence is

$$q \times q = q^2.$$

The probability of the third occurrence is

$$p \times q = pq.$$

The probability of the fourth occurrence is

$$q \times p = pq.$$

And the sum of these is equal to

$$p^2 + 2pq + q^2 = (p + q)^2 = 1,$$

as it should, since these together embrace every possible chance.

The expression $2pq$ is the measure of the probability of drawing a black and a white ball at two trials, without regard to the order in which they may be drawn. If now we consider n trials, we shall, in like manner, find

the sum of the different probabilities of all the possible chances, given by the formula

$$(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{1 \cdot 2}p^{n-2}q^2 + \&c. + q^n.$$

The first term of the second member expresses the probability of drawing a white ball at every one of the n trials; the second term expresses the probability of drawing $n-1$ white balls, and 1 black ball, without regard to the order of the occurrences; the third term expresses the probability of drawing $n-2$ white balls and two black ones, without regard to the order of their occurrence, and so on; and the last term expresses the probability of drawing n black balls, or a black ball at every trial.

In practice it is in general required to determine the probabilities that an occurrence will not be repeated less than a certain number of times in a given number of chances. In the preceding example, let it be required to find the measure of the probability that no fewer than $n-v$ white balls will be drawn in n trials. It is evident, that as the first term expresses the probability of drawing n white balls, the second term that of drawing $n-1$ white balls, the third that of drawing $n-2$, and so on; and as each of these combinations satisfies the given condition, the required probability will be found by taking the sum of the terms from the first to the $(v+1)^{\text{th}}$, inclusively.

For example: let it be required to ascertain the probability of throwing two aces in four throws of the same die. Here

$$p = \frac{1}{6}, \quad q = \frac{5}{6}, \quad n = 4, \quad \text{and} \quad v = 2:$$

hence the probability is expressed by

$$\left(\frac{1}{6}\right)^4 + 4\left(\frac{1}{6}\right)^3 \times \frac{5}{6} + 6\left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 = \frac{171}{1296}$$

or the probability lies between $\frac{1}{7}$ and $\frac{1}{8}$.

Again, suppose it were required to find the probability of throwing an ace, at least once in four throws. Here p, q, and n, are the same as before, and $v = 3$. In computing the sum of the first four terms of the development, much may be saved by recollecting that the sum of all the terms of the development is 1, and consequently, the required sum is equal to 1, diminished by the last term, or the probability is equal to

$$1 - \left(\frac{5}{6}\right)^4 = 1 - \frac{525}{1296} = \frac{671}{1296}.$$

Probability derived from experience. We have thus far supposed that the number of different ways in which an event can happen, is known; but in the greater number of the most important questions to which the method of probabilities is applied, it happens that the number of chances, favorable and unfavorable to the occurrence of the event, is unknown. In such cases, the ratio which is taken as the measure of probability can only be inferred, from considering the ways in which the event has already been observed to happen. Taking the case of the urn once more, let us suppose it to contain a certain number of balls, of different colors, but that the number of each color is unknown. Let it be required to determine the measure of the probability of drawing a ball of a particular color, at the first trial. The method is this: taking a simple case, suppose the urn to contain only four balls, two white. and two black, and that in four successive trials, (the ball being replaced at each trial,) three white ones and one black one, have been drawn. Now, three hypotheses may be formed, with respect to the number of balls, of each color, in the urn:

1st. That there are three white balls and one black one.

2d. That there are two white balls and two black ones; and

3d. That there is one white ball and three black ones.

If we denote the probability of drawing a white ball at a single trial, by p, and that of drawing a black one, by q, we shall have, under the first hypothesis,

$$p = \tfrac{3}{4}, \quad q = \tfrac{1}{4};$$

under the second hypothesis,

$$p = \tfrac{2}{4}, \quad q = \tfrac{2}{4};$$

and under the third hypothesis,

$$p = \tfrac{1}{4}, \quad q = \tfrac{3}{4}.$$

Now, calculating by the preceding principles the probability of drawing three white balls and one black one, in four successive trials, we find the following results:

On the 1st. hypothesis, $4p^3q = \frac{27}{64}$,
on the 2d. hypothesis, $4p^3q = \frac{16}{64}$,
and on the 3d. hypothesis, $4p^3q = \frac{3}{64}$.

The numerators of these results, may be taken as expressions of the relative probabilities of the observed event, on each of the hypotheses. Now, since one or the other of these hypotheses must be true, the sum of the probabilities must be 1. Whence, the probability of the truth of each hypothesis is expressed by the several fractions, $\frac{27}{46}$, $\frac{16}{46}$, $\frac{3}{46}$.

Now, in order to find the probability of drawing a white ball at the next trial, we may reason as follows:

If the first hypothesis is true, the probability of drawing a white ball is $\frac{3}{4}$; but the probability of the hypothesis being true, is only $\frac{27}{46}$, hence, the combined probability of the occurrence of the event, and the truth of the first hypothesis, is

$$\frac{3}{4} \times \frac{27}{46} = \frac{81}{184}.$$

If the second hypothesis is true, the probability of drawing a white ball is $\frac{2}{4}$; but the probability of the hypothesis being true, is only $\frac{16}{46}$, hence, the combined probability is

$$\frac{2}{4} \times \frac{16}{46} = \frac{32}{184}.$$

If the third hypothesis is true, the probability of drawing a white ball is $\frac{1}{4}$, but the probability of its being true is only $\frac{3}{46}$; hence, the combined probability is

$$\frac{3}{46} \times \frac{1}{4} = \frac{3}{184}.$$

Adding together these partial probabilities, the whole probability of drawing a white ball is

$$\frac{81}{184} + \frac{32}{184} + \frac{3}{184} = \frac{116}{184}.$$

Of course, the probability of drawing a black ball at the next trial, is

$$1 - \frac{116}{184} = \frac{68}{184}.$$

The method of reasoning in this particular case may be rendered general. Let c, c', c'', c''', &c., denote separate hypotheses, either of which may account for the occurrence of an event E. Let the probability of the truth of these hypotheses be denoted, respectively, by h, h', h'', h''', &c.. denote the probability of the occurrence of the event calculated on these hypotheses, by p, p', p'', p''', &c., then is the probability of the occurrence of the event given by the following formula,

$$P = hp + h'p' + h''p'' + h'''p'''' + \&c. \ldots.$$

The preceding course of reasoning exhibits the method of submitting the probability of the occurrence, or failure of any contingent event to numerical computation.

One of the most common and useful applications of the methods of probabilities is, in computing the elements employed in the subjects of annuities, reversions, assurances, and other interests, depending upon the probable duration of human life. See *Annuities*, *Reversions*, *Assurances*, *&c.*

Another important application is to determine the most probable *mean*, or average, of a great number of observed results, in practical Astronomy and general Physics. See *Probable Error*.

PROB'A-BLE ERROR. [L. *probabilis*, from *probo*, to prove]. When a great number of observations have been made, for the purpose of determining any element, each of which is liable to error, the element to be determined is also liable to error; the probable error is the quantity such, that there is the same probability of the true error being greater or less than it. Thus, if twenty measurements of an angle, with a theodolite, give results whose arithmetical mean is 15° 18′ 23″, and if there is the same probability that the error of this result is greater than 2″, that there is of its being less than 2″, then is 2″ the probable error of the mean result obtained.

Let l, l', l'', &c., denote the results of h observations, and let m denote their arithmetical mean, that is, their sum divided by their number. Now, if m be subtracted from each of the results l, l', &c., and the sum of the squares of these differences or errors, be denoted by

$$\Sigma (l - m)^2,$$

we shall have, if we denote the probable error by P, the formula

$$P = .674489 \frac{\sqrt{\Sigma (l - m)^2}}{h};$$

that is, the square root of the sum of the squares of the errors divided by the number of observations, and multiplied by .674489.

It is often found more convenient to find the probable error, by means of the weight. See *Weight*.

Instead of using as above, the arithmetical mean, we may use the result deduced from the method of least squares. See *Squares, least method of.*

PROB'LEM. [Gr. $\pi\rho o\beta\lambda\eta\mu\alpha$, from $\pi\rho o$, and $\beta\alpha\lambda\lambda\omega$, to throw]. A question proposed that requires solution. The solution of a problem, is the operation of finding such a value or values as will satisfy the given conditions of the problem. Problems may be algebraical or geometrical. The solution of an algebraical problem consists of two parts; the statement and the solution of the resulting equation or equations.

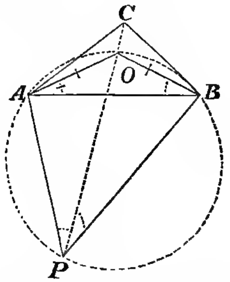
The statement of a problem, is the operation of translating the conditions of the problem into algebraic language. In order that the solution may be determinate, there must be as many independent equations as there are independent conditions. Having found the equations of the problem, their solution is effected by the usual operations of algebra, and the interpretation of the roots found, makes known the required parts of the problem.

Geometrical problems may be solved either by the constructions of Geometry directly, or more simple relations may be deduced by means of Algebra, and the results constructed by means of geometrical principles. The number of geometrical constructions is infinite, and no general principle can be laid down for the solution of problems by means of Geometry.

Geometrical problems may be solved by means of Algebra, by the following rule. Conceive the problem solved, and draw a figure which shall represent the known and required parts of the problem, and draw such auxiliary lines as may be necessary to establish the relations between the known and required parts of the problem. Denote the known parts of the problem by the leading letters of the alphabet, and the required parts by the final letters. Then consider the relations existing between the known and required parts of the problems, and express them by equations. Find as many independent equations as there are required parts of the problem. Solve these equations and find from them the values of the required parts, and construct their values; the problem will be solved.

Problem of the three points. A problem of much use in surveying, particularly in offshore surveying, for fixing the location of buoys, sounding-boats, &c.

The problem is this: From a station P, there can be seen three objects, A, B and C, whose distances from each other, viz., AB, AC and BC, are known. Having measured the angles APC, CPB and APB, it is required to find the distances AP, BP and CP. or the position of the point P.

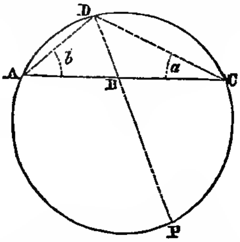


This problem may be solved, trigonometrically, as follows: Conceive a circle to be passed through the three points A, B, and P, cutting the line PC in O: draw OA and OB; then is the angle OAB equal to the angle CPB, and the angle OBA equal to the angle CPA, both of which are known. In the triangle AOB, we have given one side and two angles, and we may therefore compute the two sides OB and OA. In the triangle ABC, the three sides being given, we may compute the angles CAB and CBA. Subtracting the angle OBA from the angle CBA, we have the angle OBC; and in the triangle OBC, we shall then have the sides OB and CB, together with their included angle; and we may, therefore, compute the angle OCB. In the triangle CPB, we now know the side CB, the angle PCB, and the angle CPB, and may, therefore, compute the sides BP and CP. Subtracting the angle OCB from ACB, we have the angle OCA; and then in the triangle ACP, there will be known the sides AC and CP, and the angles ACP and APC; whence, we may compute the remaining side AP, and the problem will be completely solved.

For a general solution of this problem,

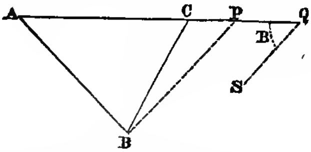
geometrically, see *Plotting*. The solution there given admits of several modifications, depending upon the relative position of the points. There may be 6 cases, which will be considered individually.

1. The three given points may be situated on the same straight line. Denote in all of the cases the measured angle APB by a, and the measured angle BPC by b. The first case admits of two constructions, or the point may be situated on either side of the line ABC.



Construction. Construct the angle ACD$=a$, the angle CAD $=b$, and through the three points A, C, and D, describe the circumference of a circle. Draw DB, and produce it till it cuts the circumference in P; then will P be the point required. A similar construction gives the other point above the line.

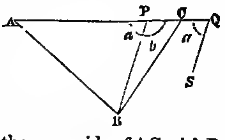
2. When the point P lies on the prolongation of the side AC. In this case, $a = b$.



Construction. Take any point Q on AC, and draw QS, making the angles AQS$=a=b$; through B draw BP parallel to SQ, intersecting AQ in P; then is P the required point.

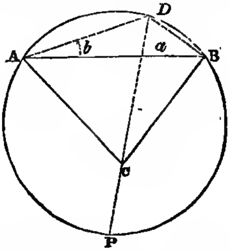
3. When the point P falls between A and C: in this case, $a = 180° - b$.

Construction. Take any point Q in AC, and through it draw QS, making the angle AQS$=a$; through B draw BP parallel to SQ, intersecting AC in P; then is P the point required.

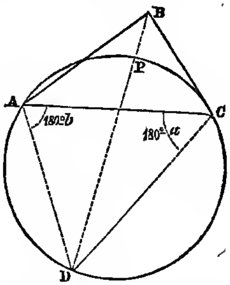


4. When P lies on the same side of AC with B.

Construction. Make the angle ACD $=a$, and the angle CAD $=b$; through the points A, C, and D, draw a circle, and draw a line DB, intersecting the circumference at P; then is P the required point. This is the same as the construction in the general case.

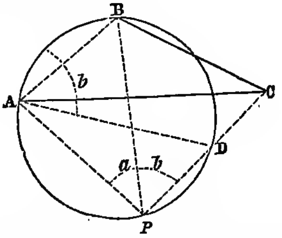


5. When the point P falls within the triangle ABC.



Construction. Make ACD $= 180° - a$ and CAD $= 180° - b$: through the points A, C and D draw a circle, and draw the line DB, cutting the circumference in P; then is the point P the point required.

6. When the point P lies on the side of AC opposite to B.

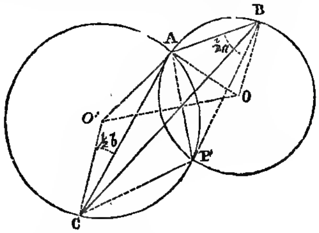


Construction. On AB construct a segment whose inscribed angle is equal to $a + b$; make BAD $= b$, and through D draw the line DC, and prolong it till it cuts the circumference at P; then is P the point required.

If the circle passes through the three points A, B, and C, the problem is indeterminate. In

the first case, if P is on the line ABC, the problem is also indeterminate.

The following analytical solution appears simpler than the one already given :



Lay off $BAO = ABO = \frac{1}{2}a$, also

$$ACO' = CAO' = \tfrac{1}{2}b,$$

with O and O′ as centres, and with radii equal to OB and O′A, describe circles cutting each other in P; draw PA, PB and PC. Then we have

$$AO = \frac{\frac{1}{2}AB}{\cos\frac{1}{2}a},\quad AO' = \frac{\frac{1}{2}AC}{\cos\frac{1}{2}b},\text{ and angle}$$

$$O'AO = CAB - \tfrac{1}{2}(a + b) = \gamma.$$

$$AO' + AO : AO' - AO :: \tan\tfrac{1}{2}(O + O') : \tan\tfrac{1}{2}(O - O').$$

$$2AO' \sin O' = AP.$$

$$O'AP = 90° - O' = \delta,\quad CAB = \tfrac{1}{2}b + \delta.$$

$$CA + AP \quad CA - AP :: \tan\tfrac{1}{2}(C + P) : \tan\tfrac{1}{2}(C - P).$$

$$O' = \tfrac{1}{2}b + O'CP\text{; whence, } O'CP = O' - \tfrac{1}{2}b,$$

$$CP = AO' \cos(O - \tfrac{1}{2}b).$$

PROC′ESS. [L. *processus*]. A course of proceeding.

PROD′UCE. In Geometry, to extend. We are said to produce a limited straight line when we prolong it in either direction. Produce, in Algebra, means to give rise to, or to generate ; thus, we say that the multiplication of two factors produces a result called the product.

PROD′UCT. [L. *productus*, brought forth]. The result obtained by taking one quantity as many times as there are units in another. The two quantities are called *factors*, and the operation is called *multiplication*.

The continued product of any number of factors is the result obtained by multiplying the first factor by the second, that result by the third, that by the fourth, and so on to the last. If the factors are equal, the product is called a power, and the degree of the power is denoted by the number of factors.

The following are some of the properties of products with respect to their forms :

1. The product of the sum and difference of two quantities is equal to the difference of their squares ; that is,

$$(x + y)\,(x - y) = x^2 - y^2.$$

2. Twice the sum of two squares is equal to the sum of two squares ; that is,

$$2(x^2 + y^2) = (x + y)^2 + (x - y)^2.$$

Consequently, the sum of two squares multiplied by any power of 2, is also the sum of two squares. Thus, $5 = 2^2 + 1^2$ and

$$8 \times 5 = 40 = 6^2 + 2^2,$$

also $\quad 5 \times 16 = 80 = 8^2 + 4^2$, &c.

3. The product of the sum of two squares by the sum of two squares, is the sum of two squares ; that is,

$$(x^2 + y^2)\,(u^2 + v^2) = (xu + yv)^2 + (xv - yu)^2 = (xu - yv)^2 + (xv + yu)^2.$$

Thus
$$5 = 2^2 + 1^2$$
$$13 = 3^2 + 2^2$$
$$65 = 8^2 + 1^2 = 7^2 + 4^2.$$

4. The product of the sum of four squares by the sum of four squares, is the sum of four squares ; that is,

$$(w^2 + x^2 + y^2 + z^2)\,(w'^2 + x'^2 + y'^2 + z'^2)$$
$$= (ww' + xx' + yy' + zz')^2 + (wx' - xw' + yz' - y'z)^2$$
$$+ (wy' - xz' - yw' + zx')^2 + (wz' + xy' - yx' - zw')^2.$$

5. The products of two numbers of the form $x^2 + ay^2$, is also of the same form ; that is,

$$(x^2 + ay^2)(x'^2 + ay'^2) = (xx' + ayy')^2 + a(xy' - yx')^2$$
$$+ (xx' - ayy')^2 + a(xy' + yx')^2 = z^2 + aw^2,$$

in accordance with the preceding principles.

6. Two numbers of the form $x^2 + y^2 + z^2$, $x'^2 + y'^2 + 2z'^2$, are such that either multiplied by 2, gives a result of the form of the other ; that is,

$$2\,(x^2 + y^2 + z^2) = (x + y)^2 + (x - y)^2 + 2z^2,$$

and,

$$2(x'^2 + y'^2 + z'^2) = (x' + y')^2 + (x' - y')^2 + (2z')^2.$$

The binomial formula, in the case of a positive exponent of the power to which the binomial is to be raised, may be deduced from the law for the formation of the continued product of any number of factors of the form of $x + a$, $x + b$, &c. The product is indicated

by the formula,

$$(x + a)(x + b)(x + e)\ldots$$
$$= x^m + Ax^{m-1} + Bx^{m-2} + \ldots Nx^{m-n} + \ldots + U.$$

The law of the exponents is, the exponent of the first term of the product is equal to the number of binomial factors employed, and it goes on diminishing by 1, in each term to the right, until the last term, where it is 0. The law of the co-efficients is this; the co-efficient of the first term is 1; that of the second term is equal to the sum of the different products of the second terms of the factors taken in sets of 1; that of the third term is equal to the sum of the different products of the second terms of the binomials, taken in sets of two, and so on; the co-efficient of the term which has n terms preceding it, is equal to the sum of the different products of the second terms of the binomials taken in sets of n, and so on; the last term, or the co-efficient of x^0, is equal to the continued product of the second terms of the binomials employed.

The terms, product of a line by a line, and of a line by a surface, are often used in a technical sense; the former signifies the operation of forming a rectangle, whose adjacent sides are equal in length, respectively, to the two lines multiplied together; the latter signifies the operation of forming a volume equivalent to the volume of a right prism, whose base is equivalent to the area of the surface, and whose altitude is equal to the length of the line. These are the technical meanings of the terms; when used in an arithmetical sense, they might be explained as follows:

To multiply a line by a line, we simply multiply the number of units in the length of one line, by the number of units in the length of the other, and the result is the number of square units in the surface of the rectangle already described. To multiply a surface by a line, we simply multiply the number of square units in the surface by the number of linear units in the line, and the result is the number of units of volume in the solid described. The idea of multiplication in these two cases, is entirely analogous to the sense in which it is constantly used in ordinary life. Thus, we say, that if we multiply the rate of travel by the time employed, we shall get the space passed over. Here, we simply mean, that if the number of units in the rate be multiplied by the number of units of time employed, the product will be the number of units in the space passed over.

PRŌ'FĬLE. In Surveying, a section of the surface of the earth, or of some ideal surface made by a vertical plane, or vertical cylinder. Profiles are made to show the irregularities of the earth's surface along a proposed line of communication, as a railroad, canal, aqueduct, &c Profiles are also made in connection with them, to show the grades of the work along different sections.

The name, profile, is applicable not only to the line of contour in the field, but also to its representation upon paper. The horizontal projection of the line of contour, along the line proposed, is called the plan. In representing a profile on paper, we suppose the projecting cylinder to be developed upon a tangent plane.

The data for making a profile drawing of any section of the earth's surface, are the heights of its different points above some assumed horizontal line, called a *datum* line, together with the horizontal distances of the same points from some fixed point of the line. The vertical distances being generally very small, compared with the horizontal ones, two different scales become necessary in plotting a profile. In order that the vertical distances may be fully exhibited in the drawing, the scale used is much larger than that used for lines in a horizontal direction. See *Leveling for Profile.*

PRO-GRES'SION. [L. *progressio*, from *progredior*, to advance]. A series in which the terms increase or decrease according to a uniform law. There are two kinds of progressions, *Arithmetical* and *Geometrical.*

An *Arithmetical progression.* is a series in which each term is derived from the preceding one by the addition of a constant quantity, called the *common difference.*

If the common difference is *positive*, each term will be greater than the preceding, and the progression is said to be *increasing*. If the common difference is *negative*, each term is less than the preceding, and the progression is *decreasing*. Thus.

$$\ldots 1,\ 5,\ 9,\ 13, \ldots$$

is an increasing progression, and the common difference is $+4$;

$$\ldots . 19,\ 16,\ 13,\ 10 \ldots$$

is a decreasing progression, and the common difference is -3. There is always an infinite number of terms in any progression, but it is customary to consider a limited number of them as constituting a progression; in this case the first and last are called *extremes*; all the other ones are called arithmetical *means*. Such a progression should be called a *limited* progression.

In any limited arithmetical progression, let us denote the first term by a, the last term by l, the common difference by d, the sum of all the terms by s, and their number by n; then will the following formulas be sufficient tc solve any problem in which three of these quantities are given, and the other two required.

TABLE.

	Given.	Required.	Formulas.
1	$a, d, n,$	$l, s,$	$l = a + (n-1)d; \quad s = \frac{1}{2}n[2a + (n-1)d].$
2	$a, d, l,$	$n, s,$	$n = \frac{l-a}{d} + 1: \quad s = \frac{(l+a)(l-a+d)}{2d}.$
3	$a, d, s,$	$n, l,$	$n = \frac{d - 2a \pm \sqrt{(d-2a)^2 + 8ds}}{2d}; \quad l = a + (n-1)d.$
4	$a, n, l,$	$s, d,$	$s = \frac{1}{2}n(a+l); \quad d = \frac{l-a}{n-1}.$
5	$a, n, s,$	$d, l,$	$d = \frac{2(s-an)}{n(n-1)}; \quad l = \frac{2s}{n} - a.$
6	$a, l, s,$	$n, d,$	$n = \frac{2s}{a+l}; \quad d = \frac{(l+a)(l-a)}{2s-(l+a)}.$
7	$d, n, l,$	$a, s,$	$a = l - (n-1)d; \quad s = \frac{1}{2}n[2l - (n-1)d].$
8	$d, n, s,$	$a, l,$	$a = \frac{2s - n(n-1)d}{2n}; \quad l = \frac{2s + n(n-1)d}{2n};$
9	$d, l, s,$	$n, a,$	$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}; \quad n = l - (n-1)d.$
10	$n, l, s,$	$a, d,$	$a = \frac{2s}{n} - l; \quad d = \frac{2(nl-s)}{n(n-1)}.$

Any two numbers may be taken as the extremes of an arithmetical progression of any number of terms: To find the common difference of the progression, subtract the first term from the last, and divide the remainder by the number of terms in the required progression less 1, the result will be the *common difference.* Thus, to insert 4 arithmetical means between 7 and 17, we have $\frac{17-7}{5} = 2$, the common difference; hence, the progression

$$7,\ 9,\ 11,\ 13,\ 15,\ 17.$$

A *geometrical progression* is a series in which each term is derived from the preceding one by multiplying it by a constant quantity called the *ratio* of the progression.

If the ratio is greater than 1, each term is greater than the preceding one, and the progression is *increasing*. If the ratio is less than 1, each term is less than the preceding one, and the progression is *decreasing*. If the ratio is negative, the terms are alternately *plus* and *minus*; the plus terms taken together form a series, whose ratio is the square of the given ratio, and the negative terms, with their signs changed, form a series whose ratio is also the square of the given ratio. Hence, in the following discussion, the ratio may always be regarded as positive:

The series . . . 3, 6, 12, ... is an increasing progression, whose ratio is 2.

The series . . . . 16, 8, 4, 2, ... is a decreasing progression, whose ratio is $\frac{1}{2}$.

If the terms of a decreasing progression be taken in an inverse order, they will constitute an increasing progression, and conversely.

If we denote the first term of a limited

geometrical progression by a, the last term by l, the number of terms by n, the sum of the terms by s, and the ratio of the progression by r, we shall have the following relations:

$$l = ar^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$s = \frac{ar^n - a}{r - 1} = \frac{lr - a}{r - 1}. \quad . \quad . \quad . \quad (2)$$

$$r = \sqrt[n-1]{\frac{l}{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Formula (1) enables us to find the value of the last term of a progression, when we know the first term, the ratio, and the number of terms; formula (2) enables us to find the sum of the terms, and formula (3) enables us to interpolate any number of geometrical means between two given numbers taken as extremes. The rule deduced from the formula for finding the ratio, is this: Divide the last term by the first, and extract the root of the quotient whose index is equal to the number of means required, plus 1. Thus, let it be required to insert 6 geometrical means between 3 and 384; we have

$$r = \sqrt[7]{\frac{384}{3}} = \sqrt[7]{128} = 2,$$

and the series 3, 6, 12, 24, 48, 96, 192, 384.

If we resume formula (2), changing the signs of both terms of the second member, it may be written

$$s = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

If the progression is a decreasing one, r is a proper fraction, and r^n is also a fraction, which diminishes as n increases. The greater the number of terms we take the more will $\frac{a}{1 - r} \times r^n$ diminish, and consequently, the nearer will the value of s approximate to the value $\frac{a}{1 - r}$, and finally, when $n = \infty$, we shall have $s = \frac{a}{1 - r}$; hence, the sum of the terms of a decreasing progression, in which the number of terms is infinite, is equal to the first term divided by one minus the ratio.

PRO-JEC'TION. [L. *projectio*, a throwing out]. The projection of a point upon a plane, in Descriptive Geometry, is the foot of a perpendicular to the plane, drawn through the point.

Projection of a Straight Line. The projection of a straight line upon a plane, is the trace of a plane passed through the line and perpendicular to the given plane.

Projection of a Curved Line. The projection of a curved line upon a plane, is the intersection of the plane with a cylinder passed through the curve, and perpendicular to the given plane. In Descriptive Geometry, points and lines are given by their projections upon two planes, taken at right angles to each other, called planes of projection. We have described the projection as being made by the projecting lines perpendicular to the plane of projection; this is called *Orthographic* or *Orthogonal Projection.* When the projection is made by oblique and parallel lines, it is called *oblique* projection; when the projection is made by drawing lines through a point, called the point of projection, it is called *divergent* projection. See *Descriptive Geometry.*

Spherical Projection. A representation of the surface of the sphere upon a plane, according to some geometrical law, so that the different points in the representation can be accurately referred to their positions on the surface of the sphere. The plane upon which the projection is made is called the *primitive plane.* The primitive plane is generally taken through the centre of the sphere, and when so taken, the great circle cut out by it is called the *primitive circle.*

When the primitive plane is taken through the centre of the sphere, there are three different kinds of projection, depending upon the position of the eye or the projecting point. 1. When the eye is taken in the axis of the primitive circle, and at an infinite distance, the projecting lines are perpendicular to the primitive plane, and the projection is called the *Orthographic projection* 2. When the eye is taken at the pole of the primitive circle, the projection is divergent, and is called the *Stereographic projection.* 3. When the eye is taken in the axis of the primitive circle, and without the surface, equal to the radius of the sphere into the sine of 45°, the projection is also divergent, and is called the *Globular projection.*

These are the only kinds of projection used for projecting the entire sphere. When only a portion of the sphere is to be projected,

other kinds of projection are used, amongst which, the most important are the

Gnomonic Projection, in which the eye is taken at the centre of the sphere, and the principal plane is tangent to the surface of the sphere at a point which is called the principal point.

Polar Projection is when the eye is taken at the centre of the sphere, and the principal plane passes through one of the polar circles.

Conic Projection is when the eye is taken at the centre of the sphere and the surface of a zone is projected either upon the surface of a cone tangent to the surface, along the middle circle of the zone, or upon a secant cone passing through two circles of the zone equi-distant from each other, and from the bases of the zone, which surface, with the projection, is developed upon a plane tangent to it, along one of the elements of the surface.

Cylindrical Projection is when the eye is taken at the centre of the sphere, and the surface of an equatorial zone is projected upon a cylindrical surface tangent to the surface of the sphere, along the equator, which cylinder, with the projection, is developed upon the surface of a plane tangent to the surface of the cylinder along one of its elements.

Mercator's Projection. A modification of the cylindrical projection.

Flansteed's Projection. A modification of the combined cylindrical and conic projections.

Besides these, several others have been proposed, but have not been very extensively employed.

We shall consider these in their order.

The principal circles of the sphere which are projected, are the *Equator*, which is a great circle, whose plane is perpendicular to the axis of the sphere.

The *Ecliptic*, a great circle, whose plane is inclined to the plane of the equator, in an angle equal to about 23° 28′. The points in which it cuts the equator are called *equinoctial* points, aud the points which are 90° distant from these are called solstitial points.

Circles of Latitude, which are small circles whose planes are parallel to that of the equator; the circles of latitude which have received particular names, are the *tropics*, which pass through the solstitial points, and the *polar circles*, which pass through the poles of the ecliptic. The northern tropic is the *Tropic of Cancer*, and the southern one is the *Tropic of Capricorn;* the northern polar circle is called the *Arctic circle*, and the southern one is the *Antarctic circle.*

Meridians are great circles, whose planes pass through the axis of the sphere. Twelve entire meridians, or twenty-four semi-meridians, called hour circles, making angles of 15° with each other, are generally projected. Two principal meridians have received particular names; the one passing through the equinoctial points is called the *equinoctial colure*, and the one through the solstitial points is called the *solstitial colure.*

The ecliptic and the colures are of particular importance in projecting the celestial sphere.

1. *Orthographic Projection.*

In orthographic projection, the eye is taken at an infinite distance. In it the projection of a point is made by drawing a straight line through the point perpendicular to the primitive plane, and finding where it pierces that plane. A limited straight line is projected, by projecting its two extremities, and joining these projections by a straight line. The length of the projection of a straight line is equal to the length of the line multiplied by the cosine of its inclination to the primitive plane. If the inclination is 0, the line is parallel to the primitive plane; the cosine of the inclination is 1, and the projection is equal in length to the line itself. If the inclination is 90°, the cosine of the inclination is 0, and the length of the projection is 0; that is, a straight line which is perpendicular to the primitive plane, is projected into a point.

Every circle of the sphere is projected into an ellipse by the following rule:

Project that diameter which is parallel to the primitive plane, and the result will be the transverse axis of the ellipse. Then project the diameter perpendicular to this one, and its projection will be the conjugate axis of the ellipse. On these axes describe an ellipse, and it will be the projection required.

The length of the transverse axis is always

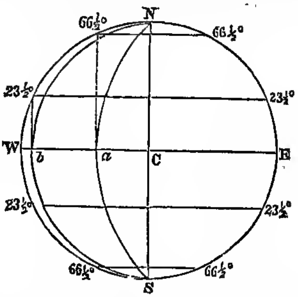
equal to the diameter of the circle to be projected, and the length of the conjugate axis is equal to that diameter into the cosine of the inclination of the plane of the circle to the primitive plane. When the inclination is 0, the plane of the circle is parallel to the primitive plane, the cosine of the inclination is 1, the axes are equal, and the projection is a circle equal to the given circle. When the inclination is 90°, the plane of the circle is perpendicular to the primitive plane, the cosine is 0, the length of the conjugate axis is 0, and the projection is a limited straight line equal in length to the diameter of the circle to be projected. These principles enable us to project every circle of the sphere under every possible supposition.

In making the projection, only one-half of the sphere is projected from one position of the sphere. In order to project the remaining hemisphere, it is revolved through an angle of 180° about a line drawn tangent to the primitive circle; the primitive circle again comes into the original position of the primitive plane, and the projection is finished. This operation is common to the orthographic, stereographic, and globular projections.

The orthographic projection is generally made either upon the plane of the equator, or upon the plane of a meridian. In the former case, the meridians are projected into straight lines, intersecting each other at the centre of the primitive circle. In the latter case the circles of latitude are projected into straight lines perpendicular to the axis of the sphere. The annexed figures show a projection of one hemisphere, made upon the plane of a meridian and upon the plane of the equator. They also show the method of making a graphical construction in each case.

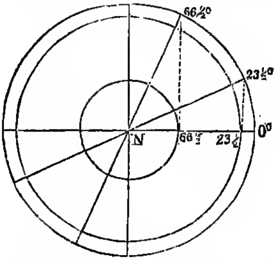
The first figure is the projection upon the plane of a meridian, made as follows:

A circle NESW is described, and two diameters, NS and EW, are drawn at right angles to each other; the first represents the axis, and the second the projection of the equator. The quadrants of the circle are next divided into spaces of 10° each, and numbered from E and W to N and S. We have only projected some of the elements, but enough to show the method of proceeding, in all cases. From the points of division draw lines parallel to EW, and they will represent the projections of parallels of latitude. From the same points let fall perpen-



diculars upon EW, and describe ellipses, having a common transverse axis NS, and semi-conjugate axes, respectively equal to Ca, Cb, &c. These will be the projections of meridians.

The second figure is the projection upon the plane of the equator. In the figure, we have only drawn some of the simplest elements to indicate the method of projection



It is customary to project circles of latitude 10° or 5° apart, and to project meridians also 10° or 5° apart. Let us suppose that circles, at distances of 10° apart, are projected; then would the quadrangular spaces, in both figures, represent spherical quadrilaterals on the surface of the sphere, 10° of latitude in length, and 10° of longitude in breadth. A simple inspection of the figure would show that, with the exception of those which occupy the centre of the projections, the several regions, particularly those nearest the circumference, would be much distorted and diminished in magnitude. This fact renders this kind of projection of little value.

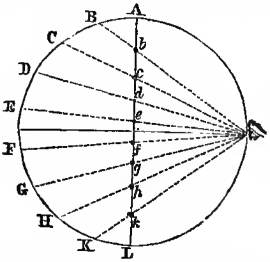
2. *Stereographic Projection.*

In this projection, the eye is taken at the

pole of the primitive circle, and each hemisphere is projected separately.

The advantages of this projection over the orthographic are: that there is not so much distortion of figure, not so much crowding of parts together, and it is easier of execution.

In the stereographic projection, every circle of the sphere is projected into a circle. In general, only one diameter of the circle is projected into a diameter, and that one is cut



out by a plane passed through the axis of the circle to be projected and the axis of the primitive circle. This plane cuts from the primitive plane a line, called the *line of measures*. In order, therefore, to project any circle, we have only to project that diameter, and on this projection, as a diameter, to describe a circle. The following rules enable us to project any circles of the sphere stereographically:

1. To project a great circle. The centre of the projection is in the line of measures at a distance from the centre of the primitive circle equal to the tangent of the inclination of the circle, and the radius of the projection is equal to the secant of the inclination, the radius of the sphere being 1.

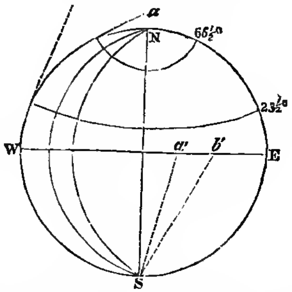
2. To project a small circle whose plane is parallel to the primitive plane. The centre of the projection is at the centre of the primitive circle, and the radius with which it is described is equal to the semi-tangent of the circle's polar distance. The *semi-tangent* of an arc, in this connection, is used to signify the *tangent of half the arc.*

3. To project a small circle whose plane is perpendicular to the primitive plane. The centre of the projection is in the line of measures of the circle, at a distance from the centre of the primitive circle equal to the secant of the circle's polar distance, and the radius of the projection is equal to the tangent of the polar distance.

4. To project a small circle whose plane is oblique to the primitive plane. The extremities of a diameter of the projection are found in the line of measures at distances from the centre of the primitive circle, one equal to the semi-tangent of the circle's inclination, *plus* its polar distance, and the other at a distance equal to the semi-tangent of the circle's inclination, *minus* its polar distance. In all cases where the plane of the circle passes through the point of sight, the projection is a straight line, being the case in which a circle passes to its limit, the radius being infinite.

In this projection the sphere may be represented either upon the plane of a meridian, the plane of the equator, or upon an oblique plane.

The figure represents a part of the projection of a hemisphere upon the plane of a

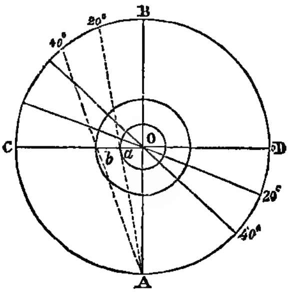


meridian constructed as follows: Describe a circle NESW to represent the meridian. Draw two diameters NS and EW at right angles to each other; the former represents the axis of the sphere, and the latter the equator. Divide the quadrant NW into equal parts and number them as indicated in the figure. At the points of division draw tangents to the meridian, cutting NS produced at a, b, &c. With these points as centres, and radii equal to the respective tangents, describe arcs of circles; these will be the projections of the corresponding parallels of latitude. Draw through S the line Sa', Sb', &c., making with NS the angles 20°, 40°, &c. With a', b', &c., as centres, and the distances from these points to S as radii, describe arcs of circles; these will be the projections of meridians making angles of 20°, 40°,

&c. with the assumed meridian. In this manner all the meridians may be projected.

The construction below represents the stereographic projection of a portion of a hemisphere on the plane of the equator.

Draw a circle ACBD to represent the equator, and in it draw two diameters, AB and



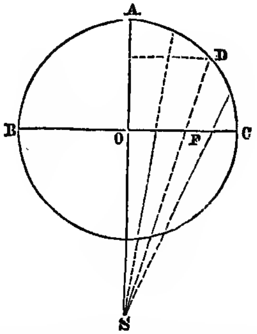
CD, at right angles. Divide each quadrant into equal parts, and number them as indicated on the figure. Through their points of division draw diameters, and they will represent the projections of meridians. Through A, and the points of division, draw straight lines, cutting CD in the points a, b, &c. With the centre of the primitive circle as a centre, and with radii respectively equal to the distances from the centre to a, b, &c., describe circles; these will be the projections of parallels of latitude 10° apart.

The disadvantages of this projection are that the projections of parts of the sphere are much crowded together, though the inconvenience from this cause is not so great as in the orthographic projection. This crowding is evidently the greatest near the centre of the primitive circle.

The sphere may be projected on the plane of any great circle, but the two cases already illustrated are those most generally used.

3. *Globular Projection.*

In this projection the eye is taken without the surface of the sphere in the axis of the primitive circle, and at a distance from its pole equal to the sine of 45°. The nature of the projection is indicated by the figure. If AD is equal to 45°, it can easily be shown that OF = FC, that is, the arc AD = 45° is projected into a line equal to the projection of its equal arc DC. If visual rays be drawn from S to the points of division of the quadrant, their intersection with the plane BC will determine spaces much more nearly equal than in either of the other projections considered.



This projection is still very defective, inasmuch as the projections of portions of the surface of the sphere are very much distorted. A modification of this projection, called the equidistant projection, is sometimes used in practice, constructed as follows:

Draw a circle, and in it two diameters, at right angles to each other. Assume the vertical diameter as the axis of the sphere, and divide the horizontal one into equal parts. Through the poles, and each point of division, draw a semicircle. These will represent the projections of the meridians. Divide each quadrant into equal parts, and number them from the equator towards each pole. Divide the vertical semidiameters into the same number of equal parts, and number them from the equator. Through corresponding points of division, on the same hemisphere, draw arcs of circles; they will represent the projections of circles of latitude.

The circles of the sphere may be projected on the plane of the equator, as follows: Draw a circle, and divide it into equal sectors by diameters; these will represent the projections of meridians. Divide any radius into equal parts, and through the points of division draw circles concentric with the assumed circle; they will represent the projections of equidistant circles of latitude.

This method, as before stated, is not strictly speaking a projection of the sphere, but it is what is usually known as the *globular* projection. With respect to the three kinds of projection considered, the following remarks will

serve to show their defects and their relative advantages.

1. In the orthographic projection upon the plane of a meridian, the parallels of latitude are projected into straight lines, and the meridians into ellipses. Equal spaces and equal distances on the surface of the sphere are represented by unequal spaces and unequal distances in projection. These spaces and distances lessen successively from the centre to the circumference of the projection. Consequently, whilst the central parts of the projection are in tolerable proportion, those at a distance from the centre are much distorted, and diminished in magnitude.

2. In the stereographic projection, parallels and meridians are all projected in circles. Equal distances and equal spaces on the surface of the sphere, are represented by unequal distances and unequal spaces in projection. These distances and spaces increase successively from the centre towards the circumference, so that the parts near the circumference are represented on a larger scale than those near the centre; but the projections of circles intersect under the same angles as the circles do upon the surface of the sphere, and consequently, the relative forms of the several regions are better preserved than in the orthographic method.

3. In the globular and in the *equi-distant* projection, (which differs from the globular chiefly in this, that in the former all circles of the sphere are projected into ellipses with small eccentricities, whereas, in the latter, they are taken to be perfect arcs of circles,) equal distances and equal spaces on the surface of the sphere, are represented by equal or nearly equal spaces in projection; and consequently, the relative dimensions of the different regions are preserved better than in either of the preceding projections. But as the projections of circles do not intersect under the same angles as the lines themselves, the forms of parts of the surface are greatly distorted, and the more so as they are more distant from the centre of the projection. The equi-distant projection is most easily executed, but on the whole, the stereographic projection seems to unite more advantages with fewer disadvantages, in a practical point of view, than either of the others, and is the one most commonly adopted.

4. *Gnomonic Projection.*

In the gnomonic projection, the eye is taken at the centre of the sphere, and the primitive plane is tangent to the surface at some point. This point is called the *principal point.* The meridian through the principal point is called the *principal meridian*; the circle of latitude through the principal point is called the *principal parallel*; and the polar distance of the principal point is called the *principal polar distance.* There are three cases.

1. *When the principal point is at the pole of the sphere.* In this case, the meridians are projected into straight lines passing through the principal point, and making angles equal to those contained between the meridians themselves. The circles of latitude are projected into circles, having their centres at the principal point, and described with radii respectively equal to the tangents of their polar distances.

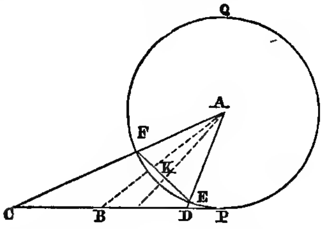
2. *When the principal point is on the equator.* In this case, the meridians are projected into straight lines symmetrically disposed on each side of the projection of the principal meridian, and at distances from it equal to the tangent of the inclinations of the meridians to the principal meridian. The circles of latitude are projected into arcs of hyperbolas, whose transverse axes are coincident with the projection of the principal meridian, and whose centres are at the principal point; the lengths of the semi-transverse axes are equal to the tangents of the latitude of the parallels. The asymptotes of these curves make with the projection of the meridian, angles respectively equal to the complements of the latitudes of the parallels.

3. *When the principal point is on the arc of any circle of latitude.* In this case, the meridians are projected into straight lines passing through a point on the projection of the principal meridian, and at a distance from the principal point equal to the tangent of the principal polar distance; the angles which these projections make with that of the principal meridian, has for tangents the quotient of the tangent of the angle which the meridians make with the principal meridian, by the tangent of the principal polar distance.

The circles of latitude, whose polar distan-

ces are less than the inclination of the axis of the sphere to the primitive plane. are projected into ellipses; the circle whose polar distance is equal to the inclination is projected into a parabola, and all other circles are projected into hyperbolas, whose principal axes are all coincident with the projection of the principal meridian. The projection of any one of the ellipses may be made graphically, as follows.

Let PQ be the principal meridian, EF the diameter which it cuts out of the circle to be projected, P the principal point, and PC the projection of the principal meridian. Draw



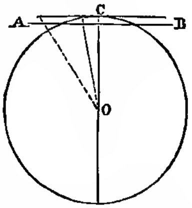
AE and AF, producing them to D and C; then will CD be the projection of the diameter FE, and also the principal axis of the projection of the circle. Bisect CD in B, and draw BA intersecting FE in K: at K draw the chord of the circle which is perpendicular to FE, and project it upon the primitive plane: this will be the remaining axis of the projection: on these describe an ellipse.

By a somewhat analogous construction, the projections of the circles giving hyperbolas may be made. The transverse axis of the projection is found as in the case of the ellipse. Then pass a plane through the centre of the sphere and find the two points in which it cuts the circle to be projected; through these and the centre of the sphere draw two straight lines, and project them upon the principal plane; these will be the asymptotes of the projection, and from these data the construction may be made.

To project the circle which gives the parabola: draw a straight line through the eye and the lowest point of the circle, and find where it pierces the primitive plane; this will be the principal vertex. Draw a straight line through the eye and one extremity of the diameter, which is parallel to the primitive plane, and project this upon the principal plane. Draw from the principal vertex a straight line perpendicular to the projection of the principal meridian, and from the point in which it intersects the first projection draw a line perpendicular to it, then will the point of intersection be the focus.

This projection is but little used, and then only for projecting a limited portion of the sphere in the neighborhood of the principal point. The only case in which it can be applied with any advantage, is when the principal point is taken at the pole, in which case it seems to supply a defect existing in Mercator's projection.

5. *The Polar Projection.*



In this projection, the eye is taken at the centre of the sphere, and the primitive plane is taken through one of the polar circles. In this case the meridians are projected into straight lines intersecting each other at the point in which the primitive plane cuts the axis of the sphere, making angles with each other equal to the angles made by the meridians themselves. The circles of latitude are projected into concentric circles, having their common centre at the same point, and with radii equal to the product of the tangents of their polar distances by the cosine of the polar distance of the polar circle, or the $\cos 23\frac{1}{2}°$.

This projection only answers for a zone lying a few degrees on each side of the polar circle, and like the preceding projection, is used as supplementary to Mercator's projection.

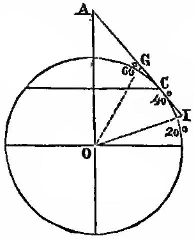
6. *The Conic Projection.*

There are two principal varieties of the conic projection. In both, the eye is taken at the centre of the sphere, and the circles are projected upon a conic which is afterwards rolled or developed upon a plane tangent to it, along one of its elements.

1st. When the projection is made upon a tangent cone.

In this case we suppose a cone to be passed tangent to the surface of the sphere

along the middle circle of latitude of the zone to be projected. In the diagram we have supposed that the zone to be projected extends from latitude 20° to 60° N., and have taken the cone tangent along the circle of 40° N. latitude. In this case the meridians are projected into right-lined



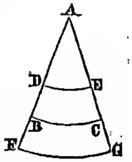
elements of the conic surface, and are developed into straight lines intersecting at the development of the vertex of the cone. The circles of latitude are projected into circles of the conic surface, and are developed into arcs of circles, whose common centre is the development of the vertex of the cone. The distances CG and CI, which measure the distances between the projection of the middle parallel and the extreme parallels, are each equal to the tangents of half the difference of latitude of the extreme parallels. In practice they are usually taken equal to half the length of the circle of longitude between the extreme parallels; and then to find the projection of the intermediate parallels of latitude these are divided into equal parts, and arcs of circles are drawn through the points of division concentric with the projection of the middle parallel. These principles indicate the following constructions:

Let l denote the latitude of the middle parallel of the zone to be projected, and d the number of degrees of longitude to be embraced in the map; then will the absolute length of the middle parallel of the map be equal to

$$\frac{d^{\circ}}{180^{\circ}}\pi \cdot \cos l,$$

the length of the tangent AC is equal to the cotangent of the latitude, and consequently, the number of degrees which the development of the arc subtends at the centre of the development, is denoted by $(d \sin l)^{\circ}$. Now if we denote the number of degrees of latitude between the extreme parallels by d', the length of the map along any meridian will be equal to $\frac{\pi d'}{180^{\circ}}$, and the length from the middle parallel to either extreme, will be $\frac{\pi d'}{360^{\circ}}$, the radius of the sphere being 1.

Construct the angle BAC equal to $(d \sin l)^{\circ}$; with A as a centre, and a radius AB equal to the cotangent of the latitude, describe an arc BC; this will be the projection of the middle parallel.



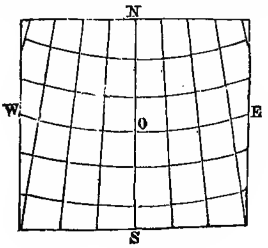
Lay off the distances BD and BF, each equal to $\frac{\pi d'}{360}$ and describe DE, FG: these will be the projections of the extreme parallels. Divide BC into any number of equal parts, and draw through A and these points straight lines: these will be the projections of meridians. Divide FD into any number of equal parts, and through the points of division describe concentric circles having their centre at A; these will be the projections of intermediate parallels.

2d. When the projection is made upon a secant cone.

In this case we suppose the cone to be passed through parallels of latitude equidistant from each other, and from the extreme parallels. The method of constructing the projections in this case are entirely the same as in the case last considered. There is a method somewhat like the conic projection, called Flamsteed's projection.

We shall explain the method of projecting a map by Flamsteed's projection, taking the circles of latitude and longitude 1° apart.

Draw a straight line NS, through the middle of the paper, to represent the middle meridian, and through the middle point of it, O, describe an arc of a circle EW, having its centre on NS produced, and having a radius equal to the cotangent of the latitude of the parallel. From O, on NS, lay off distances



equal to the length of a degree of latitude, to the scale of the map, in both directions, and through these points describe circles concentric with EW. Having computed the length of a degree of longitude for the latitude of each parallel, set this length off from NS, in each direction, and through the point thus determined, draw curved lines, and they will represent the projections of meridians. In the diagram these are sensibly straight lines.

7. *The Cylindrical Projection.*

This projection differs from the conical projection in the fact that the projection is made upon the surface of a cylinder taken tangent to the surface of the sphere along the equator, instead of the surface of a tangent or secant cone.

The pure cylindrical projection is little used, but a modification of it, called Mercator's projection, is much used in projecting sailing charts. In this projection both meridians and parallels of latitude are represented by straight lines. The meridians are represented by parallel straight lines at equal distances from each other. The parallels of latitude are represented by straight lines whose distances apart increase inversely as the lengths of a degree of longitude decreases. In this projection the loxodromic curve, or the path of a ship, when her course makes a constant angle with all the meridians crossed, is represented by a straight line, and this is the great advantage possessed by this kind of projection. See *Mercator's Projection.* Under the head of Mercator's Projection a method was given for constructing a chart on this principle. We add here an analysis of the method of computing the distance of the projection of any parallel of latitude from the equator.

Let l denote the distance of any parallel of latitude from the equator, and L the corresponding distance on the projection, and denote by dl and dL corresponding increments of their elements. Then from the law of the projection

$$dl : dL :: \cos l : 1, \quad \text{or} \quad dL = \frac{dl}{\cos l}.$$

If we denote CN by x, and MN by y, we shall have the relation,

$$x^2 + y^2 = 1.$$

We have also

$$dl = \sqrt{dx + dy^2} = \frac{dx}{\sqrt{1 - x^2}},$$

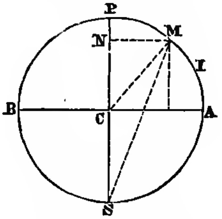
and $$\cos l = \sqrt{1 - x^2};$$

hence, $$dL = \frac{dx}{1 - x^2};$$

or $$dL = \frac{1}{2}\left(\frac{dx}{1 + x} + \frac{dx}{1 - x}\right);$$

whence, by integration,

$$L = \tfrac{1}{2}\left\{ l(1 + x) - \log(1 - x) \right\} + C.$$



But $x = 0$, when $L = 0$; hence, $C = 0$, and we have

$$L = l\left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}}.$$

But from the figure

$$MN = \sqrt{1 - x^2}, \quad SN = 1 + x;$$

consequently, $\frac{SN}{MN}$, or tangent of the angle SMN, is equal to

$$\frac{1 + x}{\sqrt{1 - x^2}}, \quad \text{or} \quad \left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}};$$

hence, $L = l(\tan SMN) = l(\cot PSM)$,

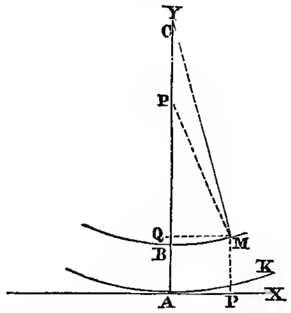
or $L = l(\cot \tfrac{1}{2} PCM)$;

that is, the projection of the length of a meridional arc measured from the equator, the latitude of the extremity being l, is equal to the Naperian logarithm of the tangent of one-half of l, the radius of the sphere being 1. The above formula may be used to compute a table of meridional parts, or in any particular case, it may be used to make a projection of a portion of the sphere.

On account of the rapidly increasing scale of the map, as we approach the polar circles, it becomes impossible to represent the polar regions on a Mercator's chart. To supply the deficiency, those regions are projected either on the plane of the polar circle, or gnomonically, on the tangent plane to the sphere at the pole.

PROJECTION OF MAPS. In addition to the

methods described under the head of spherical projections, the following method, employed by the French Topographers, seems peculiarly worthy of attention. It is a combination of Flamsteed's projection, and the conical method. In it, circles of latitude are represented by concentric circles, described from a centre situated on the central meridian. The radius of the circle of middle latitude is equal to the cotangent of the latitude, and the distances between the parallels of latitude, for each degree, are laid off on the central meridian, being made equal to the actual distance on the terrestrial spheroid, whilst the lengths of the degrees of longitude are made proportional to the cosine of the latitude.



Let A be the centre of the development, AY the central meridian, and AX perpendicular to it; AC, laid off on the central meridian, being the radius for describing the parallel of middle latitude passing through A, and equal to the cotangent of the latitude of A by the normal of the same point, terminating at the axis of the spheroid: AK a portion of the parallel of middle latitude; these distances, equal to the length of 1° or 5°, being set off from A, along the line AC, furnish points through which the concentric parallels of latitude are to be drawn. If along these parallels distances be laid off from the central meridian, equal to 1° or 5° of longitude, on the spheroid corresponding to the particular latitude, the slightly curved lines joining these points will represent the meridians. For small differences of latitude, these lines will not differ sensibly from straight lines. In order to make this projection practically, it is usual to compute the rectangular co-ordinates of the points to be laid down, these being estimated for the lines AX and AY as axes. These co-ordinates are computed from the latitudes and longitudes of the points; and conversely, if these co-ordinates are given, the latitudes and longitudes of the points may be determined.

To investigate the necessary formulas for making this projection, let r denote the radius AC for the parallel of middle latitude, equal to N cot l; let N denote the normal at A, and l the latitude of A; let s denote the distance, in feet, between two parallels of latitude, w the difference of longitude, or the angle AP′M, and θ the angle ACM, subtended at C by the difference of longitude:

$$\text{CM} = \rho, \quad \text{QM} = x = \rho \sin\theta, \quad \text{QC} = \rho\cos\theta;$$
$$\text{PM} = y = \text{BQ} + s = s + \text{BC} - \text{CQ} = s + \rho - \rho\cos\theta = s + \rho(1-\cos\theta) =$$
$$s + \rho\left(1-(1-2\sin^2\frac{\theta}{2})\right) = s + \rho 2\sin^2\frac{\theta}{2}.$$

Substituting for ρ, its value, $\dfrac{x}{\sin\theta}$,

$$y = s + \frac{x}{\sin\theta}\times 2\sin^2\frac{\theta}{2} = s + x\,\frac{1-\cos\theta}{\sin\theta}$$
$$= s + x\tan\frac{\theta}{2}.$$

Since ρ is known, being equal to $r \pm s$, it only remains to find θ, in order to be able to determine x and y for every point in the arc BM. Now, the given longitude w is an arc of the equator, and if an arc θ be taken of the same length on the circle BM, w and θ will be to each other inversely as their radii.

Now, the radius of the circular arc BM is ρ, and that for w is the radius of the parallel of latitude of B represented by λ, is

$$x' = \rho' \cos\lambda,$$

ρ' representing the normal terminating at the minor axis for the latitude λ, so that

$$w \,.\, \theta :: \rho : x', \text{ or } \theta = \frac{wx'}{\rho} = w\cos\lambda\frac{\rho'}{\rho}.$$

Hence, x and y are easily found, and may be tabulated for convenience.

Again, if the co-ordinates of any point of the plane of projection are given, the latitude and longitude of that point may readily be computed from them. Thus,

$$\text{CQ} = r - y \quad\text{and}\quad \tan\theta = \frac{x}{r-y},$$

and $\text{CM} = \rho = \text{CQ}\sec\theta = (r-y)\sec\theta$; or, otherwise,

$CM = \sqrt{(r-y)^2 + x^2}$ and $r - \rho = s$, in feet.

Then, with the given latitude of the point A, or parallel of middle latitude, convert by the aid of the requisite table the meridional distance x into seconds of arc, approximately in the first instance, and correctly after the first approximation by which the first latitude of B or M is found, and the normal ρ' corresponding to it from the tables; and then

$$\omega = \frac{\theta}{\cos\lambda} \cdot \frac{\rho}{\rho'}.$$

PROJECTING CONE. A cone whose directrix is the given line, and whose vertex is the projecting point.

PROJECTING CYLINDER. In the orthogonal projection, a cylindrical surface passing through the line, and having its elements perpendicular to the plane of projection.

PROJECTING LINE OF A POINT. In the orthogonal projection, a straight line passing through the point and perpendicular to the plane of projection. In the divergent projection a straight line drawn through the point and the projecting point.

PROJECTING PLANE OF A STRAIGHT LINE. In the orthogonal projection, a plane passing through the straight line, and perpendicular to the plane of projection. In the divergent projection, a plane passing through the line and the projecting point.

PROJECTING POINT. The assumed position of the eye.

PROLATE SPHEROID. [L. *prolatum*, drawn out]. A solid that may be generated by revolving an ellipse about its transverse axis. Its volume is equivalent to two thirds of that of its circumscribing cylinder.

PROOF. A verification of a rule or result. A converse rule for testing the accuracy of an operation. Addition may be proved on the principle that the whole is equal to the sum of all the parts, by a second addition. But this, strictly speaking, is no proof, but another way of arriving at the result, which, if found to agree with the first result obtained, is said to confirm it. Addition might be proved by the converse operation of continual subtraction, or by taking in succession each part from the sum, until the last, when the final result should be 0. Subtraction may be proved by adding the remainder, or difference, to the subtrahend, which, when the operations are correctly performed, give a result equal to the minuend.

Multiplication may be proved by Division. The quotient obtained by dividing the product by either factor should be equal to the other factor.

Division may be proved by Multiplication. The product of the divisor by the quotient, increased by the remainder, ought to be equal to the dividend.

All of these operations may be proved by the method of casting out the 9's. See *Nines*.

Raising to powers, or evolution, may be proved by the extraction of roots, or Involution. The root of a power of the same degree ought to be equal to the quantity, raised to the power.

Extraction of roots may be proved by raising to powers, or Evolution. The power of a root of the same degree ought to be equal to the quantity whose root was to be extracted.

PROP'ER-TY. [L. *proprietas*, a quality]. An essential attribute of an expression or magnitude. Thus, it is a property of a triangle that it has three sides and three angles.

A characteristic property is an attribute which characterizes the magnitude, and which, if it exists, the magnitude must be of a particular kind. Thus, it is a characteristic property of the hyperbola that the portion of a tangent to the curve at any point, which is intercepted between the asymptotes, is bisected at the point of contact. If it can be proved that a curve has two asymptotes, and that the portion of any tangent between these asymptotes is bisected at the point of contact, the curve must necessarily be an hyperbola.

PRO-PŌR-TION. [L. *proportio*; from *pro* and *portio*, a part]. The relation which one quantity bears to another of the same kind, with respect to magnitude or numerical value. This relation may be expressed in two ways: 1st, by the difference of the quantities, and 2d, by their quotient. When the relation is expressed by their difference, it is called an Arithmetical relation; when by their quotient, Geometrical Proportion, or simply Proportion. The latter method of comparison is by far the most used.

When we divide one quantity by another of the same kind, to ascertain their relative magnitudes or numerical value. one of the quantities is regarded as a standard, and the quotient, which is always a number, is then the measure of the relation which the standard bears to the quantity whose magnitude or value is to be ascertained. The quantity taken as a standard is assumed as known before the comparison is made, and is called the *antecedent;* the quantity to be determined becomes known in consequence of the comparison, and is called the *consequent;* the quotient obtained is called the *ratio* of the standard to the measured quantity; hence, the measure of proportion is *ratio.*

Four quantities are in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth; this relation is expressed algebraically thus,

$$a : b :: c : d.$$

This expression is called a *proportion;* it is read, a is to b as c is to d, and is equivalent to the expression

$$\frac{b}{a} = \frac{d}{c}.$$

Hence, *a proportion* may be defined to be the algebraic expression of equality of ratios.

Two variable quantities are in proportion when their quotient is constant. Two quantities are reciprocally or inversely proportional when the quotient of one by the reciprocal of the other is constant, or when their product is constant. Thus, if $xy=a$, a being constant, $\frac{1}{x} : y$, in the constant ratio a; hence, x and y are reciprocally proportional.

In the proportion

$$a : b :: c : d,$$

the quantities a, b, c and d are called *terms* of the proportion. The first and fourth terms of a proportion are called *extremes;* the second and third terms are called *means;* the first and third are *antecedents;* the second and fourth are *consequents;* the first and second make up the *first couplet;* the second and fourth make up the *second couplet.*

If $b = c$, b is a mean proportional between a and d, and d is a third proportional to a and b. If b and c are not equal, d is a fourth proportional to the other three taken in their order.

If antecedent be compared with antecedent, and consequent with consequent, the proportion is transformed by *alternation.* If the consequents be made antecedents, and the antecedents be made consequents, the proportion is transformed by *inversion.*

If the sum of the antecedent and consequent, in each couplet, be compared with either the antecedent or consequent in each, the proportion is transformed by *composition.* If the difference of the antecedent and consequent, in each couplet, be compared with either the antecedent or consequent in each, the proportion is transformed by *division.*

The following principles indicate the various transformations that may be made upon proportions:

1. If four quantities are in proportion, they are so by alternation; that is

If $a : b :: c : d$, then, $a : c :: b : d$.

2. If four quantities are in proportion, they are so by inversion; that is,

If $a : b :: c : d$, then, $b : a :: d : c$.

3. If four quantities are in proportion, they are so by composition; that is,

If $a : b :: c : d$, then, $a + b : b :: c + d : d$
or $a + b : a :: c + d : c$.

4. If four quantities are in proportion, they are so by division; that is,

If $a : b :: c : d$, then, $a - b : b :: c - d : d$
or $a - b : a :: c - d : c$.

5. Equimultiples of two quantities are proportional to the quantities; that is

$$ma : mb :: a : b.$$

6. In a continued proportion, the sum of all the antecedents, and the sum of all the consequents are proportional to any couplet; that is,

If $a : b :: c : d :: e : f :: g : h$, &c., then,
$a+c+e+g$ &c. $: b+d+f+h+$ &c. $:: a : b$ &c.

7. If the corresponding terms of two proportions be multiplied together, the products are proportional; that is

If $a : b :: c : d$ and $e : f :: g : h$, then,

$$ae : bf :: cg : dh.$$

It is a property of a proportion, deduced immediately from the definition, that the product of the extremes is equal to the product of the means; and conversely, if the product of two quantities is equal to the product of

two others, the first two may be taken as the extremes, and the last two as the means, of a proportion.

PROPORTION. In Arithmetic, a name for the rule of three, since the three given terms, together with the fourth term, constitute a proportion. The rule of three depends upon the principles of proportion. See *Rule of Three.*

PROPORTION HARMONIAL. Four quantities are in harmonial proportion when the first is to the fourth as the difference between the first and second is to the difference between the third and fourth. Thus 24, 16, 12, and 9 are in harmonial proportion, because

$$24 : 9 :: 8 : 3.$$

Three quantities are in harmonial proportion when the first is to the third as the difference between the first and second is to the difference between the second and third. Thus, the numbers 6, 4, and 3 are in harmonial proportion, because

$$6 : 3 :: 2 : 1.$$

PRO-PŌR′TION-AL. One quantity is proportional to another when it so increases or diminishes with it, that their ratio remains constant.

PROPORTIONAL. Relating to proportion, as proportional parts, proportional compasses, &c.

PROPORTIONAL COMPASSES. Compasses or dividers with two pairs of opposite legs, turning on a common point, so that the distances between the points, in the two pairs of legs, is proportional. They are generally constructed with a groove in each leg, so that they may be set to any ratio. They are used in reducing or enlarging drawings according to any given scale.

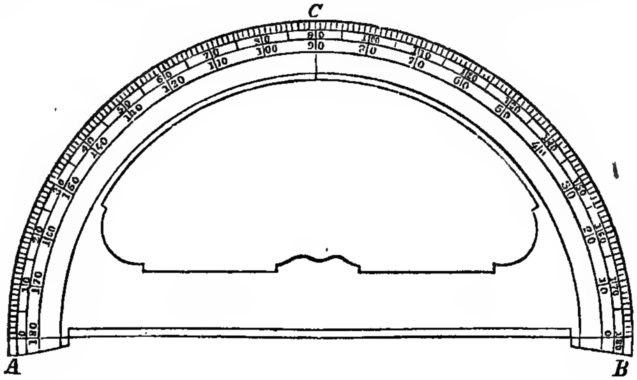
PROPORTIONAL PARTS of magnitudes, are parts such that the corresponding ones, taken in their order, are proportional; that is, the first part of the first is to the first part of the second as the second part of the first is to the second part of the second, as the third part of the first is to the third part of the second, and so on.

PROPORTIONAL SCALE. Same as logarithmic scale. It is a scale on which are marked parts proportional to the logarithms of the natural numbers. They are used in rough computations, and for solving problems graphically, the solution of which requires the aid of logarithms.

PROP-O-SĬ″TION. [L. *propositio*]. Something to be proved or to be done. When something is proposed to be proved, the proposition is called a *theorem.* When something is proposed to be done, the proposition is called a *problem.* In the former case, a principle is to be investigated; in the latter, a principle is to be applied.

PRO-TRACT′. [L. *protractus*, *pro* and *traho*, to draw]. To plot or to draw to a scale. See *Plotting.*

PRO-TRACT′OR. An instrument for laying off angles in plotting. There are three principal forms of the protractor, the *semicircular*, the *circular*, and the *rectangle*, each of which will be explained in turn.



1. The semi-circular protractor consists of a semi-circle of brass, horn, or other hard material, whose circumference is divided into degrees and half degrees, and which are numbered from 0 to 180°, in each direction from the principal diameter, AB, of the protractor, around to the same diameter. There is a small mark at the middle of the diameter AB, which indicates the centre of the graduated arc.

To use this protractor to lay off an angle from a given line at a given point: Place the diameter, AB, so that it shall coincide with the given line, and so that the centre of the protractor shall be at the vertex of the required angle. Then count the number of degrees from A towards B, or from B towards A, and mark the extremity of the arc with a pin; remove the protractor and draw a line through this point and the vertex; this will make the required angle with the given line.

To measure an angle with the protractor: place the protractor so that its centre shall be at the vertex, and so that the diameter, AB, shall coincide with one side of the angle; note where the other side cuts the graduated arc, and the reading of that point of the arc will be the angle required.

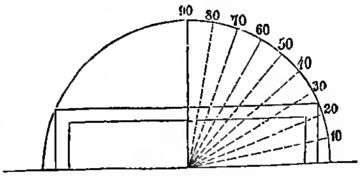
2. The circular protractor consists of a brass circular limb, about six inches in diameter, with a movable arm or index, having a vernier at one extremity and a milled screw at the other, with a concealed cog-wheel, which works into the teeth bordering the limb, by means of which the arm is moved about an axis passing through the centre of the graduated limb. At the centre of the protractor is a small circular glass plate, on which two lines are cut; their point of intersection marks the centre of the graduated circle. The limb is graduated to half degrees, and by means of the vernier, readings may be taken to minutes or half minutes. Two angular pieces of brass, each having a small steel pin at its extremity, are fastened to the index-arm, and turn freely about the axes at right angles to the index-arm. Small antagonistic screws serve to move them in the directions of these axes, for the purpose of bringing them into the prolongation of the same diameter of the graduated limb.

To adjust the steel points, lay the instrument upon a piece of paper, where it is retained from sliding by small steel points projecting from its under-surface. Take the reading of the vernier and record it, then press the points into the paper; they will be thrown out from the paper by two small springs on the under side of the triangular pieces.

Add 180° to the recorded reading, and turn the index-arm by means of the milled screw, till the reading is equal to this sum; then press the points into the paper, and if the instrument is in adjustment, they will fall into the same holes as before, if not, make the correction by means of the adjusting screws, and proceed as before till the adjustment is complete.

To lay off an angle, at a given point, from a given line: Place the instrument so that its centre is exactly over the given point, and turn the index-arm till the two steel points, when pressed down, pierce the paper in the prolongation of the given line; then take the reading of the vernier; to this add the angle to be laid off, and turn the index-arm till the reading is equal to the sum; then press the points into the paper, and removing the protractor, draw through the points a straight line; it will pass through the angular point and make the required angle with the given line. It may be used to measure a given angle on paper as follows: Place the centre over the angular point, and bring the points into the prolongation of one side of the angle and take the reading; then turn the index till the points coincide with the prolongation of the other side; then if the 0 of the vernier has not passed the 0 of the limb, the difference of the readings is the angle required. If the 0 of the vernier has passed the 0 of the limb, add 360° to the lesser reading, and subtract the greater reading from the sum; the remainder will be the angle required.

3. The rectangular protractor, is a rectan-



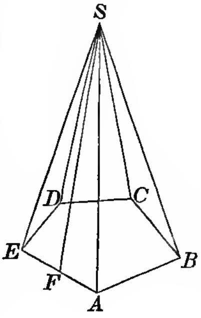
gular scale of brass or ivory, graduated on

three edges to degrees and half degrees, according to the following principles.

One edge of the scale is placed so as to coincide with a diameter of a graduated circle, the middle of the edge being at the centre of the circle ; straight lines are then drawn from the points of division to the centre, and the points in which they cut the edges of the ruler are marked and numbered, so as to accord with the numbers on the circular arc. The use of this protractor is entirely the same as that of the semi-circular protractor.

PURE MATHEMATICS. [L. *purus*, unmixed]. That portion of Mathematics which treats of the principles of the science, in contradistinction to applied Mathematics, which treats of the application of the principles to the investigation of other branches of knowledge, or to the practical wants of life.

PYR'A-MID. [Gr. πυραμις from πυρ, fire or flame, and ειδος, shape]. In Geometry, a polyhedron bounded by a polygon having any number of sides called the base, and by triangles meeting in a common point, called the *vertex*. The triangles taken together make up what is called the convex, or lateral surface of the pyramid.



Pyramids take different names according to the natures of their bases ; they may be *triangular*, *quadrangular*, *&c.*, according as their bases are *triangles*, *quadrilaterals*, *pentagons*, *&c.*

The base and lateral triangles are called *faces ;* the lines in which the faces meet are called edges ; the points in which the edges meet are called vertices of the pyramid.

A right pyramid is one whose base is a regular polygon, and in which a perpendicular let fall from the vertex upon the base, passes through its centre.

The regular pyramid is a pyramid bounded by four equal equilateral triangles ; it is called the *tetrahedron*.

The volume of a pyramid is equivalent to one-third of the volume of a prism, having an equivalent base and an equal altitude. The *altitude* of a pyramid, is the distance from the vertex to the base.

Pyramids are to each other as the products of their bases and altitudes. If their bases are equivalent, they are to each other as their altitudes. If their altitudes are equal, they are to each other as their bases ; if their bases are equivalent, and their altitudes equal, they are equivalent.

The slant height of a right pyramid is the distance from the vertex to either side of the base ; the convex surface of a right pyramid is equal to the product of the slant height by the perimeter of the base. If the number of sides of the base of a right pyramid be increased, the pyramid approaches the right cone, and when the number of sides become infinite, the pyramid becomes a cone, that is, the cone is the limit of all right pyramids inscribed in it, or circumscribed about it.

A triangular pyramid may always have a sphere inscribed within it, and a sphere circumscribed about it.

To inscribe a sphere within a triangular pyramid : Pass planes through three of the edges lying in the same plane, bisecting the diedral angles formed by the adjacent faces : the point in which they meet is the centre of the sphere, and the radius is equal to the distance from this point to either face.

To circumscribe a sphere about a triangular pyramid : Pass planes perpendicular to, and bisecting, these edges, meeting at a common vertex : the point common to these planes is the centre of the required sphere, and the radius is equal to the distance from this point to either vertex.

If a plane be passed cutting a pyramid, and parallel to the base, the portion included between the cutting plane and the base, is called a *frustum* of the pyramid. If the cutting plane is oblique, it is a *truncated* pyramid. The section made by a plane parallel to the base, is a polygon similar to the base, and is called the upper base of the frustum. The altitude of a frustum is the distance between the upper and lower bases of the frustum.

The volume of a frustum of a pyramid is equivalent to the sum of the volumes of

three pyramids, having for bases the upper base, the lower base, and a mean proportional between the two bases, and having for a common altitude the altitude of the frustum.

A spherical pyramid is that portion of a sphere included within three or more plane angles meeting at the centre of the sphere. The spherical polygon included within the plane faces of the pyramid is called the base, and the lateral faces are sectors of circles. The volume of a spherical pyramid is equal to the product of the base by one third of the radius of the sphere.

PYRAMIDAL NUMBERS. The same as figurate numbers. Pyramidal numbers are formed from polygonal numbers by the same rules that polygonal numbers are formed from arithmetical progressions. See *Polygonal Numbers*.

Thus, the series of triangular numbers

1, 3, 6, 10, 15, 21, 28, &c.,

gives rise to the series of triangular pyramidal numbers

1, 4, 10, 20, 35, 56, 84, 120, &c.

In like manner, the other series of pyramidal numbers may be derived. See *Figurate Numbers*.

Q, the seventeenth letter of the English alphabet. As an abbreviation, it stands for *question;* also for *quantity*. As a numeral, Q has been used to stand for 500; with a dash over it, $\bar{Q}$, it stands for 500,000.

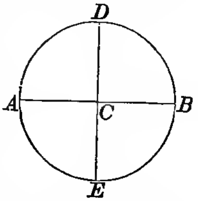
Q. E. D. An abbreviation for *quod erat demonstrandum: which was to be demonstrated;* these letters are frequently written at the end of a demonstration.

Q. E. F. An abbreviation for *quod erat faciendum: which was to be done;* these letters are frequently written at the end of a solution of a problem.

QUAD-RAN″GLE. [L. *quadratus;* from *quatuor*, four, and *angulus*, angle]. In Geometry, a figure having four angles, and consequently four sides. See *Quadrilateral*.

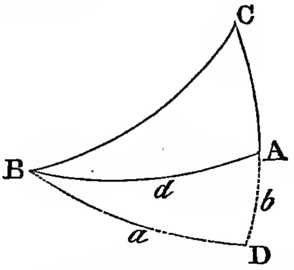
QUAD-RAN″GU-LAR. Having four angles.

QUAD′RANT. [L. *quadrans*, a fourth]. In Trigonometry, one quarter of a circumference of a circle. In trigonometry, the circle is divided by two diameters, at right angles to each other, into four equal parts, called quadrants. One of these diameters is horizontal, the plane of the circle being regarded as vertical, and the other one is vertical. The right-hand extremity of the horizontal diameter is taken as the origin of arcs, and the arcs are estimated from this point around, in a direction contrary to the motion of the hands of a watch, through any number of quadrants. The quadrants, thus passed over, are numbered in their order; the one above the horizontal or initial diameter, and at the right of the vertical diameter, is the *1st quadrant:* the arc immediately at the left of the 1st, is the *2d quadrant;* the one immediately below the 2d, is the *3d quadrant;* and the one immediately below the 1st, is the *4th quadrant*. If the arc be still extended, the 5th quadrant coincides with the 1st, the 6th with the 2d, the 7th with the 3d, &c. See *Trigonometry*.



QUADRANT. An instrument used in Navigation for measuring the apparent altitude of the heavenly bodies. Its principle is the same as that of the sextant,—which see.

QUAD-RANT′AL. A spherical triangle is quadrantal, when one of its sides is a quadrant, or 90°.



Let BAC be a spherical triangle, in which BC = 90°: then is it a quadrantal triangle: Quadrantal triangles admit of solution by means of auxiliary right-angled triangles, as follows: Produce the side CA till CD is equal to 90°, and draw the arc BD: then is BDA a right-angled spherical triangle, which may be solved when any two parts of the quadrantal triangle are given besides the quadrant. For, in it, AD is equal to 90 − CA,

the angle BAD is equal to the supplement of the angle BAC, and BD is equal to the angle C. Having found the parts of the auxiliary triangle, which we may do by the aid of Napier's circular parts, we can, from the above relations, find the corresponding parts of the given quadrantal triangle. If, in the given triangle, the side AC exceeds 90°, AD will be equal to AC − 90°.

QUAD-RAT'IC. Square, denoting a square, or pertaining to it.

Quadratic Equation. An equation of the second degree, containing but one unknown quantity. Every quadratic equation may be reduced to the form

$$x^2 + 2px = q,$$

and its roots, when thus reduced, are

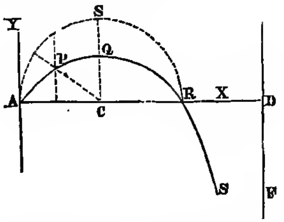
$$x = -p + \sqrt{p^2 + q} \quad \text{and} \quad x = -p - \sqrt{p^2 + q}.$$

If $q > p^2$, the roots are both real. If q is negative, and numerically equal to p^2, the roots are equal; if it is numerically greater than p^2, the roots are both imaginary. If $p = 0$, the equation is incomplete, and the roots are equal with contrary signs, or $x = \pm\sqrt{q}$. They are real when $q > 0$, or when $q = 0$, and imaginary when $q < 0$. See *Equations*.

QUAD-RĀ'TRIX. A curve first employed for finding the quadrature of other curves. The two most important curves of this class, are those of Dinostratus and Tschirnhausen.

1. *Quadratrix of Dinostratus.*

If a straight line revolve uniformly about one of its points, continuing in the same plane, and at the same time if a straight line moves uniformly in the same plane, continuing parallel to its first position, the locus of the intersection of these two lines is the Quadratrix of Dinostratus.



Let AX and AY be two straight lines at right angles, taken as co-ordinate axes, and C, a point, about which the line ACX revolves uniformly, from left to right. Whilst this line revolves through 90° to the position CQ, suppose that the line AY has moved uniformly, and parallel to its first position, to the same position CQ. If now the two motions be continued, according to the same law, the arc APQRS generated, will be an arc of the quadratrix. If CD be made equal to 2CA, and a line be drawn perpendicular to AD through D, this line will be an asymptote to the curve. If the motion be continued, there will result an infinite number of infinite branches, having asymptotes parallel to DF, and at a distance from each other equal to 2AC. If we suppose the motion to have commenced before the generating point reached it, we shall have, in like manner, an infinite number of similar branches on the left. It will be sufficient to consider the arc AQR. The point C is called the pole.

If we denote the abscissa and ordinate of any point P, by x and y, the distance AC by a, we shall find the equation of this arc to be

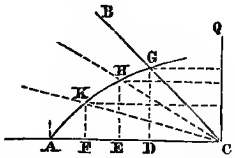
$$y = (a - x) \tan \frac{\pi x}{2a}.$$

It is a property of this quadratrix that CA is a mean proportional between AS, the quadrant of the circumference of the circle, described with the radius CA, and the line CQ: that is

$$\text{AS} = \frac{\text{AC}^2}{\text{CQ}}.$$

It is this property that would enable us to express the circumference of a circle in exact terms of its radius, and consequently to construct a square equivalent to a given circle, were it possible to construct the point Q geometrically.

This curve may be used to trisect an angle, as follows: Let AC be the axis of X, and AKG an arc of the quadratrix. Make the angle ACB equal to the angle to be trisected.

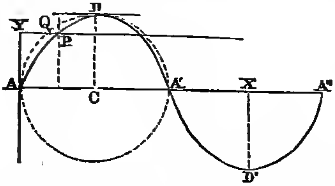


Through G, the point in which the side BC cuts the quadratrix, draw GD parallel to QC,

and divide the line DA into three equal parts. Through E and F, the points of division, draw EH and FK parallel to CQ, and from H and K, where they cut the curve, draw HC and KC; they will trisect the angle ACB. In like manner a given angle may be divided into any number of equal parts, or *multisected.*

2. *The Quadratrix of Tschirnhausen.*

Let AX and AY be two rectangular axes; take a point C upon the axis of X, and with CA as a radius, describe a circumference, and call it the directing circle. If now we suppose the line AY to move uniformly to the



right, remaining parallel to its first position, and at the same time if we suppose the line AA′ to move parallel to its first position, so that its point of intersection with the circumference shall move uniformly from A towards D, then will the locus of the intersection of these two lines be the quadratrix of Tschirnhausen. The two motions are regulated so that whilst the point Q describes a quadrant, the line AY describes a distance equal to the radius of the directing circle.

If now we suppose the line AY to continue moving to the right, the point Q will describe, in succession, the different quadrants, and the curve will take the form APDA′D′A″, &c., having an infinite number of points of intersection, A, A′, A″, &c., with the axis of X, which are at distances from each other equal to the diameter of the directing circle. Its applications are nearly the same as those of the quadratrix of Dinostratus. It may be used for multisecting an angle in the same manner.

Its equation, referred to AX and AY, as axes, is

$$y = a \sin \frac{\pi x}{2a},$$

in which a is the radius of the directing circle.

QUAD′RA-TURE. [L. *quadratura*, squaring]. The operation of finding an expression for the area, bounded by the curve, the axis of X, and any two ordinates. Whenever a square can be constructed, equivalent in area to this portion of the curve, the curve is said to be quadruble. We assume that a square can always be constructed equivalent to the area of a definite portion of a curve, when we can find an expression for it in algebraic terms.

The best method of finding an expression for the area of a portion of the curve, is by means of the Integral Calculus. The formula for a plane area, limited by the curve and axis of X, is

$$A = \int y dx \quad . \quad . \quad . \quad . \quad (1).$$

To apply this formula to any particular case; Find, from the rectangular equation of the curve, the value of y in terms of x; substitute this in the formula, and perform the integration indicated; the resulting integral expresses the area bounded by the curve, the axis of X and any two ordinates whatever.

If we wish to commence estimating the area from any particular ordinate, we can do so by means of an arbitrary constant C which enters the integral. To find the proper value of the arbitrary constant, put the integral equal to 0, and in it substitute for x its value, corresponding to the assumed ordinate; then, from the resulting equation deduce the value of C, and substitute it in the indefinite integral. The value of C being determined, the integral is called a particular integral, and expresses the area bounded by the curve, the axis of X, the assumed ordinate, and any other ordinate whatever.

If we wish now to find the area up to a second assumed ordinate, we substitute, in the particular integral, for x, its value corresponding to this assumed ordinate. The result is the definite integral, and expresses the area of a definite portion of the curve.

For example, let it be required to find the area of a portion of the common parabola. The equation of the curve is

$$y^2 = 2px; \quad \text{whence,} \quad y = \sqrt{2px};$$

and this, in equation (1), gives

$$A = \int \sqrt{2px}dx = \sqrt{2p} \int x^{\frac{1}{2}} dx = \frac{2\sqrt{2p}}{3} x^{\frac{3}{2}} + C.$$

Or, by reduction,

$$A = \frac{2}{3}x\sqrt{2px} + C = \frac{2}{3}xy + C.$$

If now we wish the area to be estimated from the principal vertex, where $x = 0$, $y = 0$, we shall find $C=0$, and denoting the particular integral by A', we shall have

$$A' = \frac{2}{3}xy;$$

that is, the area of any portion of the parabola, estimated from the vertex, is equal to $\frac{2}{3}$ of the rectangle of the abscissa and ordinate of the extreme point.

If it is required to find the area up to the double ordinate through the focus, we have for this limit, $x = \frac{1}{2}p$, $y = p$; whence, denoting the definite integral by A'' we have

$$A'' = \tfrac{1}{3}p^2.$$

This denotes the area between the curve, the axis, and the focal ordinate; hence, if we double it, we shall find the desired area, or

$$A''' = \tfrac{2}{3}p^2 = \frac{4p^2}{6} = \tfrac{1}{6}(2p)^2;$$

that is, the area is equal to $\frac{1}{6}$ of the square described on the parameter of the curve. The curve is, therefore, quadruble. As a general rule, all the parabolas, whose equations can be reduced to the general form,

$$y^m = p'x^n,$$

are quadruble; and if the area is estimated from the vertex of the curve, we have

$$A' = \frac{np'x^{\frac{m+n}{n}}}{m}.$$

All hyperbolas whose equations can be reduced to the general form,

$$y^m x^n = a,$$

are quadruble, except the common hyperbola, in which $m = n = 1$. The expression for the area, in this case, involves a logarithmic expression, which cannot be constructed geometrically.

To find an expression for the area of a surface of revolution, we have the formula,

$$A = 2\pi y\sqrt{dx^2 + dy^2} \dots (2).$$

To apply this in any given case, differentiate the equation of the meridian curve; from the equation and differential equation find expressions for y and dy, in terms of x and dx; substitute these in equation (2), and perform the integration indicated: the resulting indefinite integral will represent the area of a portion of the surface included between any two planes perpendicular to the axis of revolution, taken as the axis of X. The limits of the area may be fixed, as in the case of a plane area, by attributing suitable values to C and x.

No surface of revolution is quadruble, because the expression for the surface always involves π, which is transcendental, and cannot be constructed geometrically.

The *quadrature of the circle* is a famous problem which has probably been the subject of more discussion and research than any other problem within the whole range of mathematical science.

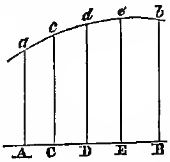
The area of the circle being equal to a rectangle described upon the radius and half of the circumference, it follows that the quadrature would be possible if an algebraic expression, with a finite number of terms, could be found for the length of the circumference. Hence, the problem is reduced to finding such an expression, or to finding an exact expression in algebraic terms for the ratio of the diameter to the circumference. No such expression has yet been found, and it is by no means probable that such an expression will ever be found. The problem may safely be pronounced impossible, and all attempts at the solution of the quadrature of the circle have long been abandoned by every one having the least pretension to mathematical knowledge. It is true, pretenders to the discovery of the quadrature of the circle occasionally present themselves, but they are confined to the list of what may be called mathematical quacks, and their reasoning, when intelligible, is always based upon some absurd hypothesis, or involves some mathematical absurdity easily pointed out by any one having even a smattering of Geometry. Long since the learned societies of Europe have refused to examine any paper pretending to a discovery of the quadrature of the circle, classing it with the problems for the geometrical tri-section of an angle, the duplication of the cube, &c., all of which are now regarded as beyond the power of exact geometrical construction.

QUADRATURES, METHOD OF. A name given

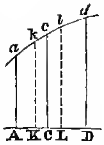
to a peculiar process of the ancient Geometry, for finding an approximate expression for the area included within a given curve. The method consists in drawing ordinates of the bounding curve, at equal distances, between the proposed limits, and then uniting the points in which these ordinates meet the curve, thus formiog an inscribed polygon made up of trapezoids; by taking the sum of these trapezoids as the true area of the curve, an approximate result will be obtained which may be made as nearly equal to the true result as is desirable, by taking the ordinates sufficiently near together. The areas of the trapezoids form a series, the law of which can generally be determined, and the sum of the areas may be found by the ordinary rules for summing series. The exact area may be found by considering the equal distances between the assumed ordinates as arbitrary, then finding a general expression for the sum of the trapezoids, and passing to the limit of this expression by making the arbitrary distance equal to 0. Most of the cases to which the method of quadrature was formerly applied, may be more readily solved by the integral Calculus. See *Quadrature*.

The method of quadratures just considered, affords approximations which, for practical purposes, are sufficiently exact, and which, in many instances, are more readily attained than by the aid of the Calculus. We subjoin a praotical method of finding the area of a given curve.

Let AabB represent any area bounded by the curve $acdeb$, the ordinates Aa, Bb, and the axis AB. Divide the distance AB, into any *even* number of equal parts, AC, CD, &c., and at the points of division draw the ordinates Cc, Dd, &c. Find the lengths of these ordinates by means of a scale of equal parts, then add together *the extreme ordinates, four times the sum of the even ordinates*, and *twice the sum of the odd ordinates;* multiply the result by *one-third of the distance between any two consecutive ordinates*.



To deduce the preceding rule: take two consecutive areas AacC and CcdD; draw two auxiliary ordinates, Kk and Ll, dividing AD into three equal parts.



The areas of the three new trapezoids are,

$$\frac{2}{3}AC\left\{\frac{Aa + Kk}{2}\right\}$$

$$\frac{2}{3}AC\left\{\frac{Kk + Ll}{2}\right\} \text{ and } \frac{2}{3}AC\frac{Ll + Dd}{2},$$

and their sum is,

$$\frac{1}{3}AC\left\{Aa + 2Kk + 2Ll + Dd\right\};$$

but, $2Kk + 2Ll$ is nearly equal to $4Cc$; hence, the area of AadD is equal to

$$\frac{1}{3}AC\left\{Aa + 4Cc + Dd\right\};$$

the area of the next pair of trapezoids is, in like manner, expressed by

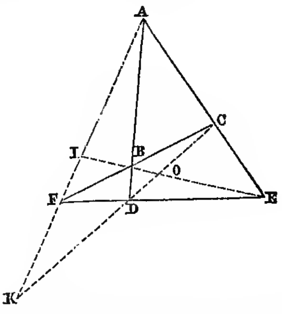
$$\frac{1}{3}AC\left\{Dd + 4Ee + Bb\right\},$$

and so on. Finally, if we take the sum of all the expressions, we shall deduce for the expression of the entire area, denoted by A,

$$A = \frac{1}{3}Ac\left\{Aa + 4Cc + 2Dd + 4Ee + \&c.\right\};$$

whence, the rule above enunciated.

QUAD-RI-LAT′ER-AL. [L. *quatuor*, four, and *latus*, a side]. A polygon of four sides, or four angles. In general, we understand by the term, a salient polygon of four sides, as BCED.



The term, complete quadrilateral, has been applied to the figure formed by drawing straight lines through four points, B, C, D and E, no three of which are in the same straight line. These lines will, in most cases, intersect in two additional points, A and F.

The complete quadrilateral embraces the following, as particular cases :

1. The salient quadrilateral BCED, whose diagonals are CD and BE.

2. The single re-entering quadrilateral ABFE, whose diagonals are AF and BE.

3. The double re-entering quadrilateral ACBFD, whose diagonals are AF and CD.

It will be perceived that the complete quadrilateral has three diagonals, viz. : BE and CD, which are called *interior diagonals*, and AF, which is called an *exterior diagonal.*

In the complete quadrilateral, the diagonals divide each other, so that the parts bear to each other the following relations, viz. :

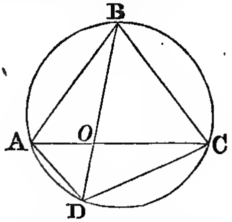
EO : BO : : EI : BI; CO : DO : : CK : DK;
and AI : FI : : AK : FK.

We shall only consider the salient quadrilateral in the following remarks :

General Properties of the Quadrilateral.

1. The sum of its interior angles is equal to four right angles.

2. If the sum of two angles, diagonally opposite to each other, is equal to two right angles, the figure may be circumscribed by a circle, and is called inscriptible.



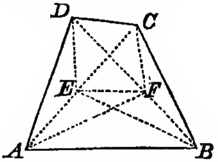
3. In every inscriptible quadrilateral, the rectangle of the two diagonals is equivalent to the sum of the rectangles of the opposite sides, two and two ; that is,

$$AC \times BD \simeq AB \times DC + BC \times AD.$$

4. The area of any quadrilateral is equal to the rectangle of its diagonals multiplied by the sine of the angle included between them ; that is, the area of ABCD is equal to $AC \times BD \sin AOD$.

5. In any quadrilateral, the sum of the squares of the four sides is equivalent to the sum of the squares of the diagonals, plus four times the square of the distance between the middle points of the diagonals ; that is,

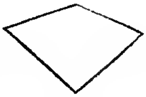
$$CD^2 + CB^2 + AD^2 + AB^2 \simeq BD^2 + AC^2 + 4EF^2.$$



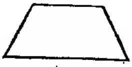
If the quadrilateral is a parallelogram, $EF = 0$, and the sum of the squares of the four sides is equivalent to the sum of the squares of the two diagonals.

Quadrilaterals are divided into three classes, depending upon the relative directions of the sides :

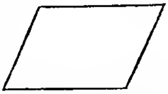
1. The *trapezium*, which has no two sides parallel.



2. The *trapezoid*, which has only two of its sides parallel.

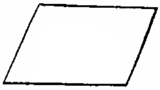


3. The *parallelogram*, whose opposite sides are parallel.

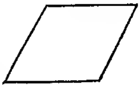


Parallelograms are classed, according to the nature of their angles, into two species ;

1. The *rhomboid*, which is an oblique-angled parallelogram.

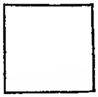


The *rhombus* is an equilateral rhomboid.



2. The *rectangle*, which is a right-angled parallelogram.

The *square* is an equilateral rectangle.



For the properties of these figures, see the articles under the proper headings.

QUAD-RI-NŌ'MI-AL. [L. *quatuor*, four, and *nomen*, name]. A polynomial of four terms.

QUAD'RU-PLE [L. *quadruplus*; from *quatuor*, four, and *plico*, to fold]. Four times; as the quadruple of a given square, is a square having four times the area of the given one.

QUAD-RŬ'PLI-CATE. The fourth power. The quadruplicate ratio of two quantities is the fourth power of their ratio: in a geometrical progression, the ratio of the first term to the second, is the ratio of the progression; and the ratio of the first term to the fifth, is the quadruplicate ratio of the progression, or the fourth power of this ratio.

QUAN'TI-TY. [L. *quantitas*; from *quantus*, how much]. Any thing that can be increased, diminished and measured. Thus, number is a quantity; time, space, weight, &c., are also quantities.

In Mathematics, quantities are represented by symbols; and for convenience, these symbols themselves are called quantities. No error can arise from this conventionality; since we always refer to the quantities which they represent, whenever it is necessary to interpret any result. By a gradual extension of the meaning of the term quantity, it has at last come to be applied to any expression, to which the rules of mathematics are applicable. It is in this sense, that expressions of the form $\sqrt{-1}$, $\sqrt[4]{-1}$, &c., are called quantities. Strictly speaking, the operations indicated by these expressions cannot be performed; they may, nevertheless, be regarded as within the province of Analysis, and by so regarding them, important results are often deduced. We have, then, the following enlarged definition of quantity, viz.: It is anything which may be made the subject of mathematical investigation, or to which the processes of mathematics are applicable.

In this acceptation, the term *quantity* becomes technical, and ceases to have the same meaning, as in ordinary language. This enlarged signification of terms is, by no means, uncommon in mathematics. As an example of the extension of the ordinary meaning of terms, when made mathematical, or of the extension of the meaning of mathematical terms, as the science progresses, we may refer to the term *power*, which originally meant —the product resulting from taking a quantity a certain number of times, as a factor. It has now come to signify any expression affected with an exponent, without reference to the nature of the quantity, or of the exponent. It is, therefore, highly important in mathematical language, to acquire not only the ordinary, but also the technical, meaning of every term employed; as, otherwise, many simple processes would be unintelligible.

Quantities are distinguished, as *known* and *unknown*; *real* and *imaginary*; *constant* and *variable*; *rational* and *irrational*. *Known* quantities are those whose values are given; *unknown* quantities are those whose values are sought: *real* quantities are those which do not involve any operation impossible to perform; *imaginary quantities* are those which involve operations impossible to perform; such as extracting an even root of a negative quantity. *Constant* quantities are those that retain the same value in the same expression; *variable* quantities are those which admit of an infinite number of values in the same expression; *rational* quantities are those which do not involve any radicals; *irrational* quantities are those that involve radicals.

QUÄRT. [L. *quartus*, a fourth]. A unit of measure, equivalent to one-fourth of a gallon. See *Gallon*.

QUAR'TER. [L. *quartus*, a fourth part]. In avoirdupois weight, a quarter is 25 pounds. Dry measure. English, eight bushels of corn.

QUARTER POINT. In Navigation, a fourth part of a point, equivalent to 2° 48′ 45″ of arc.

QUARTER SQUARES. A table of the fourth part of the squares of numbers. It may be used in lieu of a table of logarithms. The formula

$$\frac{(a+b)^2}{4} - \frac{(a-b)^2}{4} = ab,$$

may be proved true, by performing the operations indicated in the first member, and reducing the result. The rule deduced from this formula for multiplying two numbers by addition and subtraction, is this:

Add the factors together and subtract the one from the other: find from the table the quarter square of each of the results, and take the latter from the former; the remainder is the product required.

For example, let it be required to multiply 24 and 16 together; the sum of the factors is 40, and their difference is 8; the quarter square of 40 from the tables is 400, and the quarter square of 8 is 16; taking 16 from 400 there remains for the required product of 24 and 16, the result 384. A table of quarter squares has been published, but is more curious than useful.

QUIN-DEC′A-GON. [L. *quinque*, five; Gr. *δεκα*, ten, and *γωνια*, angle]. A polygon of 15 sides.

QUIN-QUAN″GU-LAR. [L. *quinque*, five, and *angulus*, angle]. Having five angles, and consequently, five sides.

QUINT′AL. A hundred weight. In England, it is 112 pounds. In most countries, is only 100 pounds.

QUIN-TIL′LION. A unit of the 19th order, and expressed by 1, followed by eighteen 0's, thus, 1,000,000,000,000,000,000.

QUIN′TU-PLE. [L. *quinque*, five, and *plico*, to fold]. Five times a thing or quantity.

QUŌ′TIENT. [L. from *quoties*, how many]. The result obtained by dividing one quantity by another. When the dividend and divisor are both quantities of the same kind, the quotient is an abstract number, or a ratio. When the divisor is an abstract number, the quotient is of the same kind as the dividend. See *Division*.

R. The eighteenth letter of the English alphabet. As a numeral, R has been used to stand for 80, with a dash over it, thus, $\overline{\text{R}}$, it has been used to denote 80,000.

RAD′I-CAL. [L. *radicalis*, from *radix*, a root]. An indicated root of an imperfect power of the degree indicated. An indicated root of a perfect power of the degree indicated, is not a radical, but a rational quantity under a radical form.

Radicals are divided into orders, according to the degree of the root indicated.

An indicated square root of an imperfect square, is a radical of the *second degree;* an indicated cube root of an imperfect cube, is a radical of the *third degree;* an indicated fourth root of an imperfect fourth power, is a radical of the *fourth degree;* in general, an indicated n^{th} root of an imperfect n^{th} power, is a radical of the n^{th} *degree.* Thus, $\sqrt{2}$ is a radical of the second degree, $\sqrt[3]{6}$ is a radical of the third degree, and so on; the index of the radical indicates the degree of the radical.

The following principles serve to make many useful transformations of radicals.

1. The product of the n^{th} roots of two quantities, is equal to the n^{th} root of the product of the two quantities.

2. The quotient of the n^{th} roots of two quantities is equal to the n^{th} root of the quotient of those quantities.

The following are some of the most important transformations to which radicals may be subjected, without changing their values.

1. A factor may be removed from under the radical sign, and placed as a co-efficient, thus:

Resolve the quantity under the radical sign into two factors, one of which is the greatest perfect n^{th} *power which enters as a factor; extract the* n^{th} *root of this factor, and place it as a co-efficient before the radical sign, under which place the other factor.* This transformation reduces the radical to its simplest form, and enables us to compare radicals of the same degree, as to their similarity.

2. The converse operation enables us to introduce a co-efficient under the radical sign, as a factor.

Raise the co-efficient to the n^{th} *power, and introduce it as a factor under the radical sign.* This serves to simplify the operation of finding the numerical values of radicals.

The following principle gives rise to an important transformation.

The m^{th} root of the n^{th} root of a quantity, is equal to the n^{th} root of the m^{th} root of the quantity, or to the mn^{th} root of the quantity.

3. To reduce radicals having different indices to equivalent ones, having a common index.

Find the least common multiple of all the indices; this will be the common index required: then raise the quantity under each radical sign, to a power denoted by the quotient of the common index, by the index of the radical; the resulting radicals will be the required form.

The following are the rules for addition, subtraction, multiplication, &c., of radicals:

1. *To add radicals.*

Reduce them, if possible, to similar radicals, that is, to those of the same degree, and

having the same quantity under the radical sign, then add their co-efficients for a new co-efficient; after this, write the common radical part. If they cannot be reduced to similar radicals, they cannot be added except by indicating the operation. A similar rule may be given for subtracting one radical from another.

2. *To multiply radicals together.*

Reduce them to equivalent radicals having the same index: multiply the co-efficients together for a new co-efficient; after this, write the common radical sign, under which place the product of the quantities under the radical signs in the two factors.

3. *To divide one radical by another.*

Reduce them to equivalent radicals having the same index; divide the co-efficient of the dividend by the co-efficient of the divisor for a new co-efficient; after this, write the common radical sign, and under it the quotient of the quantity under the radical sign in the dividend, by that in the divisor.

4. *To raise a radical to any power.*

Raise the co-efficient to the required power for a new co-efficient; after this, write the radical sign with its index unchanged; and under it the required power of the quantity that was under the given radical sign.

If the exponent of the power is a factor of the index of the radical, instead of raising the quantity under the radical sign to the required power, divide the index of the radical by the exponent, and leave the quantity under the sign unchanged. If the index of the radical, and the exponent of the power have a common factor, resolve the exponent into two factors, divide the index of the radical by one of them, and raise the quantity under the radical sign to a power indicated by the other.

5. *To extract any root of a radical.*

Extract the required root of the co-efficient for a new co-efficient; write after this, the radical sign, under which place the required root of the quantity under the radical sign in the given expression.

If the quantity under the radical sign is not a perfect power of the degree indicated, instead of extracting the root, *multiply the given index by the index of the required root, and leave the quantity under the radical sign unchanged.* If the quantity under the radical sign is a perfect power, indicated by a factor of the index of the required root, *extract the root indicated by that factor, and multiply the index by the other factor.*

These rules are sufficient for performing any algebraic operation upon any radicals whatever, except in the case of imaginary quantities. For the rules employed in these cases, see *Imaginary Quantities.* For a conventional method of representing radicals, see *Exponents Negative.*

RADICAL SIGN. The sign $\sqrt{\ }$, when written over a quantity, denotes that its root is to be extracted. The *degree* of the root is indicated by a figure written over it, called the index. Thus, the expression $\sqrt[3]{10}$ indicates that the cube root of 10 is to be extracted, and 3 is the index of the radical. When a root of a polynomial is to be indicated, it is done either by inclosing the polynomial in a parenthesis, and writing the radical sign before it, with the proper index, or the radical sign terminates in a vinculum covering the polynomial. Thus, the indicated cube root of $4a^2 + 2ab - c^2$, may be expressed thus,

$$\sqrt[3]{4a^2+2ab-c^2}\text{; or, thus, }\sqrt[3]{}(4a+2ab-c^2).$$

RĀ'DI-US. [L. *radius*, a ray or spoke]. Half a diameter of a circle, or the distance from the centre to any point of the circumference. All radii of the same circle, or of equal circles, are equal. The radius of a sphere is half a diameter, or it is the distance from the centre to any point of the surface. In the same, or equal spheres, all radii are equal.

RADIUS OF CURVATURE of a curve at any point, is the radius of the osculatory circle at that point. It is so called because its reciprocal is taken as the measure of the curvature at the point The osculatory circle coincides so nearly with the curve, in the immediate neighborhood of the point, that they may be regarded as coincident from a very small space, so that the curvature of the one may be taken for the curvature of the other. Since the curvature of a circle diminishes as the radius increases, it follows that the reciprocal of the radius of curvature is a proper measure of the curvature of a curve.

The formula for the radius of curvature of any curve, at a point whose co-ordinates are x, y and z, is

$$R = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2}} \quad (1)$$

in which s represents the length of any arc of the curve. In this formula neither variable has been designated as the independent variable. If we take s as the independent variable, $d^2s = 0$, and formula (1) becomes

$$R = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}} \quad . \quad . \quad . \quad (2)$$

If we suppose the curve to be a plane curve, and its plane to be the plane of XY, $d^2z = 0$, and formula (1) becomes

$$R = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 - (d^2s)^2}} \quad . \quad . \quad . \quad (3)$$

If we take s as the independent variable, $d^2s = 0$, and formula (3) becomes

$$R = \frac{ds^2}{\sqrt{(d^2x)^2 + d^2y^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we take x as the independent variable, we shall have

$$d^2x = 0,\ ds^2 = dx^2 + dy^2,\ d^2s = \frac{dy}{ds}d^2y, \text{ or}$$

$$(d^2s)^2 = \frac{dy^2}{dx^2 + dy^2}(d^2y)^2;$$

and formula (3) reduces to

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)} \quad . \quad (5)$$

In a polar system the formula for the radius of curvature of a plane curve, is

$$R = \frac{rdv}{dp}; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which r is the radius vector, and p the perpendicular distance from the pole to the tangent to the curve at the point of osculation.

In order to apply these formulas in any given case, they must be combined with the equation and differential equations of the curve, so as to eliminate the differentials, whence the value of R will be obtained in terms of the variables, or in terms of the independent variable, when the independent variable is designated. We shall illustrate the general manner of applying the formulas by considering the case of the radius of curvature of the conic sections. Formula (5) is applicable in this case.

The equation of the conic sections referred to the principal vertex is

$$y^2 = 2px + r^2x^2;$$

whence,

$$\frac{dy}{dx} = \frac{p + r^2x^2}{y} \text{ and } \frac{d^2y}{dx^2} = \frac{r^2y^2 - (p + r^2x)^2}{y^2};$$

and these, combined with equation (5), give, after reduction,

$$R = \frac{\sqrt{(2px + r^2x^2 + (p + r^2x)^2)^3}}{p^2};$$

that is, the radius of curvature is equal to the cube of the normal divided by the square of half of the parameter. From the principle already laid down, the curvature at any point is measured by the square of half the parameter divided by the cube of the normal. The curvature will be a maximum when the normal is a minimum, which is at the principal vertex; and it will be a minimum when the normal is a maximum. There is no such point in either the parabola or hyperbola, but the curvature continually diminishes, and finally becomes 0 at an infinite distance. In the parabola the curve approaches parallelism with the axis, and may very soon be taken as a straight line. In the hyperbola the curve approaches coincidence with the asymptote, and may very soon be regarded as coinciding with it; in both cases the curvature ought to be regarded as 0.

In the ellipse the normal is a maximum at the vertex of the conjugate diameter, and in this case the curvature is a minimum.

Besides the formulas for the radius of curvature, already given, the following are also used in some cases:

$$R = \frac{ds^3}{d^2ydx}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is deduced immediately from formula (5) by substituting ds for $\sqrt{dx^2 + dy^2}$,

$$R = \frac{ds}{d\phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which $d\phi$ is the angle included between the consecutive normals drawn at the extremities of the arc ds.

Radius Vector of a point, in any system of polar co-ordinates, is the distance from the pole to the point. It may be shown analytically, that the radius vector must always be positive, as it should be, since the distance is

conventionally reckoned positively outwards from the pole, in every direction. See *Polar Co-ordinates.*

RAIL'ROAD CURVES. Curves laid out upon the ground for connecting two straight reaches of a railroad track, which make an angle with each other.

The connecting curves are generally arcs of circles, tangent to the two straight portions of the track. Let AB and DC (next figure) represent two straight portions of a track, to be connected by a circular arc. The problem is to lay out this arc upon the ground. There are two principal methods employed for the solution of this problem, which we shall briefly indicate.

1st. *The method by deflections:*

This method depends upon the following elementary principles of Geometry:

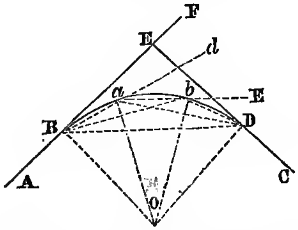
1. The angle formed by a tangent and a chord is equal to half the angle at the centre of the circle subtended by the chord.

2. The angle of deflection formed by any two equal chords, meeting on the circumference, is equal to the angle at the centre subtended by either chord.

3. A line bisecting the angle of deflection formed by any two equal chords, is tangent to the circumference at the point where the two chords meet.

4. If an arc of a circle be divided into any number of equal parts, and lines be drawn from the points of division, meeting each other at any point of the circumference, these lines will form equal angles at the point of meeting, and the angles thus formed will be measured by half of one of the divisions of the arc.

To apply these principles in constructing or laying out the circle on the ground:



Denote the total angle of deflection FED by a, the radius of the arc by r, and the length of the chord BD by C. The angle DOB at the centre is equal to FED, or equal to a, and since BOD is isosceles, we shall have

$$r = \frac{c}{2 \sin \frac{1}{2}a}.$$

Now conceive the angle BOD divided into any number of equal parts, as n; then will each angle, as BOa, be equal to $\frac{a}{n}$, which denote by a'. If a is a point of the circumference, and if we denote the length of the chord Ba by c', we shall have

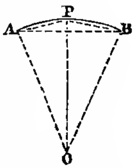
$$c' = 2r \sin \tfrac{1}{2}a' = c\,\frac{\sin \frac{1}{2}a'}{\sin \frac{1}{2}a}.$$

Set up a theodolite at A, and lay off the angle of deflection, FBd, equal to $\frac{1}{2}a'$, and on the line Bd take a distance Ba equal to c', and drive a stake at a. Set up the instrument at a, and lay off the line of deflection daE equal to a', and lay off the distance

$$ab = c',$$

and drive a stake at b; in like manner, continue to determine points and drive stakes till the point D is reached, which should be the position of the last stake driven, if the work is correct. By taking the angle a' small enough, the points a, b, c, &c., may be determined as close together as may be desirable.

Having found some of the principal points in this manner, the curve may be traced in several ways: one of which, by the method of ordinates, will be described hereafter. Another method is, where the chords determined are long, to fill in points by the method of deflections, just described, till the chords are reduced to distances of about 50 feet, then the curve may be traced as follows:



Suppose AB to be one of the short chords of about 50 feet. Denote the angle at the centre of the subtended arc by β. Let P be the middle point of the arc, and PA a chord. Then is

$$\text{PA} = 2r \sin \tfrac{1}{4}\beta \quad \text{and} \quad \text{PA} + \text{PB} = 4r \sin \tfrac{1}{4}\beta.$$

Fasten a chain equal in length to $4r \sin \frac{1}{4}\beta$, so that its two extremities shall be at A and B; then take any link of the chain and stretch the two branches, and drive a stake at the vertex of the angle: this will be in

the circumference, and in this manner any number of points may be found. The chord AB, just considered, is called a sub-chord, and it is often desirable that the sub-chords should be of a given length, say exactly 50 feet or 100 feet. Let us take the latter case; we shall have

$$100 = c\frac{\sin\frac{1}{2}a'}{\sin\frac{1}{2}a};\text{ whence, } \sin\tfrac{1}{2}a' = \frac{100\sin\frac{1}{2}a}{c},$$

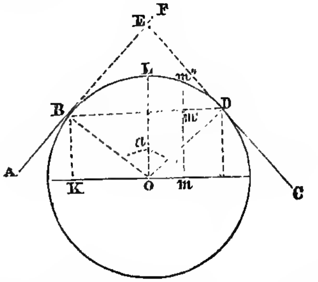
and from this the value of a' may be determined, so that the sub-chords shall be equal to 100 feet. The last sub-chord may be less than 100 feet. If we replace 100 by 50, the formula will give the angle of deflection, so that the sub-chords shall be equal to 50 feet.

These formulas may be tabulated for practical use in the field.

2. *The method by ordinates:*

This method consists in computing the lengths of the offsets or ordinates at points of the sub-chord, taken at equal intervals along that line, and then setting off these ordinates and driving stakes at their extremities. These points are points of the curve.

To investigate a formula for computing the ordinates:



Find, as before, the value of r, and we shall have, from the figure,

$$BK = mm' = r\cos\tfrac{1}{2}a.$$

The equation of the arc BD, referred to KO and OL, as axes, is

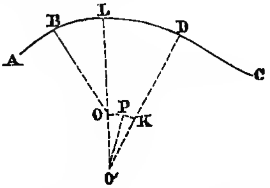
$$y = \sqrt{r^2 - x^2};\text{ hence,}$$

$$m'm'' = y - r\cos\tfrac{1}{2}a = \sqrt{r^2 - x^2} - r\cos\tfrac{1}{2}a,$$

is the required formula. If in the preceding formula we assume values for x, equal to 5, 10, 15, &c., feet, and deduce the corresponding values of $m'm''$, these will be the ordinates or offsets required.

This formula may be tabulated for different values of the chord BD and different angles a, for practical use in the field.

It may be desirable to connect the two straight branches of the track by two arcs of circles tangent respectively to the straight branches and to each other. The construction is as follows:



Let AB and DC be the two straight lines, and B and D the points of contact. Draw BO and DO′ perpendicular to AB and DC: assume O as a centre of the first curve, and make DK = BO; draw OK, and bisect it by a line PO′ perpendicular to it, intersecting DK at O′. Draw OO′ and produce it. With O as a centre, and OB as a radius, describe the arc BL; and with O′ as a centre, and O′L = O′D as a radius, describe the arc LD: it will be tangent to BL at L, and also to DC at D. The two arcs will be tangent to the straight lines and to each other.

RATE PER CENT. In Interest, the part of one dollar which is paid annually for the use of the money. Thus, if 6 dollars are paid per annum for the use of 100 dollars, the rate per cent is $\frac{6}{100}$, or 6 per cent. See *Interest*.

RĀ′TIO. [L. *ratus*, *reor*, to establish, to set; *reor*, is contracted from *redor* or *retor*, and primarily signifies to set in the mind; and *setting* gives the sense of a fixed rate or rule]. The measure of the relation which one quantity bears to another of the same kind; that is, it is the number of times that one quantity contains another regarded as a standard. The only way in which two quantities can be compared with each other, so as to afford an idea of the measure of their relative values, is by dividing the one by the other. The quotient or ratio thus obtained is the proper measure of the relation of the two quantities. Many writers, when comparing two quantities by means of their difference, call this difference arithmetical ratio, to dis-

tinguish it from ratio, as we have defined it above. In no proper sense of the term is this difference a ratio, nor does a comparison of two terms, by means of their difference, convey the remotest idea of their relation obtained by measuring one by the other. To say that one line is an inch longer than another, shows nothing, with regard to how many times one line is longer or shorter than the other, and it is difficult to perceive any appropriateness in calling one inch a ratio. We shall accordingly reject this definition of ratio, and only consider the definition that has been given.

In every ratio there are two quantities compared, one of which is supposed known, and is assumed as a standard; the other is to be determined in terms of this standard These quantities are called *terms* of the ratio: the first one, or that which is antecedently known, is called the *antecedent*, and that whose value is to be measured by the antecedent, is called the *consequent*.

Thus, the ratio a to b is $\frac{b}{a}$; a and b are *terms* of the ratio, a is the *antecedent* and b is the *consequent*. Hence, we say that the ratio of one quantity to another of the same kind, is the quotient arising from dividing the second by the first.

Writers have not agreed in their definition of *the ratio of one quantity to another*. One class have considered it as the quotient of the second quantity by the first, whilst others. of equal or perhaps higher authority, have regarded it as the quotient of the first by the second. We have adopted the former view, only after the most careful consideration of the arguments existing in favor of each.

The following reasons for taking this view of the question, appear to us to be conclusive in the case.

First: This use of the term *ratio* is perfectly consonant with its employment in ordinary language. It is no uncommon thing to hear it said that the population of the country is increasing in a rapid ratio; if this expression has any meaning, it is, that if we assume several epochs equidistant in time, from each other, and then divide the number of inhabitants at any epoch by the number at the preceding one, the quotient becomes greater as we pass from epoch to epoch. At each step of the continued comparison the number of inhabitants at any epoch is taken as a standard in passing to the next. But all authors concur in regarding the standard as the divisor, and in this case the popular idea of the ratio is plainly the quotient of the *consequent* by the *antecedent*. In comparing numbers, the mind necessarily fixes upon 1 as a standard, and all subsequent numbers are regarded as formed from it; therefore, when we inquire what is the relation between 1 and 8, the mind naturally goes through the process of dividing 8 by 1, and the ratio thus found is regarded as the true measure of the relation.

Secondly: It is more convenient in practical operations of arithmetic to regard the ratio as the quotient of the second by the first. Every example in the Rule of Three is a proportion, of which the answer sought is the fourth term. In order to find this fourth term we have only to multiply the third by the ratio of the first to the second. Since the first term is always to be the divisor, the simple rule just given would not apply, were we to regard the ratio of the first term to the second as the quotient of the first term by the second. Hence convenience as well as general analogy indicate the propriety of the definition adopted.

Thirdly: A concrete quantity can only be expressed numerically by the quotient obtained from dividing such quantity by its unit of measure, whatever that unit may be. Every such process is expressed by the following proportion.

unit of meas. : quantity : : 1 : num. value;

when we divide the second term by the first.

But if we divide the first by the second, we must write

quantity : unit of meas. : : num. value : 1;

thus placing the *divisor* in the second place. and the *required* number (the numerical value) in the third; and consequently, reversing all the processes of the Rule of Three.

Fourthly: The adoption of this definition insures uniformity, the first requisite in the use of mathematical terms. All writers concur in regarding the ratio of a geometrical progression as the quotient of the second term by the first, for they all define a progression to be a series of terms each of which is de-

rived from the preceding one by multiplying it by a constant quality called the *ratio* of the progression, or else give some equivalent definition.

Now, the ratio of a progression is only the ratio of one term to the succeeding one, and those writers who adopt the second definition of ratio, are obliged to depart from it here, and adopt one exactly contrary ; thus introducing confusion and breaking up the uniformity of meaning of the term. The inconsistency here referred to will be rendered plainer by an example. In the proportion

$$2 : 4 :: 4 : 8,$$

the ratio, according to the second view of the case is $\frac{1}{2}$; but the advocates of this view, as well as those of the opposite one, admit that the proportion may be written

$$2 : 4 : 8,$$

and that when thus written it is a progression, and all admit in this case that the ratio is 2. But it is plain that the ratio has not changed, and therefore we meet with the absurdity of the ratio of two numbers in the same case, being at one and the same time $\frac{1}{2}$ and 2. According to the view we have adopted, the ratio is 2 in both cases. The last consideration alone, in the absence of any opposing ones, ought to be sufficient to settle the question. The considerations of analogy, convenience and uniformity, taken together, leave no room for the adoption of a contrary definition.

Ratio. A name sometimes given to the Rule of Three in Arithmetic. See *Rule of Three.*

Ratio of a Geometrical Progression. The constant quantity by which each term is multiplied to produce the succeeding one. To find the ratio of a given progression, divide any term by the preceding one.

RA'TION-AL FRACTIONS. [L. *rationalis*, rational]. Fractions in analysis, in which the variable is not affected with any fractional exponents. The co-efficients may be rational or irrational. See *Fractions Rational.*

RA'TION-AL QUANTITY. A quantity which involves no radicals. They are called rational in contradistinction to radical quantities, which are irrational, that is, they cannot be expressed in exact parts of 1.

RAY OF LIGHT. In Shades and Shadows the line of direction along which light is supposed to proceed. A plane of rays is a plane parallel to a ray. A cylinder of rays is a cylinder whose elements are parallel to a ray. See *Shades and Shadows.*

Visual Ray. In Perspective, a straight line drawn through the eye. In divergent projections the projecting line of any point is called a ray.

RĒ'AL QUANTITY. One which does not involve any operations impossible to be performed ; such, for instance, as the extraction of an even root of a negative quantity. The term stands opposed to *imaginary quantity.* See *Imaginary Quantity.*

RE-CIP'RO-CAL. [L. *reciprocus*, returning upon itself] The reciprocal of a quantity is the quotient arising from dividing 1 by the quantity ; thus, the reciprocal of a is $\frac{1}{a}$. The product of a quantity, and its reciprocal, is always equal to 1. The reciprocal of a vulgar fraction is the denominator divided by the numerator.

Reciprocal Equations. A reciprocal equation is one which remains unchanged in form, when the reciprocal of the unknown quantity is substituted for that quantity.

Every equation of the form

$$\left.\begin{array}{l} x^m + px^{m-1} + qx^{m-2} + \ldots \\ \quad + qx^2 + px + 1 = 0 \end{array}\right\} \quad . \quad . \quad (1)$$

is a reciprocal equation ; for, if we substitute in it $\frac{1}{x}$ for x, there results

$$\frac{1}{x^m} + \frac{p}{x^{m-1}} + \frac{q}{x^{m-3}} + \ldots + \frac{q}{x^2} + \frac{p}{x} + 1 = 0;$$

whence, by multiplying both members by x^m and reversing the order of the terms, we find the equation

$$x^m + px^{m-1} + qx^{m-2} + \ldots + qx^2 + px + 1 = 0,$$

an equation which is, in all respects, of the same form as the given equation. Hence, it follows in a reciprocal equation, that if a is a root, the reciprocal of a, or $\frac{1}{a}$, is also a root.

There may be two cases ; the reciprocal equation may be of an odd degree, or it may

be of an even degree. If it is of an odd degree, the co-efficients of the terms, at equal distances from the extremes of the first member, are *equal with the same sign*, or *equal with contrary signs*; an equation of an odd degree can only be reciprocal in these two cases. If it is of an even degree, and complete, the co-efficients of the terms at equal distances from the extremes of the first member, are *equal with the same sign*; if the middle term is wanting, the co-efficients of the terms at equal distances from the extremes of the first member, are *equal with the same or with contrary signs*; an equation of an even degree can only be reciprocal in these two cases.

We shall discuss these four cases separately.

1. *When the reciprocal equation is of an even degree, and the co-efficients of the terms at equal distances from the extremes are respectively equal.*

Let the equation be

$$\left.\begin{matrix} x^{2n} + px^{2n-1} + qx^{2n-2} + \dots \\ + qx^2 + px + 1 = 0 \end{matrix}\right\} \cdot (1)$$

Dividing both members of the equation by x^n, and taking the terms at equal distances from the extremes, in pairs, we have,

$$\left.\begin{matrix} \left(x^n + \frac{1}{x^n}\right) + p\left(x^{n-1} + \frac{1}{x^{n-1}}\right) \\ + q\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \text{\&c.} = 0 \end{matrix}\right\} \cdot (2)$$

If now we make

$$x + \frac{1}{x} = z,$$

we shall have

$$x^2 + \frac{1}{x^2} = z^2 - 2$$

$$x^3 + \frac{1}{x^3} = z^3 - 3z$$

$$x^4 + \frac{1}{x^4} = z^4 - 4z^2 + 2$$

" "

" "

$$x^n + \frac{1}{x^n} = z^n + 5z^{n-2} + \text{\&c.}$$

The second members form a recurring series of the second order, whose scale is $(z, -1)$.

If we substitute these values for

$$x^n + \frac{1}{x^n}, \quad x^{n-1} + \frac{1}{x^{n-1}}, \text{ \&c.}$$

in (2), and reduce, we shall have an equation of the form

$$z^n + p'z^{n-1} + q'z^{n-2} + \dots + t'z + n' = 0 \;.\,.\; (3)$$

an equation which is of a degree only half as great as that of the given equation.

2. *When the reciprocal equation is of an odd degree, and the co-efficients of terms at equal distances from the extremes are respectively equal.*

In this case, the equation is of the form

$$x^{2n+1} + px^{2n} + qx^{2n-1} + \dots + qx^2 + px + 1 = 0.$$

If we make $x = -1$, the first member reduces to 0; therefore, -1 is a root of the equation, and the first member is divisible by $x+1$. Performing the division, the resulting equation will be of an even degree, and reciprocal, having the co-efficients of terms at equal distances from the extremes, respectively equal to each other. By the preceding principle, this equation may be reduced to one of the n^{th} degree in terms of z. Hence, in the case under consideration, the equation can be reduced to one of the n^{th} degree.

3. *When the reciprocal equation is of an even degree, and the co-efficients of terms at equal distances from the two extremes, are equal with contrary signs.*

It may be shown, as before, that both members are divisible by $x^2 - 1$, and the resulting equation will be of the first form considered. Hence, in this case, the reciprocal equation of the $2n^{th}$ degree may be reduced to one of the $(n-1)^{th}$ degree, in terms of z.

4. *When the reciprocal equation is of an odd degree, and the co-efficients taken at equal distances from the extremes, are equal with contrary signs.*

It may be shown by a course of reasoning similar to that employed above, that both members can be divided by $x - 1$, and that the resulting equation will be of the form of the one first considered; hence, in this case, the reciprocal equation of the $(2n+1)^{th}$ degree can be reduced to one of the n^{th} degree in terms of z.

These properties aid in solving reciprocal equations, and have been applied to the case of binomial equations of the form

$$x^n \pm 1 = 0,$$

with much success. See *Binomial Equation.*

Reciprocal Ratio. The same as the reciprocal of a ratio.

Reciprocal Rectangles, in Geometry, are those which are not equal, but whose

areas are equivalent. The base is reciprocally proportional to the altitude, and the reverse.

RECIPROCALLY PROPORTIONAL. Two quantities are reciprocally proportional when both being variable the ratio of the one to the reciprocal of the other, is constant. This requires that their product should be constant. In the equation

$$xy = m,$$

x and y are reciprocally proportional.

RE-CI-PROC'I-TY. [F. *reciprocité*, mutual]. In prime numbers, a certain relation that exists between the remainders resulting from performing the division indicated by the expressions

$$\frac{m^{\frac{n-1}{2}}}{n} \text{ and } \frac{n^{\frac{m-1}{2}}}{m},$$

when m and n are prime. If we designate the remainder in the first case by R, and in the second by R', then, when m and n are both of the form $4a - 1$, we shall have

$$R' = -R,$$

and in all other cases

$$R = R' \ldots.$$

RECK'ON. To compute, to calculate by figures.

DEAD RECKONING. In Navigation, the method of determining the place of a ship from a record kept of the courses sailed and the distance made on each course. This record is called the log book. The courses sailed are determined by the compass, and the distances made on each course by the *log* and *line*. The leeway should be added to or subtracted from the course sailed, as the case may be. The term reckoning is sometimes applied to designate the record kept of the courses, distances, &c.

RE-CLIN'ING DIAL. [L. *re* and *clino*, to lean]. A dial whose plane is inclined to the vertical line through its centre.

RE-CON'NOIS-SANCE. [Fr.] A preliminary, or rough survey of a portion of the country, sometimes undertaken for the purpose of selecting suitable points for trigonometrical stations, preparatory to a more accurate survey; sometimes for the purpose of ascertaining the relative advantages and disadvantages of two or more proposed routes of communication, preparatory to locating a line of railroad, canal, or aqueduct; and sometimes for the purpose of acquiring a general idea of the features of an unexplored country.

A reconnoissance of a portion of country may be undertaken with a view of ascertaining its resources and facilities of transportation, with reference to conducting a military campaign.

In reconnoitering for the purpose of locating points of triangulation of a Geodesic Survey, the essential conditions to be satisfied are, that the selected points should be so chosen that when united by straight lines, the triangles formed shall be well conditioned, that is, shall have no very acute angles; as many of them as possible should be visible from each station, and also from the extremities of the base line; the triangles should be as large as possible, the sides increasing in length from the base to the longest admissible line. A proper reconnoissance for a geodetical survey, is a work of great delicacy, and its successful performance requires a combination of sound judgment and high scientific qualifications. See *Geodesy*.

In reconnoitering for location of a road, canal, or aqueduct, the objects to be attained are, to find the most direct route between the points to be connected, with the most uniform grades and fewest curves. Attention should also be paid to economy of construction, facilities for obtaining materials, and a proper equalization of cutting and embankment. There is another element which, in most cases of reconnoissance, exercises more or less influence, which is, giving such a location to the line of communication as shall not only accommodate the people of the extreme points, but also the greatest number in the general direction of the line. No rules can be laid down for conducting such a reconnoissance, within the narrow limits devoted to this article.

In a reconnoissance for determining an outline of the geographical features of an unexplored country, two sets of operations are generally carried on by the same party. *First*, a set of astronomical operations, which serve to fix, with considerable accuracy, the latitudes and longitudes of the principal points: *secondly*, a running survey intended

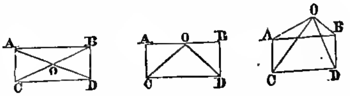
to fill in the astronomical outlines, conducted by means of the pocket-compass, or sextant, and a common watch. The astronomical points are fixed by means of the sextant, transit and zenith sector. The details of these operations do not fall within the proposed limits of this work.

The compass survey is conducted as follows: On starting from a station, the bearing of the prominent objects are carefully measured with the compass, and noted in a book kept for the purpose; they are likewise roughly plotted in a sketch-book, carried by the surveyor. The bearing of the course to be followed is then taken and entered in the note-book, together with the time of starting, and the bearing is also plotted in the sketch-book. After having traveled at as nearly a uniform rate as possible, for a suitable length of time, the surveyor comes to a halt, again notes the time and enters it in his book, and again takes the bearings of the same objects as before, together with other prominent ones, in the general direction of the line of operations. He should note the time of crossing streams, and should take the bearing of their general course, all of which are to be entered in the note-book. A distance may be laid off in the plot of the course followed, proportional to the time occupied in passing over the course, and the new bearings may be plotted on the sketch. The points determined by intersections of the lines whose bearings have been taken, are then fixed upon the plot. The crossings of streams and their directions may also be laid down, and the remaining features necessary to link these outlines together may be sketched in pencil by the surveyor, according to his judgment of their relative positions. Starting from the second station as from the first, the same or similar operations are performed, and so on till the next astronomical point is reached. This is called a line of reconnoissance. A sufficient number of these lines should be run in all directions through the territory to be surveyed, so that by their combination, the geography of the whole territory may be determined. Other operations are generally carried on in connection with a reconnoissance of this nature, having for their end to ascertain the laws of temperature, and the variations of temperature, the natural history of the region, or the nature of the plants and animals inhabiting it, &c.

In noting the time occupied in traversing each course, if the rate of travel is not uniform, or if it varies in passing over different courses, the rate or change of rate should be entered in the note-book as nearly as can be estimated. A good check on the estimation of distances, is a *viameter* attached to a wheel of a wagon, and by a little practice and a careful comparison of estimated results, with measured distances, the surveyor may easily acquire a very correct habit of estimating distances, and also rates of travel.

In plotting a reconnoissance, the circles of latitude and longitude are first projected, and all the points determined by astronomical observation are carefully plotted. Next, the courses connecting these points are separately plotted on a scale larger than that of the general map, and afterwards reduced to the scale of the map. Lateral bearings are next plotted and reduced in like manner, after which the topographical details noted in the note-book and contained in the sketch-book, are laid down; the map is then completed. Maps of this kind are very useful guides to future travelers, and also serve as guides in making subsequent and more detailed surveys.

RECT′AN″GLE. [L. *rectangulus*; from *rectus*, right, and *angulus*, angle]. A parallelogram whose angles are all right angles. The equilateral rectangle is a square. Rectangles having equal bases, are to each other as their altitudes; having equal altitudes, they are to each other as their bases: generally, any two rectangles are to each other as the product of their bases and altitudes. The area of a rectangle is equal to the product of its base and altitude. The area of a rectangle is also equal to the product of its diagonals multiplied by half the sum of their included angle. See *Quadrilateral.*



It is a property of the rectangle, that if any point be taken in its plane, and straight lines be drawn to the vertices of the four angles, the sum of the squares of two lines drawn to

the vertices of the two angles diagonally opposite, is equivalent to the sum of the squares of the lines drawn to the remaining vertices, that is

$$OA^2 + OD^2 \simeq OB^2 + OC^2.$$

The term, rectangle, is sometimes employed for product. Thus, we often say, the rectangle of a and b, meaning thereby their product. This form of expression is of frequent use in analysis.

RECT-AN″GU-LAR. Having right angles. Thus, a parallelopipedon is rectangular, when all its angles are right angles. A system of co-ordinates is rectangular, when the axes of the system are at right angles to each other.

REC-TI-FI-CĀ′TION. [L. *rectus*, right, and *facio*, to make]. The rectification of a curve, is the operation of finding an expression for the length of a definite portion of the curve. When a straight line can be constructed equal in length to any definite portion of a curve, that curve is said to be *rectifiable*. We assume it possible to construct a right line represented by any algebraic expression having a finite number of terms, and we therefore say, that a curve is rectifiable when we can find an expression for the length of any definite portion of it in a finite number of algebraic terms. The most convenient method of rectifying a curve, is by means of the differential and integral Calculus. The formnla for the length of an elementary arc of a plane curve, is

$$ds = \sqrt{dx^2 + dy^2},$$

in which s represents the length of any arc, x any y being the co-ordinates of every point of it.

By integrating, we have

$$s = \int \sqrt{dx^2 + dy^2} \ldots\ldots\ (1).$$

In any particular case, differentiate the equation of the curve, and from this and the given equation find the values of dy, in terms of x and dx: substitute it in the formula, after which perform the operations indicated; the resulting formula will express the length of any portion of the curve. By means of the arbitrary constant which is added, in integrating, we can commence to estimate the length from any point of the arc, and by means of the variable which enters the expression for the particular integral, we can estimate the length up to any point whatever.

To apply these principles to an example, let it be required to rectify the semi-cubic parabola, whose equation may be reduced to the form of

$$y^2 = px^3.$$

By differentiation, we have,

$$2y\,dy = 3px^2\,dx,$$

whence,

$$4y^2\,dy^2 = 9p^2x^4\,dx^2,$$

or, by substitution and reduction,

$$dy^2 = \frac{9}{4}px\,dx^2,$$

which in the integral formula gives

$$s = \int dx\left(1+\frac{9}{4}px\right)^{\frac{1}{2}} = \frac{8}{27p}\left(1+\frac{9}{4}px\right)^{\frac{3}{2}} + C;$$

this is the indefinite integral. Let it be required to estimate the length of the curve from the vertex. For this point,

$$s = 0, \quad x = 0,$$

consequently,

$$C = \mp \frac{8}{27p}.$$

Denoting the particular integral by s', we have

$$s' = \frac{8}{27p}\left[\left(1+\frac{9}{4}px\right)^{\frac{3}{2}} \mp 1\right].$$

This is the particular integral, and if from it we wish to find to length of the curve up to a point whose abscissa is 4, we shall have, denoting the definite integral by s'',

$$s'' = \frac{8}{27p}\left[\left(1+9p\right)^{\frac{3}{2}} \mp 1\right].$$

This is expressed in a finite number of algebraic terms; the semi-cubical parabola is therefore rectifiable.

When the plus sign is given to the value of the binomial part within the parenthesis, the minus sign before 1 must be used: when the minus sign is given to the binomial part, the plus sign before 1 must be used. The numerical value of the length is the same in either case.

It has been ascertained, that all parabolas are rectifiable whose equations can be reduced to the general form,

$$y^m = 2px^n,$$

whenever the exponents m and n are consecutive whole numbers, the odd one being the greatest. The cycloid is also a rectifiable curve, but the circle, ellipse, hyperbola, and parabola, are not rectifiable. Although a curve may not be rectifiable, it is often possible to find an expression for its length in transcendental terms.

The following formula expresses the length of any portion of the arc of a common parabola, estimated from the vertex.

$$s' = \frac{y\sqrt{p^2+y^2}}{2p} + \frac{p}{2}\left[l(\sqrt{p^2+y^2}+y) - lp\right],$$

in which p denotes half of the parameter of the curve.

To rectify the ellipse, we have

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y},$$

this, in the formula, gives

$$s = \int dx\sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}};$$

in which e denotes the eccentricity. Since $x^2 < a^2$, we may write

$$x = a\sin\phi,$$

whence,

$$dx = a\cos\phi\, d\phi,$$

and

$$s = \int a\,d\phi\,(1 - e^2\sin^2\phi)^{\frac{1}{2}};$$

which may be developed into a series and then integrated; this will give

$$s = a\left[\phi - \frac{1}{2}e^2\int\sin^2\phi\,d\phi - \frac{1}{2}\cdot\frac{1}{4}e^2\int\sin^4\phi\,d\phi\right.$$
$$\left. - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}e^2\int\sin^6\phi\,d\phi - \&c.\right].$$

Integrating between the limits

$$\phi = 0, \quad\text{and}\quad \phi = \frac{\pi}{2},$$

we have for the length of one-fourth of the circumference of the ellipse,

$$s'' = \frac{\pi a}{2}\left[1 - \left(\frac{1}{2}e\right) - \frac{1}{3}\left(\frac{1}{2}\cdot\frac{3}{4}e^2\right)^2\right.$$
$$\left. - \frac{1}{5}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}e^3\right)^2 - \&c.\right]$$

If $e = 0$, the ellipse becomes the circle, and we have

$$s'' = \frac{\pi a}{2}.$$

To rectify the hyperbola, we have, as before,

$$s = \int dx\sqrt{\frac{e^2x^2 - a^2}{x^2 - a^2}};$$

and, making $x = \frac{a}{\cos\phi}$, we have

$$s = \int ae\frac{d\phi}{\cos^2\phi}\sqrt{1 - \frac{\cos^2\phi}{e^2}}.$$

Developing and integrating by series, we have

$$s = ae\tan\phi - \frac{a\phi}{2e} - \frac{a}{e}\int d\phi\left[\frac{1}{2}\cdot\frac{1}{4}\frac{\cos^2\phi}{e^2}\right.$$
$$\left. + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\frac{\cos^4\phi}{e^4} + \&c.\right].$$

If we take any point of the curve whose abscissa is x, the length of the corresponding portion of the asymptote is $\frac{ae}{\cos\phi}$, and if we subtract the corresponding integral from this expression, and integrate to the limit $\phi = \frac{\pi}{2}$, which gives $x = \infty$, we shall find for the difference,

$$s''' = \frac{\pi a}{4e}\left[1 + \frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{e}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{e^2}\right)^2\right.$$
$$\left. + \frac{1}{4}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{e^3}\right)^2 + \&c.\right].$$

If a plane curve is given by its polar equation, the formula for the length of any portion of the arc is

$$s = \int(dr^2 + r^2dv^2)^{\frac{1}{2}}.$$

If the curve is one of double curvature, the formula for rectification, when the curve is given by rectilinear equations, is

$$s = \int\sqrt{dx^2 + dy^2 + dz^2};\ \text{or,}$$

$$s = \int dx\left(1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}}$$

When the curve is given by its polar equations, it is

$$s = \int(dr^2 + r^2dv^2 + r^2\sin^2v\,du^2)^{\frac{1}{2}}.$$

REC-TI-LIN'E-AR, REC-TI-LIN'E-AL. [L. *rectus*, right, and *linea*, line]. Appertaining to right lines. A rectilinear polygon is one bounded by straight lines. A rectilinear system of co-ordinates is a system in which points are referred to right lines, as axes.

RE-CUR'RING. [L. *recurro*, to run back from *re* and *curro*]. Returning at intervals.

RECURRING DECIMALS. See *Repeating Decimals.*

RECURRING SERIES. A series in which each term is equal to the algebraic sum of the products obtained by multiplying one or more of the preceding terms by certain fixed quantities. These quantities, taken in their order, are called the *scale of the series.*

Recurring series are classed in orders, the order being determined by the number of terms in the scale. When the scale contains one term, the series is of the first order; when it contains two terms, the series is of the second order; and, in general, when the scale contains n terms, the series is of the n^{th} order.

When the scale is given, and as many consecutive terms from the beginning of the series as there are units in the number indicating the order of the series, the subsequent terms may be successively deduced by multiplying the term immediately preceding the required one by the first term of the scale; the second preceding one by the second term of the scale, and so on, and then taking the algebraic sum of all the products. In this manner, any number of terms may be found.

Recurring series arise from the development of fractions of the form

$$\frac{a + bx + cx^2 + \cdots + kx^{m-1}}{a' + b'x + c'x^2 + \cdots + k'x^{m-1} + l'x^m},$$

and the scale of such a series is

$$\left(-\frac{b'x}{a'},\quad -\frac{c'x^2}{a'},\quad -\frac{d'x^3}{a'} \cdots \frac{l'x^m}{a'}\right)\cdots$$

The fraction $\dfrac{a}{a' + b'x}$,

gives rise to a recurring series of the first order, whose scale is

$$\left(-\frac{b'x}{a'}\right);$$

the first term of the series is $\dfrac{a}{a'}$, and the series is given by the equation,

$$\frac{a}{a'+b'x} = \frac{a}{a'} - \frac{ab'}{a'^2}x + \frac{ab'^2}{a'^3}x^2 - \frac{a'b'^2}{a'^4}x^3 + \&c.$$

The fraction, $\dfrac{a + bx}{a' + b'x + c'x^2}$,

gives rise to a series of the second order, whose scale is

$$\left(-\frac{b'x}{a'},\quad -\frac{c'x^2}{a'}\right);$$

the first two terms of the series are

$$\frac{a}{a'} \text{ and } \frac{ba' - ab'}{a'^2}x.$$

The following terms may easily be deduced from the law of the series:

If we have given a recurring series, we can find the fraction from which the series may have been derived; for, let us take the fraction,

$$\frac{a + bx + \cdots kx^{n-1}}{a' + b'x + c'x^2 + \cdots l'x^n}.$$

This may be written

$$\frac{\dfrac{a}{a'} + \dfrac{b}{a'}x + \cdots \dfrac{k}{a'}x^{n-1}}{1 + \dfrac{b'}{a'}x + \dfrac{c}{a'}x^2 \cdots + \dfrac{l'}{a'}x^n},$$

in which the terms of the denominator, after the first, are equal to the terms of the scale of the series taken in their order, with their signs changed. Hence, we may get the denominator of the required fraction by writing 1, and subtracting from it the algebraic sum of the terms of the scale. To find the numerator, we can assume it of the form

$$P + Qx + Rx^2 + \&c.,$$

in which P, Q, R, &c., are quantities to be determined. To find their value, write the resulting fraction equal to the sum of a sufficient number of terms of the given series, then clear of fractions and equate the co-efficients of the like powers of x in the two members; from these find the values of P, Q, R, &c. For example, let there be given the series,

$$1 - 2x - x^2 - 5x^3 + 4x^4 - \&c.,$$

whose scale is

$$(-2x, \quad +4x^2, \quad +x^3).$$

By the rule, we shall have

$$\frac{P + Qx + Rx^2}{1 + 2x - 4x^2 - x^3} = 1 - 2x - x^2 - 5x^3,$$

clearing of fractions,

$$P + Qx + Rx^2 = 1 + \left.\begin{matrix}2\\-2\end{matrix}\right|\, x - \left.\begin{matrix}4\\-4\\-1\end{matrix}\right|\, x^3 \;\&c.;$$

whence, by equating the co-efficients of the like powers of x in the two members, we have

$$P = 1, \quad Q = 0, \quad R = -9;$$

hence, the fraction is

$$\frac{1 - 9x^2}{1 + 2x - 4x^2 - x^3}.$$

In like manner, we may pass back to any fraction when we have given its development and the scale of the series. It is to be observed, that this converse operation is equivalent to finding the sum of an infinite number of terms of the series.

RE-DUC'TI-O AD AB-SUR'DUM. [L]. The name given to a method of reasoning, often employed in Mathematics. It consists in assuming some hypothesis, which must be either true or false, and then combining this assumption with known truths: the reasoning is continued according to known processes until a result is reached which either corresponds with, or contradicts some known truth; in the former case, the hypothesis is said to be proved; in the latter case, the contrary of the hypothesis is proved, or the hypothesis is proved to be absurd. In this latter case, the contrary of an hypothesis is said to be proved by the method of *reductio ad absurdum*. See *Demonstration*.

RE-DUC'TION. [L. *reductio*, from *re*, again, and *duco*, to lead]. In Arithmetic and Algebra, the operation of changing the form of an expression without changing its value; or, having an expression in terms of one unit of measure, it is the operation of finding an equivalent expression in terms of a different unit. Thus, £10 is equivalent to 200*s*., which is equivalent to 2400*d*., which is again equivalent to 9600 farthings. When we pass, as in the example just given, from a unit of any order to one of a lower order, the reduction is said to be *reduction descending*. When we pass to a unit of a higher order, the operation is called *reduction ascending*. Thus, 9600 farthings is equivalent to 2400*d*., which is equivalent to 200*s*., which is equivalent to £10: this operation of conversion is an example of reduction ascending.

In the general case, the object of reduction descending is to convert a denominate number expressed in a varying scale to another expressed in a uniform scale. To reduce denominate numbers to equivalent ones expressed in units of a lower order: multiply the number of units of the highest order by the number of units of the next lower order, which make one unit of this order, and add the number of units of this inferior order; continue this operation till the required order of units is reached.

Example: Required the number of pence in £50 7*s*. 6*d*. Multiply 50 by 20, and add 7 to the product; this gives 1007: multiply by 12 and add 6 to the product; we thus have 12090 for the required number of pence.

When it is required to reduce numbers expressed in terms of any unit to equivalent ones expressed in terms of a unit of a higher order, divide in succession by the numbers expressing the number of units of each order which make one of the next higher order, beginning at the lowest, and going on to the highest.

Thus, to find how many pounds there are in 12090 pence, we divide by 12 and find for a quotient 1007, with a remainder 6; dividing 1007 by 20, we find 50 for a quotient with a remainder 7; hence, the result is £50 7*s*. 6*d*. Were it required to express the result solely in pound units, we should reduce 7*s*. 6*d*. to decimals of a pound. To do this, write down the number of the units of the different orders in a column, the highest being at the bottom, and write after each unit figure a decimal point; then beginning at the top, divide by the number of units which make one of the next higher order in succession, and continue the operation to any required number of decimal places. Thus, in the above example we have the following operation:

$$\begin{array}{r|l} 12 & 6. \\ \hline 20 & 7.5 \\ \hline & 50.375 \end{array}$$

which gives £50.375.

The preceding principles indicate the method of reducing numbers to fractional forms.

Multiply the entire part by the denominator of the fractional part; to this product add the numerator of the fractional part, and write this sum as a numerator over the denominator of the fractional part. Thus:

$$2\tfrac{7}{16} = \frac{32+7}{16} = \frac{39}{16}.$$

By adopting the converse process an improper fraction may be reduced to the form of a mixed number. To reduce a fraction to a decimal form, place a decimal point after the numerator, and annex any number of 0's: then divide the result by the denominator, and continue the operation to any desired degree of accuracy. Thus $\frac{5}{9} = .555555 +$

The reduction of vulgar fractions may be

either descending or ascending; descending, when we reduce from a greater to a less fractional unit, and ascending when we reduce from a less to a greater. In reduction descending, we multiply both terms of the fraction by the same number, thus

$$\frac{1}{2}=\frac{2}{4}=\frac{8}{16}, \text{\&c.}$$

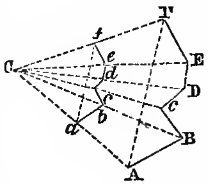
In reduction ascending we divide both terms of the fraction by the same number, thus:

$$\frac{8}{16}=\frac{2}{4}=\frac{1}{2}.$$

For other reductions of fractional expressions, See *Fractions*.

Reduction in Geometry. The operation of constructing a figure similar to the given figure, either greater, less, or equivalent. In the first case, the reduction is ascending, in the second descending. The principle of the reduction is the same in both. Plane figures, as maps, drawings, &c., may be reduced by means of the pantograph, and other instruments contrived for the purpose, or they may be reduced by geometrical construction. For the method of reducing by means of the pantograph, See *Pantograph*.

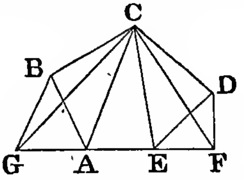
To reduce a figure by geometrical construction. Let it be required to reduce the plot of a road, ABCDEF, to one half the original scale. Draw AF, and draw also a line, *af*, parallel to AF, and equal to one-half of it. Through F and *f*, also through A and *a*, draw straight lines, meeting at O; and from O, draw lines OB, OC, OD and OE, to the angular points of the plot.



Through *f* draw *fe* parallel to FE; through *c* draw *ed* parallel to ED; through *d* draw *dc* parallel to DC, and so on, till the point *a* is reached; then will *abcdef* be the reduced plot. In this manner any plot may be increased or diminished, in accordance with any required condition.

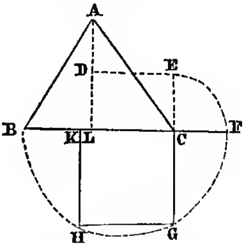
Reduction of a figure of one form to an equivalent figure of another form. This is performed according to various rules of geometry, depending upon the nature of the particular problem. We shall illustrate by one or two examples.

Let it be required to reduce a polygon to an equivalent triangle. Let ABCDE be the given polygon. Produce one of the sides of the polygon, as AE. Draw the diagonal CE, and draw DF parallel to it; draw also CF;



then is the triangle CEF equivalent to the triangle EDC. Draw CA, and also BG, parallel to it, also CG; then is the triangle CGA equivalent to the triangle CBA; hence the triangle CGF is equivalent to the given polygon. In a similar manner any polygonal area may be reduced to an equivalent triangular area.

To reduce a triangle to an equivalent square. Let ABC be the given triangle.



Draw AL perpendicular to BC, and bisect it in D. Prolong BC, making CF = LD; then upon BF as a diameter, describe a semicircle, and through C draw CG perpendicular to BK till it meets the circumference in G, on CG as a side, construct the square CGKH and it will be equivalent to the given triangle By the aid of these two problems any polygonal area may be converted into an equivalent square.

Reduction, in Algebra, sometimes signifies the same as solution. Thus we speak of reducing an equation, meaning the solution, or finding the values of the unknown quantities. The term thus used is liable to be misinterpreted, and such use should therefore be avoided, on the principle that all mathemati-

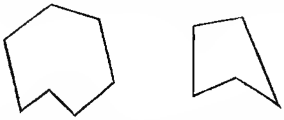
cal terms should have a clear, fixed, and definite meaning.

RE-DUN'DANT HYPERBOLA. Are hyperbola having more than two assymptotes.

RE-EN'TER-ING. Entering again; returning.

Re-entering Angle of a polygon, is an interior angle greater than two right angles.

Re-entering Polygon. A polygon containing one or more re-entering angles. The term *re-entering* stands opposed to *salient*. It



is a property of a salient polygon that no straight line can be drawn which will cut the perimeter in more than two points; whilst in a re-entering polygon such line may cut it in more than two points.

RE-FLEC'TION. [L. *reflecto*, from *re*, again, and *flecto*, to bend]. If a ray of light fall upon a polished surface which is not transparent, a large portion of the ray is driven back into the same medium, but by a different path. This change of direction of the path of the ray is called reflection. The ray, before it strikes the surface, is called the incident ray, and after it leaves the surface it is called the reflected ray. The point at which the reflection takes place is called the *point of incidence*. The plane of the incident and reflected ray is called the *plane of incidence*, and it has been found to be normal to the surface at the point of incidence.

It is a property of the incident and reflected rays that they make equal angles with the normal to the reflecting surface at the point of incidence.

RE-GRES'SION POINT. [L *regredior*, to return; from *re*, and *gredior*]. The same as cusp point, which see.

REG'U-LAR. [L. *regularis*; from *regula*, a rule]. Conformed to a rule.

Regular Polygon. In Geometry, a polygon which is both *equilateral* and *equiangular*. A regular polygon may have any number of sides, from *three* upwards.

Let us denote the number of sides of any regular polygon by n, the length of one side by a, its area by A, the radius of the circumscribed circle by R, and that of the inscribed circle by r; then will the following formulas express the relations existing between these several quantities; they also enable us to determine the remaining three, when any one of them is given:

$$l = 2R\sin\frac{180^\circ}{n} = 2r\tan\frac{180^\circ}{n} \quad \cdot (1)$$

$$A = \frac{na^2\cot\dfrac{180^\circ}{n}}{4} \quad \cdot \cdot \cdot (2)$$

These formulas have been tabulated so as to afford the ready means of determining any required element for any regular polygon up to the dodecagon inclusive. We annex the tables:

I.

When the length of the side of the regular polygon is equal to 1.

No of sides.	Rad. of cir. circle.	Rad. of in. circle.	Area in square units.
3	0.5773503	0.2886751	0.4330127
4	0.7071068	0.5000000	1.0000000
5	0.8506508	0.6881910	1.7204774
6	1.0000000	0.8660254	2.5980762
7	1.1523825	1.0382617	3.6339124
8	1.3065630	1.2071068	4.8284271
9	1.4619022	1.3737387	6.1818242
10	1.6180340	1.5388418	7.6942088
11	1.7747329	1.7028437	9.3656404
12	1.9318516	1.8660254	11.1961524

II.

When the radius of the circumscribed circle is equal to 1.

No of sides.	Length of sides.	Rad. of in. circle.	Area.
3	1.7320508	0.5000000	1.2990381
4	1.4142136	0.7071068	2.0000000
5	1.1755705	0 8090170	2.3776412
6	1.0000000	0.8660254	2.5980762
7	0.8677674	0 9009689	2.7364102
8	0.7653668	0.9238795	2 8284271
9	0.6840403	0.9396926	2.8925437
10	0.6180340	0.9510565	2.9389263
11	0.5634651	0.9594931	2.9735250
12	0.5176381	0.9659259	3 0000000

III.

When the radius of the inscribed circle is equal to 1.

No of sides.	Length of side.	Rad. of cir. circle.	Area.
3	3.4641016	2.0000000	5.1961524
4	2.0000000	1.4142136	4.0000000
5	1.4530851	1.2360680	3.6327128
6	1.1547005	1.1547005	3.4641016
7	0.9631491	1.1099160	3.3710222
8	0.8284271	1.0823919	3.3137084
9	0.7279405	1.0641776	3.2757315
10	0.6498394	1.0514622	3.2491970
11	0.5872521	1.0422172	3.2298913
12	0.5358984	1.0352760	3.2153904

IV.

When the area is equal to 1 square unit.

No of sides.	Length of side.	Rad. of cir. circle.	Rad. of in. circle.
3	1 5196716	0.8773827	0.4386912
4	1.0000000	0.7071068	0.5000000
5	0.7623870	0.6485251	0.5246678
6	0.6204033	0.6204033	0.5372849
7	0.5245813	0.6045183	0.5446520
8	0.4550899	0.5946034	0.5493420
9	0.4201996	0.5879764	0.5525172
10	0.3605106	0.5833184	0.5547687
11	0.3267617	0.5799148	0 5564242
12	0.2988585	0.5773503	0.5576775

When the length of the given side, given radius of the inscribed or circumscribed circle, or the given area is not 1, these tables may be employed by recollecting the principle that in similar polygons homologous lines are proportional, and that homologous areas are proportional to the squares of the homologous lines. For example:

Let it be required to construct a regular heptagon, whose area is 225, by the aid of the tables. Extracting the square root of 225, we get 15. Multiplying each of the elements taken from table IV, opposite 7, by 15, we have

$$l = 0.5245813 \times 15 = 7.8687,$$

required side; also,

$$R = 0.6045183 \times 15 = 9.0678,$$

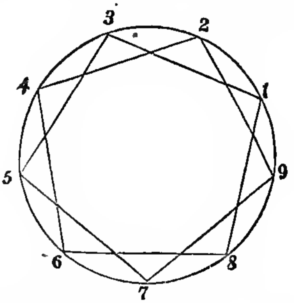
radius of circumscribed circle; also,

$$r = 0\ 5446520 \times 15 = 8.1698,$$

radius of inscribed circle.

With these elements the polygon is easily constructed. The remaining tables are used in a manner entirely analogous to the one which we have just considered.

In the preceding discussion we have supposed that the polygons are taken in such a manner that no two sides of the same polygon cross each other; this supposition excludes a species of star shaped regular polygons which conform to the general definition. As an example, take the polygon constructed as follows: Divide the circumference of a circle into 9 equal parts, and commencing at



1 unite it with 3, and 3 with 5, 5 with 7, 7 with 9, 9 with 2, and so on around. A polygon with 9 sides will be constructed, which is equilateral and equiangular, but it is not a regular polygon because the sides cross each other.

Regular Polyhedron. A polyhedron whose faces are equal regular polygons. Its polyhedral angles are all equal to each other.

There are but five regular polyhedrons, as follows:

1. The *tetrahedron*, or *regular pyramid*, bounded by four equilateral triangles.

2. The *hexahedron*, or *cube*, bounded by six squares.

3. The *octahedron*, bounded by eight equilateral triangles.

4. The *dodecahedron*, bounded by twelve regular pentagons.

5. The *icosahedron*, bounded by twenty equilateral triangles.

A sphere may always be inscribed within, and may always be circumscribed about a regular polyhedron, which will have a common centre. This is called the centre of the polyhedron.

The following tables have been constructed, showing the relations between the edge, the radii of the inscribed and circumscribed spheres, the entire surface, and the volume of each regular polyhedron:

I.

When the edge equals 1.

Name.	Rad. of cir. sphere	Rad. of in. sphere	Surface.	Volume.
Tetraehedron	0.6123724	0.2041241	1.7320508	0.1178511
Hexahedron	0 8660254	0.5000000	6.0000000	1.0000000
Octahedron	0.7071068	0.4082483	3.4641016	0.4714045
Dedecahedron	1.4012585	1.1135164	20.6457280	7.6631188
Icosahedron	0.9510565	0.7557613	8.6602540	2.1816951

II.

When the radius of inscribed sphere equals 1.

Name.	Edge.	Rad. of cir. sphere	Surface.	Volume.
Tetrahedron	1.6329932	0.3333333	4.6188023	0.5132002
Hexahedron	1.1547005	0.5773503	8.0000000	1.5396006
Octahedron	1.4142136	0.5773503	6.9282032	1.3333333
Dodecahedron	0.7136442	0.7946545	10.5146223	2.7851639
Icosahedron	1.0514622	0.7946545	9.5745413	2.5361507

III.

When the radius of the circumscribed sphere equals 1.

Name.	Edge.	Rad. of cir. sphere	Surface.	Volume.
Tetrahedron	4.8989795	3.0000000	41.5692192	13.8564064
Hexahedron	2.0000000	1.7320508	24 0000000	8.0000000
Octahedron	2.4494897	1.7320508	20.7846096	6.9282032
Dodecahedron	0.8980560	1.2584086	16 6508731	5.5502910
Icosahedron	1.3231691	1.2584086	15.1621684	5.0540561

IV.

When the surface equals 1 square unit.

Name.	Edge.	Rad. of cir. sphere	Rad. of in. sphere	Volume.
Tetrahedron	0.7598357	0.4653025	0.1551008	0.0517003
Hexahedron	0.4082483	0.3535534	0.2041241	0.0680413
Octahedron	0.5372850	0.3799178	0.2193457	0.0731152
Dodecahedron	0.2200822	0.3083920	0.2450651	0.0816884
Icosahedron	0.3398080	0.3231774	0.2568144	0.0856048

V.

When the volume equals 1 cubic unit.

Name.	Edge.	Rad. of cir. sphere	Rad. of in. sphere	Surface.
Tetrahedron	2.0395489	1.1547006	0.4163417	7.2056240
Hexahedron	1.0000000	0 8660254	0.5000000	6.0000000
Octahedron	1 2848990	0.9080604	0 5245576	5.7191069
Dodecahedron	0.5072221	0.7107492	0.5648000	5.3116140
Icosahedron	0.7710254	0.7332887	0.5827111	5.1483486

If, in applying these tables to cases in which the edge, radius of the inscribed or circumscribed spheres, surface, or volume, is not equal to 1, we have simply to remember that the surfaces of similar polyhedrons are to each other as the squares of their homologous lines, and that their volumes are to each other as the cubes of their homologous lines.

The following table shows the value of the diedral angle between two adjacent faces in any regular polyhedron :

Tetrahedron, . .	70°	31′	42″
Hexahedron, . .	90		
Octahedron, .	109	28	18
Dodecahedron, . .	116	33	54
Icosahedron, . .	138	11	23

RE-LĀ′TION. [L. *relatio*, bringing back]. Two quantities are said to be related to each other when they have anything in common, by means of which they may be compared with each other. Quantities of the same kind may always be compared with each other, and in such comparison they may be found equal to each other, or they may be unequal; hence, the two fundamental relations of *equality* and *inequality*. The relations of equality and inequality are generally expressed by means of symbols; those for equality being = and ⊂⊃, that for inequality $>$. Sameness, in every respect is *identity*; sameness in one respect only, is simply *relation*. Two triangles may be capable of superposition, so as to coincide throughout their whole extent; in which case they are absolutely *equal*; equal when not superposed. identical when superposed. The symbol of this kind of relation is =. But they may not be capable of superposition, and yet they may contain the same number of units of surface. This is a species of relation resembling that of equality, which is called *equivalency*, and is denoted by the symbol ⊂⊃. The difference between the symbols = and ⊂⊃, then, is that the former implies complete identity, whilst the latter implies absolute equality in one respect only. With respect to the symbol of inequality, it is to be observed that the opening is always turned to the greater quantity.

In ordinary language all relations between magnitudes may be expressed by means either of affirmative or negative propositions. In algebraic language these relations may be expressed symbolically by *equations*, *equivalencies*, or by *inequations*.

The relation expressed by means of equations or equivalencies, is *absolute* or *definite*; that by means of inequations is *vague* or *indefinite*. Perhaps the latter ought not, in strictness, to be regarded as a relation, but rather as expressing the fact that the relation of equality does not exist.

In the higher branches of mathematics we meet with expressions of the form

$$y = \phi(x), \quad y = f(x), \quad \phi(x, y) = 0, \text{ \&c.}$$

The entire expressions in these cases are indicative of a relation existing between x and y, but the nature of that relation is not expressed. The symbols ϕ, f, and all the various symbols of functions may be called symbols of *implied* relation.

RE-MĀIN′DER. [L. *remaneo*, to remain behind]. What remains, after taking away a part. In Arithmetic, the remainder is what remains of the subtrahend, after taking away the minuend.

In general, the remainder is such a quantity as, being added to the subtrahend, will produce the minuend. See *Subtraction*.

RE-PĒAT′. [L. *repeto*, to utter again]. To do again.

Repeating Decimal. A decimal, in which the same figures occur in the same order, at successive and equal intervals.

Thus, 3.646464 . . . is a repeating decimal. See *Circulating Decimals*.

REP-E-TEND′. [L. *repetendus; from repeto*]. That part of a repeating decimal, which is continually repeated. In the decimal 3.646464, the expression, 64, is the repetend.

REP-RE-SENT-Ā′TION. [L. *repræsento*, to present, to exhibit]. The representation of an object, is a drawing which presents to the mind, through the eye, an idea of the object. Sometimes, the drawing is made, so that it shall present to the eye, taken at a certain point, the same appearance as the object itself would present, were the drawing removed and the object placed in its stead, as in Perspective; sometimes the drawing or representation is purely conventional.

REP-RE-SENT′A-TĬVE. That which

stands for, or represents, some thing. Thus, all the symbols of Analysis are representatives of quantities, or of operations to be performed.

RE-SID'U-AL ANALYSIS. A branch of Analysis that has sometimes been employed in the solution of problems. It has met with very little favor; because all problems that can be solved by it, are more readily solved by means of the Calculus. The Residual Analysis proceeds by taking the difference of a function in two different states, and then expressing the relation between this difference and the difference of the corresponding states of the variable.

This relation is first expressed generally, and is then considered under the supposition that the difference of the two states of the variable is 0.

The general outline of the fundamental idea of this branch of analysis is closely assimilated to the method of limits, which has now come to form the basis of the science of Differential and Integral Calculus.

RES-O-LU'TION [L. *resolutio*, loosening, untying].

Resolution of a Quantity into its Factors. The operation of separating any expression into factors; that is, the operation of finding two or more expressions such, that their product is equal to the given expression. To resolve a number into its prime factors by means of a table of prime numbers: Commence with 2, and divide successively by it as often as possible; then, divide the result successively by the next highest prime number, as often as possible, and so on, till the final quotient is a prime number: the different divisors used, together with the last remainder, constitute all the prime factors of the given number.

Resolution of Equations and Problems. The same as their *solution*; that is, it is the operation of finding, in the case of an equation, such values for the unknown quantities which enter it, as will satisfy the equation, when substituted for the unknown quantities: in the case of a problem, it is the operation of finding such values for the unknown quantities as will satisfy the conditions of the problem. See *Equation* and *Problem*.

RE-SULT'. [L. *resulto*, to rebound]. That which is obtained by performing an operation upon any quantity: the conclusion arrived at by a course of reasoning. Thus, the result of an addition is the sum of the quantities added. The result of the demonstration of the binomial theorem, is the binomial formula. The result of translating a formula into common language is a rule, and the reverse.

RE-VERSE'. [L. *reversus*; *re*, and *verto*, to turn]. To turn back.

Reverse Bearings. In Surveying, the bearing of a course, taken from the second end of the course, looking backwards. The number of degrees of a reverse bearing ought always to be equal to the number of degrees in the direct bearing; but the meridional letters, as well as those of departure, are different in the two cases. Thus, if a direct bearing is N. 23° E., the reverse bearing ought to be S. 23° W.

The reverse bearing of every course ought to be taken as a check on the accuracy of the work, and if the number of degrees in it is not the same as in the direct bearing, both should be taken over, until they are found to agree. If they cannot be made to agree, the inference is, that there is some local attraction which deflects the needle, at one or both stations.

Reverse Operation. An operation, in which the steps are the same as in a direct operation, but taken in a contrary order. Thus, Division is the reverse of Multiplication; and the extraction of a root is the reverse of the operation of raising a quantity to a power.

RE-VER'SION. [L. *reversio*, a returning]. In Annuities, a payment not due till the occurrence of some contingent event, as the death of a person now living. Payments due at, or after, a specified period of time, are called deferred payments. The method of calculating the present values of reversions has been explained under the head of *Annuities*. A set of tables is generally used for computing these present values.

Let A denote the value of an annuity on a life of a given age, V the present value of \$1, to be received at the end of the year in which the life fails, r the rate of interest, and

$v = \dfrac{1}{1+r}$; then,

$V = v(1+A) - A$, or, $V = v - (1-v)A$.
Suppose, for example, that on the death of A, whose present age is 55, the sum of $5000 is to revert to B, or his assignee, and that B proposes to sell his interest in this reversion: Required the value of that interest, allowing the purchaser interest, at the rate of 4 per cent.

From the annuity table the value of an annuity of $1, on a life aged 55, is $11.0392. We have

$$r = 0.04, \quad \text{and} \quad v = \frac{1}{1.04};$$

whence

$$v = \frac{12.0392}{1.04} - 11\,0392 = 0.537.$$

This is the value of the reversion of $1; hence, the value of the reversion of $5000 is $\$5000 \times 0.537 = \$2685$.

When an annuity is to commence at the death of one individual and terminate at the death of another, the simple annuity tables will not answer, but recourse must be had to tables of annuities on joint lives. Thus, if A, on the death of B, is entitled to an annuity of $1, to continue for the remainder of his life, the present value of A's interest is

$$B - AB,$$

in which B denotes the present value of the annuity on the life of B, and AB the present value of the annuity on the joint lives of A and B; that is, to continue as long as both shall continue alive.

The following formulas are sufficient to solve all problems of reversionary interests, so far as three lives are concerned, and these embrace a vast majority of all cases which arise in practice. In the formulas, AB denotes the value of an annuity for the joint lives of A and B, APB denotes the present value of an annuity for the joint lives of A, B and P, &c., and R denotes the present value of the reversionary interest.

1. For a single life, after the longest of two lives, P and Q,

$$R = A - AP - AQ + APQ.$$

2. For the longest of two lives, A and B, after a single life P,

$$R = A + B - AB - AP - BP + APB.$$

3. For a single life A, after two joint lives, P and Q,

$$R = A - APQ.$$

4. For two joint lives, A and B, after a single life, P,

$$R = AB - ABP.$$

Reversion of Series. When one quantity is expressed in terms of another, by means of a series, the operation of finding the value of the second in terms of the first, by means of a series, is called the *reversion* of the series.

The reversion of a series is effected by means of the principle of indeterminate coefficients, as follows:

Let there be a general series, expressing the value of y,

$$y = ax + bx^2 + cx^3 + dx^4 + \&c. \quad (1),$$

and let it be required to find the values of A, B, C, D, &c, in the expression

$$x = Ay + By^2 + Cy^3 + Dy^4 + \&c. \quad (2),$$

Squaring, cubing, &c., the value of y, we have

$$y^2 = a^2x^2 + 2abx^3 + \left.\begin{array}{l} b^2 \\ +2ac \end{array}\right| x^4 + \left.\begin{array}{l} 2ad \\ +2bc \\ +\cdots \end{array}\right| x^5, \&c.$$

$$y^3 = a^3x^3 + 3a^2bx^4 + \left.\begin{array}{l} 3ab^2 \\ +3a^2c \\ +\cdots \end{array}\right| x^5 + \&c.$$

$$y^4 = a^4x^4 + 4a^3bx^5 + \&c.$$

$$y^5 = a^5x^5 + \&c.$$

Substituting these expressions in equation (1), arranging and transposing, we have

$$0 = \left.\begin{array}{l} Aa \\ -1 \end{array}\right| x + \left.\begin{array}{l} Ab \\ +Ba^2 \end{array}\right| x^2 + \left.\begin{array}{l} Ac \\ +2Bab \\ +Ca^3 \end{array}\right| x^3 + \left.\begin{array}{l} Ad \\ +Bb^2 \\ +2Bac \\ +3Ca^2b \\ +Da^4 \end{array}\right| x^4 + \&c.$$

This equation being an identical one, the coefficients of the different powers of x are separately equal to 0. Equating them separately with 0, we have

$$Aa - 1 = 0, \quad \text{whence}, \quad A = \frac{1}{a},$$

$$Ab + Ba^2 = 0, \quad \text{whence}, \quad B = -\frac{b}{a^3},$$

$$Ac + 2Bab + Ca^3 = 0, \quad \text{whence} \quad C = \frac{2b^2 - ac}{a^5},$$

$$Ad + Bb^2 + 2Bac + 3Ca^2b + Da^4 = 0,$$

whence,

$$D = \frac{5abc - 5b^3 - a^2d}{a^7},$$

&c., &c., &c.

Substituting these in equation (2), we have

$$x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 - \frac{5b^3 - 5abc + a^2d}{a^7}y^4 + \text{\&c.}, \quad \text{\&c.}, \quad \text{\&c.}$$

If we have a development of y of the form

$$y = a + bx + cx^2 + dx^3 + \text{\&c.},$$

it is impossible to develop x in terms of y, but we can place $y - a = z$, and there will result

$$z = bx + cx^2 + dx^3 + \text{\&c.},$$

and then we may find the values of A, B, C, &c., in the development,

$$x = Az + Bz^2 + Cz^3 + Dz^4, \text{\&c.}$$

Having made the development, we can replace z by its value $y - a$, and there will result a development of x in terms of $y - a$.

Let it be required, as an example, to reverse the series,

$$y = x + x^2 + x^3 + x^4 + x^5, \text{\&c.} \ldots (1)$$

Assume

$$x = Ay + By^2 + Cy^2 + Dy^4 + \text{\&c.} \,. \, (2).$$

Comparing this with the preceding case, we see that

$$a = b = c = d = \text{\&c.} = 1\,;$$

whence,

$$A = 1, \ B = -1, \ C = +1, \ D = -1, \text{\&c.}$$

hence, the required series is

$$x = y - y^2 + y^3 - y^4 + y^5 - \text{\&c.}$$

Let it be required to reverse the series,

$$y = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1\,2.3} + \frac{x^4}{1.2.3.4} + \text{\&c}\,;$$

we first make $y - 1 = z$; whence,

$$z = \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{\&c.} \ldots$$

Proceeding as before, we find,

$$A = 1, \ B = -\tfrac{1}{2}, \ C = +\tfrac{1}{3}, \ D = -\tfrac{1}{4}, \text{\&c.}\,;$$

whence,

$$x = \frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \text{\&c.},$$

and replacing z by its value $(y - 1)$, we have the series,

$$x = \frac{y-1}{1} - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} + \text{\&c.}\,.$$

The method of reversion of series is little used on account of the difficulty of recognizing the law for the formation of the series; sometimes it is necessary to determine a great number of the co-efficients before the law of the series is manifest.

RE-VERT'. [L. *reverto*; from *re*, and *verto*, to turn]. To turn back; to reverse. To revert a series is to take its terms in an inverse order, making the terms follow each other in a contrary order.

REV-O-LŪ'TION. [L. *revolvo*, to turn around]. When one line moves about a straight line, called the axis, in such a manner that every point of the moving line generates a circumference of a circle, whose plane is perpendicular to the axis, that motion is called *revolution*, and the surface is called a *surface of revolution*. Every plane through the axis is called a meridian plane, and the section which this plane cuts from the surface, is called a meridian curve. Every surface of revolution can be generated by revolving one of its meridian curves about the axis.

RHOM-BO-HĒ'DRAL. Relating to a rhomboid.

RHOM-BO-HĒ'DRON. A polyhedron bounded by six equal rhombuses. When the rhombuses are squares, the rhombohedron becomes a cube.

RHOM'BOID. Gr. ρομβος, rhomb, and ειδος, form]. A parallelogram, all of whose sides are not equal. The rhombus is but a particular form of the rhomboid, in which the sides are all equal.

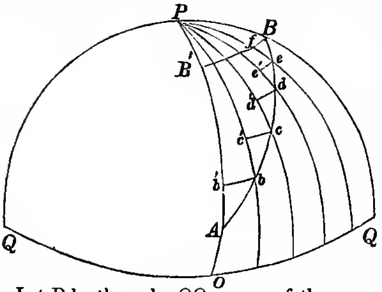
RHOM'BUS. [Gr. ρομβος, rhomb]. A parallelogram whose sides are equal to each other. If the angles of a rhombus become each equal to 90°, the rhombus becomes a square. The diagonals of a rhombus bisect each other at right angles. The area of a rhombus is equal to half the product of its diagonals.

RHUMB. [From *rhombus*]. In Navigation, the track of a ship sailing on the same point of the compass. The rhumb line, also called the *loxodromic curve*, cuts all of the meridians which it crosses under the same angle. The curve is a kind of spiral approaching nearer and nearer the pole, but only reaching it after an infinite number of turns. The angle under which the rhumb line cuts the meridian is called the *angle of the rhumb*, and the angle which it makes with the prime vertical is the *complement of the rhumb*.

If we conceive a logarithmic spiral to be constructed on the plane of the equator, hav-

ing its pole at the centre of the sphere, and then erect perpendiculars at every point, these perpendiculars will pierce the surface of the sphere in a loxodromic curve, or rhumb line; that is, the projection of a rhumb line upon the plane of the equator is a logarithmic spiral.

When a ship sails on a rhumb line, the distance sailed is computed by considering successive small portions of the arc and computing them as arcs of circles, as follows:

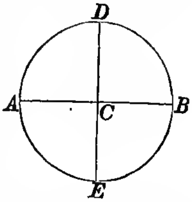


Let P be the pole, QQ an arc of the equator, AB a rhumb line. Suppose it divided into any number of equal parts at the points *b*, *c*, *d*, &c.; through these let meridians and parallels of latitude be drawn; draw also the parallel BB'; then as the ship sails along the rhumb line towards the pole from A to B, the distance sailed is made up of the parts *Ab*, *bc*, *cd*, &c., and the sum of all the small differences of latitude. *Ab'*, *bc'*, *cd'*, &c., makes up the total difference of latitude, and the sum of all the parallels, *bb'*, *cc'*, *dd'*, &c., makes up what is called the departure in plane sailing. BB' is the meridional distance or distances between the first and last meridians measured in the last parallel, and OQ is the difference of longitude.

If the ship sails from B to A, the departure, difference of latitude, and difference of longitude, are the same as before, but the meridional distance is that portion of a parallel through A intercepted by the extreme meridian.

The departure is nearly a mean proportional between the two meridional distances. The distances *Ab*, *bc*, &c., being taken very small, the elementary triangles *Abb'*, *bcc'*, &c., are then all equal, and the departure is equal to the departure in one triangle, multiplied by their number; the difference of latitude is equal to the difference in one triangle, multiplied by their number, and the distance sailed is equal to the hypothenuse of one of these triangles multiplied by their number. See *Navigation*.

RIGHT ANGLE. In Plane Geometry, if one straight line, AB, meet another straight line, DE, making the adjacent angles, BCD, DCA equals, both angles are right angles, and the two lines are perpendicular to each other. If with the point of intersection, C, as a centre, and a distance equal to 1, a circumference of a circle be described, one-fourth of this will be intercepted between the two sides of each right angle, and this is taken as the measure of a right angle. But the quadrant contains 90°; hence a right angle is the same as an angle of 90°. All angles, greater or less than 90°, are oblique angles.



RIGHT ANGLED. Having a right angle. A *right angled triangle* is a triangle having one right angle. No plane triangle can have more than one right angle. A spherical triangle may have two or three right angles; in the former case it is called a *birectangular triangle*, and in the latter case it is a *trirectangular triangle*.

RIGHT DIEDRAL ANGLE. The angle included between two planes which are perpendicular to each other. The measurement of diedral angles is reduced to that of plane angles, by considering a diedral angle to be measured by the angle included between two straight lines, one in each plane, and both perpendicular to the edge of the diedral angle at the same point. All diedral angles greater or less than a right angle, are oblique.

RIGHT POLYEDRAL ANGLE. A polyedral angle contained within three planes taken at right angles to each other. If with the vertex of the polyedral angle as a centre, and a radius equal to 1, the surface of a sphere be described, there will be one-eighth of this surface included within the faces of the angle, and this is taken as the measure of the right polyedral angle, and is the unit of measure of polyedral angles. If with the vertex of a

polyedral angle, having more than three faces, as a centre, and a radius equal to 1, a surface of a sphere be described, and if the portion of this surface intercepted between the faces of the angle is equal to one-eighth of the surface, the angle is sometimes called a right polyedral angle.

Right Cone. A cone whose axis is perpendicular to the base. See *Cone*.

Right Conoid. A conoid in which the rectilineal directrix is perpendicular to the plane directer. In this case this directrix is called the line of *striction*. because the elements are nearer to each other along this line than at any other point. See *Conoid*.

Right Cylinder. A cylinder whose elements are perpendicular to the plane of its base. See *Cylinder*.

Right Line. A straight line; that is, one which does not change its direction between any two of its points.

Right Prism. A prism whose lateral edges are perpendicular to the plane of its base. See *Prism*.

Right Pyramid. Is one whose base is a regular polygon, and in which the perpendicular let fall from the vertex on the base, passes through the centre of the base.

Right Sphere. In Spherical Projections is that position of the sphere in which the primitive plane coincides with the plane of the equator.

Right Spherical Angle. A spherical angle included between arcs of two great circles whose planes are at right angles to each other.

Right Angled, or Rectangular Cone. Is a cone such that a plane passed through its axis cuts out two elements at right angles to each other. If a right angled triangle be revolved about the bisecting line of the right angle, the volume generated is a right angled cone.

ROD. A scale of wood or metal employed in measuring distances. The rod may be of any length, and is generally divided into a certain number of equal parts.

ROD. A unit of lineal measure used in land surveying. It is equal to $5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet.

ROOD. A unit of superficial measure, used in land surveying: it is equivalent to one-fourth of an acre, or to 40 perches.

ROOT. [L. *radix*]. The original or cause of anything: the base or element.

Root of an Equation. Any quantity, whether real or imaginary, which, being substituted for the unknown quantity, will satisfy it; that is, make the two members equal. It has been shown, under the article *Equation*, that every equation, entire with respect to, and involving but one unknown quantity, has as many roots as there are units in the number expressing the degree of the equation. See *Equation*.

Root of a Quantity. Any quantity which, being taken a certain number of times as a factor, will produce the quantity. The *square root* of a quantity is a quantity which, being taken twice as a factor, will produce the quantity: the *cube root* of a quantity is a quantity which, being taken three times as a factor, will produce the quantity: in general, the n^{th} root of a quantity is a quantity which, being taken n times as a factor, will produce the quantity.

A root of a quantity may be real, or it may be imaginary.

If we square $+a$ and $-a$, the result is in both cases $+a^2$; hence it follows that every quantity has two square roots, both real, and numerically equal, but having contrary signs.

Let us take the equation $x^3 = p^3$, which may be put under the form

$$x^3 - p^3 = 0.$$

It is evident that every value of x which will satisfy this equation, must necessarily be a cube root of p^3. The equation may be written

$$(x - p)(x^2 + px + p^2) = 0;$$

an equation which can be satisfied by placing either factor equal to 0. Setting the factors separately equal to 0, we have

$$x - p = 0 \quad \text{and} \quad x^2 + px + p^2 = 0.$$

The former gives $x = p$; the latter,

$$x = p\left(\frac{-1+\sqrt{-3}}{2}\right) \text{ and } x = p\left(\frac{-1-\sqrt{-3}}{2}\right)$$

Hence, we see that p^3 has three cube roots, and only three, viz.:

$$p,\ p\left(\frac{-1+\sqrt{-3}}{2}\right) \text{ and } p\left(\frac{-1-\sqrt{-3}}{2}\right),$$

the first of which is real, the second and third imaginary.

In like manner, if we take the equation,

$$x^4 = p^4, \quad \text{or} \quad x^4 - p^4 = 0,$$

it may be written

$$(x^2 - p^2)(x^2 + p^2) = 0;$$

and, placing each factor separately equal to 0, we get

$$x^2 - p^2 = 0, \quad \text{and} \quad x^2 + p^2 = 0,$$

from which we find

$$x = +p, \quad x = -p, \quad x = p\sqrt{-1},$$

and $$x = -p\sqrt{-1}.$$

Hence, we infer that every quantity has four fourth roots, and no more: the first two are real and numerically equal, but have contrary signs; the last two are imaginary.

In general, let us take the equation,

$$x^n = p^n, \quad \text{or} \quad x^n - p^n = 0,$$

in which n is any whole number. There are two cases: 1st, when n is an even number; 2d, when n is odd. 1st, when n is even, the equation may be written,

$$(x^2 - p^2)(x^{n-2} + x^{n-4}p^2 + x^{n-6}p^4 + x^{n-8}p^6 + \cdots + p^{n-2}) = 0.$$

Placing the factors of the first member separately equal to 0, we have

$$x^2 - p^2 = 0, \quad \text{and}$$
$$x^{n-2} + x^{n-4}p^2 + x^{n-6}p^4 + \text{\&c.} + p^{n-2} = 0.$$

The first of these equations gives

$$x = +p \quad \text{and} \quad x = -p.$$

With regard to the second equation, it is plain, since all of the co-efficients are positive, and since it involves only the even powers of x, that any real quantity substituted for x will make the first member positive, and, consequently, cannot satisfy it; hence, all its roots which are $n-2$ in number, must be imaginary.

When n is odd, the equation may be written

$$(x-p)(x^{n-1} + x^{n-2}p + x^{n-3}p^2 + \text{\&c.} + p^{n-1}) = 0,$$

and by placing the factors separately equal to 0, we have the equations

$$x - p = 0 \quad \text{and}$$
$$x^{n-1} + x^{n-2}p + x^{n-3}p^2 + \cdots + p^{n-1} = 0.$$

The first of these equations gives $x = p$; the second gives only imaginary values for x.

From this discussion, we conclude that every quantity has n, n^{th} roots, and only n; that when n is even, two of the roots are real, numerically equal, but have contrary signs, and all the remaining roots imaginary. When n is odd, one of the roots is real, and all the rest imaginary. It is also seen that *there is but one numericol root that is real in any case; this is the root that is generally referred to in speaking of the root of a quantity.*

It is further to be observed, that all of the n^{th} roots of any quantity may be obtained by multiplying the numerical n^{th} root by the n^{th} roots of 1 respectively. Thus, the cube roots p^3 are shown above to be equal to p, the numerical cube root, multiplied respectively by

$$\frac{-1+\sqrt{-3}}{2} \quad \text{and} \quad \frac{-1-\sqrt{-3}}{2},$$

which are the cube roots of 1; and so on, for the other roots.

It has now come to be conventional with most mathematicians, that when the simple numerical root is meant, it is indicated by the radical symbols

$$\sqrt{\ }, \ \sqrt[3]{\ }, \ \sqrt[4]{\ }, \ \text{\&c.},$$

but when the general root, or the root which includes all possible values, is meant, it is indicated by the fractional indices, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c.; thus, $\sqrt[3]{a}$, stands simply for the numerical cube root of a, whilst $(a^3)^{\frac{1}{3}}$ stands equally for

$$a, \ a\left(\frac{-1+\sqrt{-3}}{2}\right) \text{ and } a\left(\frac{-1-\sqrt{-3}}{2}\right).$$

This convention is sometimes departed from, but it is gradually being adopted, and will eventually become universal.

ROUND. A term applied indiscriminately in common language to the shape of cylindrical, conical. spherical, spheroidal, and annular bodies; in short, to any bodies which approach regularity, and admit of an oval section.

In Geometry, the three round bodies are the right cone, the right cylinder and the sphere.

RŬLE. A direction or set of directions given for performing the operations necessary to obtain a certain result.

A rule differs from a formula only in the language by which it is expressed. A rule is always expressed in ordinary language; a formula, in algebraic or symbolical language.

If a rule is translated into algebraic lan

guage, the result is a formula; and, conversely, if a formula is translated into ordinary language, the result is a rule.

A rule, with its connecting explanations, should embody a description of the object to be attained, and the means by which it is to be attained; it should, also, point out the means of determining when it is attained. It should specify the case in which it is to be used, or when it is preferable to any other, and should be so complete that any reader, of the class to whom it is addressed, may learn all that it professes to teach without the necessity of studying the processes by means of which it has been deduced. Rules are for practical application, and are often used by those who are not familiar with the more abstruse mathematical processes, and should be specially framed to this end. For those who are thoroughly conversant with mathematical language, and specially when many rules are to be learned, it will generally be found more convenient to make use of formulas. In all cases in which it is practicable, great advantage will arise from cultivating a habit of translating formulas into rules, and rules into formulas.

RULE, RULER. A mathematical instrument, employed in drawing straight lines. It consists of a bar of metal or wood, straight on one edge, for the purpose of guiding a pencil or pen.

RULE OF SIGNS. In Algebra, a rule for determining the sign of a product or quotient. If two quantities be multiplied together, or if one be divided by the other, the sign of the result will depend upon the signs of both quantities: *if the signs are alike, that of the result is always positive; if they are unlike, the result is always negative.* When any number of factors are multiplied together, if an *odd* number of them is *negative*, the result is *negative*; if an *even* number is *negative*, the result is *positive*. This follows at once from the rule as above enunciated.

RULE OF THREE. In Arithmetic, a rule for finding from three given numbers a fourth, to which the third shall have the same ratio as the first has to the second. Hence, it is an application of the principles of proportion, and embraces that class of questions in which three of the terms of a proportion are known or given, and the fourth required.

The given and required numbers, taken in order, from a proportion, and consequently taken two and two, they must be of the same name or kind; hence, of the three given numbers, two must always be of the same name or kind, and the third must necessarily be of the same name or kind as that sought. This fact indicates the method of stating the proportion, the solution of which follows directly from the rules for solving proportions. The rule is as follows:

Whatever produces *effects*, as men at work animals eating, time, goods purchased or sold, money lent, and the like, may be regarded as *causes*. Causes are of two kinds—simple and compound.

A simple cause has but a single element, as men at work, a portion of time, goods purchased or sold, and the like.

A compound cause is made up of two or more simple elements, such as men at work, taken in connection with time, and the like.

The *results* of causes, as work done, provisions consumed, money paid, cost of goods, and the like, may be regarded as *effects*. A simple effect is one which has but a single element; a compound effect is one which arises from the multiplication of two or more elements.

Causes which are of the same kind, that is, which can be reduced to the same unit, may be compared with each other; and effects, which are of the same kind, may likewise be compared with each other. From the nature of causes and effects, we know that

1st. Cause : 2d. Cause : : 1st. Effect : 2d. Effect; and
1st. Effect : 2d. Effect : : 1st. Cause : 2d. Cause.

SINGLE RULE OF THREE. Simple causes and simple effects give rise to simple ratios. All questions involving simple ratios, are classed under the Single Rule of Three, for which we have the following rule:

1. *Write the number which is of the same kind, with the answer for the third term, its corresponding cause or effect for the first term, and the remaining cause or effect for the second term.*

2 *Multiply the second and third terms together, and divide their product by the first*

term; or, multiply the third term by the ratio of the first to the second.

Double Rule of Three. Compound causes or compound effects, give rise to compound ratios, and these to compound proportion. The double rule of three is an application of the principles of compound proportion. It embraces all that class of questions in which the causes are compound, or in which the effects are compound, and is divided into two parts:

1. *When the compound causes produce the same effects.*

The first embraces all that class of questions which has been arranged under "Rule of Three Inverse." Here, since the effects are equal, the causes are equal; hence, the products of their elements are equal; therefore,

Make the element of that cause which contains the unknown element, the first term of the proportion; the corresponding element of the other cause the second term, and the remaining element the third term: then multiply the second and third terms together, and divide the product by the first.

If 4 men can dig a ditch in 9 days, how many days will it require 18 men to dig it?

The elements of one cause are 4 and 9, and of the other 18 and the required time: hence,

$$18 \;:\; 4 \;::\; 9 \;:\; 2 \text{ days.}$$

2. *When the compound causes produce different effects.*

In this class of questions, either a cause or a single element of a cause may be required; or an effect, or a single element of an effect may be required. Denote the required cause or element by x: then,

1. *Arrange the terms of the statement so that the causes shall compose one couplet, and the effects the other.*

2. *Then if* x *fall in one of the extremes, make the product of the means a dividend, and the product of the extremes a divisor; but if* x *fall in one of the means, make the product of the extremes a dividend, and the product of the means a divisor.*

It is to be observed, that all questions under the rule involves the element of time, and further that questions of this nature may involve not only five, but also 7, 9, 11, &c., terms giving rise to what might be called triple, quadruple, &c., rule of three. It will not be necessary to discuss these cases, as their solution may be effected by a simple extension of the rule just laid down.

S. The 19th letter of the English alphabet. As a numeral it has been used for 7: with a dash over it, thus, $\overline{\text{S}}$, it stood for 7000. As an abbreviation, it stands for *South.*

SĀIL ING. The operation of conducting a ship on the ocean, from port to port, together with the necessary computations for determining her place at any time, the distance sailed, and the course necessary to steer, so as to reach a desired port. Sailing is the same as *Navigation*, which see.

Sailing is distinguished, according to the methods employed in solving the different problems that arise.

Globular Sailing is that in which the problems are solved by the principles of Spherical Trigonometry.

Great Circle Sailing, the same as globular sailing.

For the method of solving the problems in these several cases, see *Navigation.*

Mercator's Sailing, is that in which the problems are solved according to the principles used in making Mercator's projection. See *Mercator's Projection.*

Middle Latitude Sailing, is that in which the problems are solved by means of the middle latitude; that is, the half sum of the latitudes of the extreme points of a course.

Parallel Sailing, is when a ship sails on a parallel of latitude. The distance sailed in nautical miles multiplied by the cosine of the latitude, gives the number of minutes of longitude made by the ship.

Plane Sailing, is that in which the problems are solved, on the supposition, that the surface of the earth is plane. The results are very erroneous, when any great distances are considered; but it is extremely simple, and in some cases affords sufficiently accurate results.

Spheroidal Sailing, is that in which the problems are solved, on the supposition, that the face of the earth is a spheroid.

SĀ'LI-ENT. [L *saliens*; from *salio*, to leap]. Projecting outwards. Opposed to re-entering.

Salient Angle of a polygon, is an inte

rior angle less than two right angles. A polygon which has all of its angles salient, is called a *salicnt polygon*, and a straight line cannot be drawn so as to cut the perimeter in more than two points.

SALIENT POLYHEDRON is a polyhedron, no interior angle of which is measured by more than four trirectangular triangles. A cutting plane can only intersect a salient polyhedron in a salient polygon.

SAT'IS-FȲ. [L. *satis*, enough, and *facio*, to make]. An equation is said to be *satisfied*, when after the substitution of any expressions for the unknown quantities which enter it, the two members are equal. The values found for the unknown quantities of a problem, are said to *satisfy* the conditions of the problem, when being operated upon in accordance with those conditions, the result conforms to the enunciation of the problem.

SCĀLE. [L. *scala*, a ladder]. A line drawn upon any solid substance, as wood, ivory, paper, &c., and divided into parts equal or unequal, which may be transferred by means of the dividers, to aid in geometrical construction.

The manner in which the scale is divided, depends upon the nature of the algebraic or trigonometric function to be represented. When the subdivisions are all equal, the method of constructing the scale is similar to that described under the head of *Graduation*.

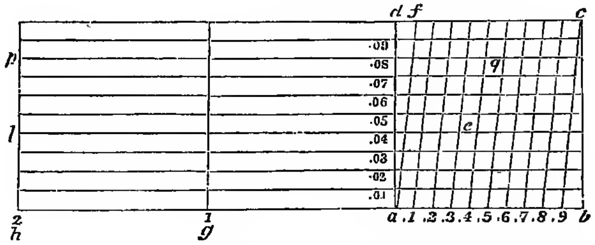
The most simple scale is that in which the subdivisions are all equal, called a scale of equal parts. The scale of equal parts is not only the most easily constructed, but it is also the most useful, for by the aid of trigonometrical tables, it may be made to supply the place of all the trigonometrical scales.

For example, let it be required to lay off an angle of 25°. With a radius equal to 50 equal parts of the scale, and from the vertex of the angle as a centre, let the arc of a circle be described, cutting the line from which the angle is to be laid off; from the tables, find the natural sine of 12° 30′ and multiply it by 100; with this taken from the scale as a radius, and from the point last determined as a centre, describe a second arc cutting the first; through this point let a line be drawn to the vertex of the angle, and the angle constructed will be the angle required. In this case, the scale of equal parts has taken the place of a scale of chords, and in like manner, it may be made to take the place of a scale of secants, &c.

In the scales that are formed upon wood or ivory, and that are contained in the ordinary boxes of instruments, the divisions are of various lengths. It is not found advantageous to divide a wooden or ivory scale into parts less than one-fiftieth of an inch in length.

Diagonal Scale of Equal Parts.

The most important scale, viz. the diagonal scale of equal parts, is thus constructed:



Take ab for the unit of the scale, which may be $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, or any part of an inch. On it construct a square $abcd$. Divide ab and dc each into 10 equal parts. Draw of and the other lines, as in the figure. Produce ba to the left, and lay off the unit from it, any number of times, and number the scale, as in the figure.

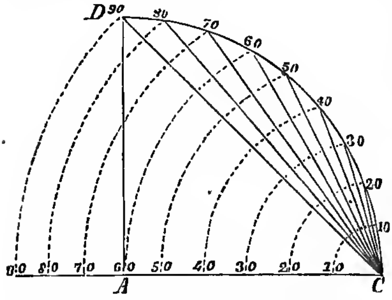
Now, the small divisions of the line ab, are each one-tenth of ab; they are, therefore, $\frac{1}{10}$th of ad, or of ag, or gh. The distance on the first line above ab, from the line ad to the line of, is equal to $\frac{1}{10}$th of df, or $\frac{1}{100}$th of the unit of the scale: the corresponding distance measured on the second line above ab, is equal to $\frac{2}{100}$ths of the unit of the scale; the

corresponding distance measured on the third line above ab, is the $\frac{8}{100}$ths of the unit of the scale, and so on. The use of the scale is evident from the description just given.

Suppose it were required to take, in the dividers, the distance 2.34 from the scale: we place one foot of the dividers at l, and open them till the other foot is at e, the distance between the points of the dividers is then equal to 2.34 units of the scale. To take off the distance 2.58: we place one foot at p, and open them till the other foot comes to q, and so on for any other distance.

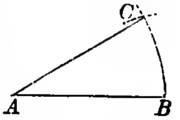
We have only represented a portion of the scale; it may be continued to any distance to the left. If a line is too long to be laid down at a single operation, let it be divided into parts, and each part taken in succession.

Scale of Chords. A scale, in which the chords of all arcs, from 0° to 90°, are laid down.



To construct a scale of chords graphically: With a point A, as a centre, and a radius 1, describe an arc of a circle equal to a quadrant, and divide it into 90 equal parts, beginning at C; through C, and each point of division, draw straight lines: they will be the chords of the corresponding arcs. with C, as a centre, and these chords as radii, describe arcs of circles intersecting AC, and number these points, as in the figure. The scale, thus formed, is a scale of chords. The chord of 60° will be equal to the radius of the circle, in which the chords are taken.

To lay off an angle, say 30°, by means of a scale of chords: Take the chord of 60°, from the scale, as a radius, and from the vertex of the required angle, as a centre, describe the arc of a circle cutting one side AB of the required angle, in B; take this point of intersection, as a centre, and with the chord of 30°, as a radius, describe an arc cutting the first arc in C; join this point with the vertex A. the angle CAB will be the required angle.



To measure an angle plotted on paper, by means of a scale of chords: With the vertex of the given angle, and with the chord of 60° as a radius, describe an arc cutting the sides of the angle; take the distance between these points of intersection in the dividers, and apply it to the scale of chords, from C: the reading on the scale will indicate the number of degrees contained in the given angle.

In like manner, we may construct scales of sines, cosines, secants, cosecants, &c.

A better method of constructing a scale of chords is by means of a table of natural sines. The chord of an arc is equal to twice the sine of half the arc. Take a table of natural sines, and from it find the sines of $\frac{1}{2}$°, $\frac{2}{2}$°, $\frac{3}{2}$°, $\frac{4}{2}$°, &c.; double each, and lay off the resulting distance on the line CA from C by means of an accurate scale of equal parts.

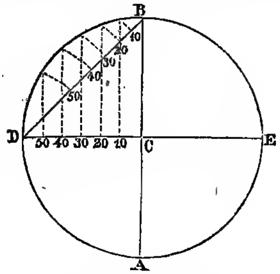
These scales may all be constructed graphically; but it is better to construct them by means of the table of sines, &c., by the aid of a nicely constructed scale of equal parts. A scale of semi-tangents may also be constructed either graphically, or by means of a table, recollecting that the semi-tangent of an arc is the same as the tangent of half the arc.

These scales of sines, secants, tangents, semi-tangents, &c., are used principally in making the projections of the circles of the sphere.

Scale of Longitudes. A scale used for determining graphically the number of miles in a degree of longitude, in any latitude. It may be constructed as follows:

Describe a quadrant BD with a radius equal to the chord of 60°, taken from the scale of chords; divide the radius DC into 60 equal parts. Draw lines through the points of division perpendicular to DC, cutting the circumference; then, with D as a centre, and distances equal to the distances from D to the points of division, describe arcs cutting the chord DB in the points 10, 20, 30, &c. Now, if this scale be laid upon the scale of

chords inverted, so that 60 shall fall upon 0 in each case; then, if any degree of latitude



be counted upon the scale of chords, there will stand opposite to it, on the scale of longitude, the number of miles in a degree of longitude for that latitude.

Plotting-Scale. See *Plotting*.

There is a variety of useful scales, some of which will be found described under their respective heads,—which see.

Scale of Numbers. A conventional expression of the law of relation between units of different orders. There are two kinds of scales of numbers, the *uniform*, and the *varying* scale. In the uniform scale, the law of relation between the units of different orders is, that a unit of any order is equal to the product obtained by multiplying a unit of the next lower order by a fixed number. This fixed number is the *modulus* of the scale, and gives name to the scale.

In numbers constructed by such a scale, if one unit of each successive order be taken, they will constitute a geometrical progression. In the varying scale, the law of relation between units of different orders is not subject to any uniform law, but to a law which varies in each particular case. The uniform scales apply to the methods of writing abstract numbers, and aid in writing denominate numbers, though they are not necessary to such expressions. The varying scales apply to denominate numbers exclusively. We shall illustrate each case separately.

Uniform Scales. In a uniform scale, the abstract number, 1, is taken as the base of every system. This is a unit of the first order. A unit of the second order is found by multiplying 1 by the modulus of the scale; a unit of the third order is found by multiplying a unit of the second order by the modulus, or by multiplying 1 by the square of the modulus, and so on. A unit of the n^{th} order is found by multiplying 1 by the $(n-1)^{\text{th}}$ power of the modulus. Thus, the units of the different orders are as indicated blow; r being the modulus of the scale.

Ascending Scale.							Descending Scale.					
n^{th} order.	5^{th} order.	4^{th} order.	3d order.	2d order.	1st order.		1st order.	2d order.	3d order.	4^{th} order.		n^{th} order.
$\cdot\cdot\, r^{n-1}\,\cdot\cdot$,	r^4,	r^3,	r^2,	r,	1	.	$\frac{1}{r}$,	$\frac{1}{r^2}$,	$\frac{1}{r^3}$,	$\frac{1}{r^4}$,	$\cdots$	$\frac{1}{r^n}\cdot\cdot$

The law of the scale being determined, the conventional method of writing the scale is as follows:

	n^{th} order.	....	4^{th} order.	3d order.	2d order.	1st order.		1st order.	2d order.	3d order.	4^{th} order.	....	n^{th} order.
..	0	....	0	0	0	0	.	0	0	0	0	....	0 ...

It is not customary to write the name of the order upon that 0 which indicates the place of a unit, but we have done so, the more clearly to indicate the nature of the conventional system

Now, if any number be written in the place occupied by any 0 in the scale, it will indicate that number of units of the order occupied by the 0. The number written must not be greater than $r-1$. In order to write any number in a uniform scale, as many separate characters are requisite as there are units in the modulus, including amongst them the character 0. The point placed on the line of 0's, marks the origin of the scale. If the modulus is 2, the scale is called a *binary* scale; if it is 3, it is called a *ternary* scale; if it is 4, the scale is *quarternary*; if 5, *quinary*, if 6, *sexenary*; if 10, *decimal*; if 12, *duodenary*, &c.

When numbers are written in any scale, the convention adopted implies, that the sum of the numbers indicated by all the numbers of the different orders, is to be taken. In the decimal scale, the number 235 is equivalent to

$$200 + 30 + 5 = 235.$$

The number 221.75 is equivalent to

$$200 + 20 + 1 + \frac{7}{10} + \frac{5}{100}$$

and so on.

In the binary scale, the number written 101110, is equivalent to

$$1 \times 2^5 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2 + 0,$$

which is equivalent to 46, as expressed in the decimal scale.

The following table shows how numbers may be written in the different scales.

Scale.	Modulus.	Number.	Development.	Decimally expressed.
Binary,	2	101110	$1\times 2^5+1\times 2^3+1\times 2^2+1\times 2+0$	46
Ternary,	3	121201	$1\times 3^5+2\times 3^4+1\times 3^3+2\times 3^2+1$	451
Quarternary,	4	123013	$1\times 4^5+2\times 4^4+3\times 4^2+1\times 4+3$	1735
Quinary,	5	413402	$4\times 5^5+1\times 5^4+3\times 5^3+4\times 5^2+2$	13602
Sexenary,	6	532412	$5\times 6^5+3\times 5^4+2\times 5^3+4\times 5^2+1\times 5+2$	43352
Decimal,	10	17844	$1\times 10^4+7\times 10^3+8\times 10^2+4\times 10+4$	17844
Duodenary,	12	7846	$7\times 12^3+8\times 12^2+4\times 12+6$	13302
&c.	&c.	&c.	&c.	&c.

The decimal scale is the only uniform scale that is of importance, the others possessing interest only as matters of curiosity. In the decimal scale, the point which marks the division of the ascending and the descending scale, is called the *decimal point*. The law of the scale from this point downwards is the same as from any preceding point downward; that is, a unit of any order is equal to one of the preceding order divided by the modulus of the scale, which, in the decimal system, is 10.

Varying Scales. In varying scales, the base is some unit of measure arbitrarily assumed, and the law of the scale, or the modulus of the scale, ceases to be uniform. The law of any particular scale is assumed, generally, in accordance with some mercantile custom, and the nature of the units of the different orders are indicated by writing over each place in the scale, some symbol to indicate the order. Thus, in the mercantile scale for writing British currency, the conventional relation of the different units is given by the following table :

4 farthings make 1 penny,
12 pence " 1 shilling,
20 shillings " 1 pound.

And the conventional scale is thus expressed;

£ *s.* *d.* *far.*

To write 10 pounds, 11 shillings, 4 pence, and 2 farthings, in this scale, we simply write the corresponding numbers in their proper places, thus,

£	*s.*	*d.*	*far.*
10	11	4	2

We are obliged to employ the decimal scale to write the numbers, 10, 11, &c.

The nature of other varying scales is the same as that just described, and their number is very great. For a further account of this subject, see *Arithmetic*, *Notation*, *&c.*

Scale of a Series. In Algebra, a succession of terms, by the aid of which, any term of a recurring series may be found, when a sufficient number of the preceding ones are given.

If the fraction,

$$\frac{a + bx}{a' + b'x + c'x^2},$$

be developed into a series, it will be found that each term, after the second, can be obtained by multiplying the one that next precedes it by

$$-\frac{b'}{a'}x,$$

and the second preceding term by

$$-\frac{c'}{a'}x^2,$$

and then taking the algebraic sum of the products: these two terms, taken in their order, and separated by a comma, thus,

$$\left(-\frac{b'}{a'}x, \quad -\frac{c'}{a'}x^2,\right)$$

form what is called the *scale of the series.* In this case, the scale contains *two* terms, and the series is called a recurring series of the *second* order.

The development of the fraction

$$\frac{a + bx + cx^2}{a' + b'x + c'x^2 + d'x^3},$$

gives rise to a recurring series of the *third*

order, whose scale is

$$\left(-\frac{b}{a'}x, \quad -\frac{c'}{a'}x^2, \quad -\frac{d'}{a'}x^3\right).$$

In general, the development of the fraction,

$$\frac{a + bx + cx^2 + \ldots + hx^{n-1}}{a' + b'x + c'x^2 + \ldots + h'x^n},$$

gives rise to a recurring series of the n^{th} order, whose scale is

$$\left(-\frac{b'}{a'}x, \quad -\frac{c'}{a'}x^3, \ldots -\frac{h'}{a'}x^n\right)\ldots$$

The scale of the series, taken in connection with the definitions of *recurring series* and of *scale*, furnish the *law of the series*.

The *scale* of co-efficients, is a scale from which the co-efficients may be formed, in a manner entirely analagous to that of forming the terms from the scale of the series. The scale of co-efficients can always be obtained from the scale of the series, by dividing each term by the first power of the leading letter of the series. The scale of co-efficients of the most general case of recurring series, is

$$\left(-\frac{b'}{a}, \quad -\frac{c'}{a'}x, \quad -\frac{d'}{a'}x^2\ldots-\frac{b'}{a'}x^{n-1}\right),$$

as given above.

SCA-LĒNE' TRIANGLE. [Gr. σκαληνος, oblique]. A triangle, whose sides are all unequal. This distinguishes this class of triangles from isosceles triangles, in which two of the sides are equal.

SCALENE CONE. A cone, such that a section made by a plane through the axis perpendicular to the plane of the base, is a scalene triangle. The term, *scalene*, is equivalent to oblique, in the last case.

SCHŌ'LI-UM. [L. *scholion;* Gr. ςχολιον, a remark]. In Geometry, a remark made upon one or more preceding propositions, which tends to point out their connection, their use, their restriction, or their extent.

SCĪ'ENCE. [L. *scientia;* from *scio*, to know]. In its general sense signifies knowledge reduced to order; that is, knowledge so classified and arranged as to be easily remembered, readily referred to, and advantageously applied. More strictly speaking, it is a knowledge of laws, principles, and relations. The basis of all science is the immutability of the laws of nature and of events. Assuming that all the phenomena of the physical, mental, or moral world are consequences of general and unchanging laws, we may define SCIENCE to be a knowledge of those laws, embracing the connected processes of observation and reasoning, by means of which they are discovered; and also the processes of reasoning, by means of which their operation is made known in the production of phenomena.

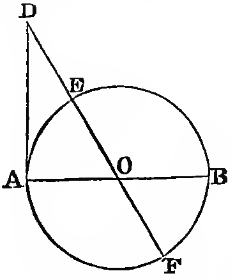
Pure science is based on self-evident truths, and laws of relation are deduced by demonstration—of this nature is mathematical science. Natural science is based upon experiment and observation: its fundamental laws are deduced by inductive reasoning; that is, they are general conclusions derived from classifying and comparing particular experiences.

Knowledge of the relations of quantity constitutes *abstract science;* that of causes and effects *physical science.*

Science is the result of general laws, and is sometimes called *theory*, as correlative with *art*. Art is the application of knowledge to practice. A principle of science is a rule in art. Science is knowledge. Art is skill in using it.

SĔ'CANT. [L. *seco*, to cut, or to cut off]. A straight line cutting a curve in two or more points. If a secant line be revolved about one of its points of secancy until the other point of secancy coincides with it, the secant becomes a tangent. If it be still further revolved, it again becomes a secant on the other side; hence, a tangent to a curve, at any point, is a limit of all secants through that point. A secant plane is one which intersects a surface or solid.

SECANT IN TRIGONOMETRY. A straight line drawn from the centre of a circle through



the second extremity of an arc, and termina-

ted by the tangent through the first extremity; thus, the straight line OD is the secant of the arc EA; if OA is equal to 1, OD is the secant of the angle AOD. In numerical value, the secant of an angle is equal to the reciprocal of the cosine.

SEC′OND. [L. *secundus*; from *sequor*, to follow]. A unit of measure employed in estimating time. It is equivalent to the 60th part of a minute, the 3600th part of an hour, or the 86400th part of a day. In trigonometry it is the 60th part of a minute, the 3600th part of a degree, or the 1296000th part of a circumference.

SEC′TION. [L. *sectio*; from *seco*, to cut off]. A part separated from the rest.

SECTION OF A SURFACE BY A PLANE. The line cut out of the surface by a plane passed so as to intersect the surface. Thus, we speak of *conic sections*, meaning the curves cut out of the surface of a right cone, with a circular base, by a secant plane. The section of a surface of revolution made by a plane passing through the axis, is called a *meridian section*. The section by a plane perpendicular to the axis is a circle.

SECTION OF LAND. A tract of land one mile square, containing 640 acres. The public domain of the United States is divided by north and south lines, six miles apart, into strips called ranges; these are again divided by east and west lines, six miles apart, into squares of 36 square miles, called townships. These ranges are numbered both east

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

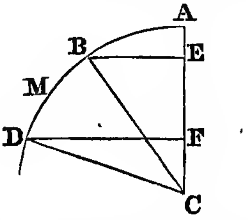
and west from some principal meridian, and the townships in each range are numbered both north and south from some principal east and west line, for the purpose of easy reference in the land offices. Each section is divided by east and west, and by north and south lines, one mile distant from each other, into squares of a mile on each side; these are called sections. The sections in each township are numbered as shown in the annexed diagram. Sections are sometimes subdivided into half sections, quarter sections, and even into eighths of a section. In order to describe a section accurately, we say, for example, section 21, township 4 north, range 3 east, land district of Arkansas. Quarter sections are described as the N. E., N. W., S. E., or S. W. quarter sections; and eighth sections are described as the east or west half of the N. E, N. W., &c., quarter section.

SECT′OR. [L. *seco*, to cut]. That portion of the area of a circle included between two radii and an arc. The area of a sector is equal to the product of the arc of the sector by half of the radius. If the angle at the centre is given, the length of the arc of the sector may be found, since it is equal to π multiplied by the radius into the ratio of 180° to the number of degrees of the sector; that is

$$S = \pi r \frac{n^\circ}{180^\circ} \quad \text{and} \quad A = \pi r^2 \frac{n^\circ}{360^\circ},$$

for the area.

A SPHERICAL SECTOR is a volume or solid that may be generated by revolving a sector of a circle about a straight line drawn through the vertex of the sector as an axis. The arc of the spherical sector generates the surface of a zone, called the base of the spherical sector, and the two radii generate the surfaces of two cones, having a common vertex at

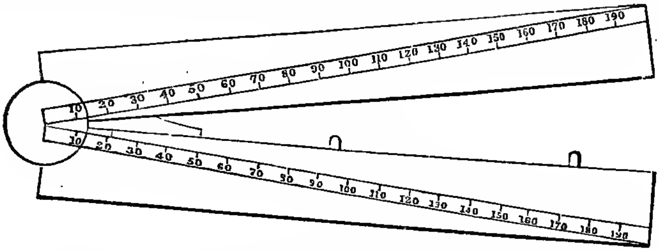


the centre of the sphere. The volume of the spherical sector is equal to the zone

which forms its base, multiplied by one-third of the radius of the sector.

Let CA represent the axis of revolution, and BCD the revolving sector, then is the surface generated by the arc BD, the base of the sector, and the volume is equal to a zone whose altitude is EF multiplied by $\frac{1}{3}$ CD.

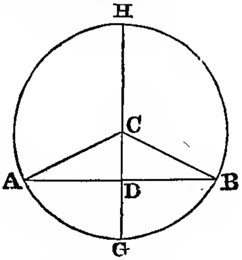
Sectoral Scale of Equal Parts. A scale of wood, brass, or ivory, consisting of two arms, which open by turning round a joint or hinge at their common extremity. There are several scales drawn on the two arms, but we shall only consider the scales of equal parts. On each arm there is a diagonal



line drawn through the point about which the arms revolve; these are divided into equal parts, which are numbered from the common point outwards. In the instrument represented in the figure, they are divided into 200 parts each. The instrument is used for plotting to any scale.

To illustrate, let it be required to lay off any distance, say 46 rods on a scale of 20 rods to the inch. Take an inch in the dividers, and applying one foot at the 20th division, on one arm, open the sector till the other foot will fall upon the 20th division on the other arm; then open the dividers till one foot being at the 46th division on one arm, the other foot will fall at the 46th division on the other arm; then will the distance in the dividers represent the distance 46 rods to the required scale; and in like manner, any other distance may be taken to the same scale.

SEG'MENT OF A CIRCLE. [L. *segmentum*, from *seco*, to cut off]. A part of the area of a circle included between a chord and



the arc which it subtends. Thus AGB, or AHB, is a segment. To find the area of a segment of a circle; find the area of the sector having the same arc, then find the area of the triangle formed by the chord and the two radii of the sector; if the greater segment is required, take the sum of their areas; if the lesser segment is required, take their difference, and the result will be the area required.

A Spherical Segment, is a portion of a sphere bounded by a secant plane and a zone of the surface. If a circular segment be revolved about a radius drawn perpendicular to the chord of the segment, the volume generated is a spherical segment. Thus if the circle AHBG be revolved about HG, perpendicular to AB, either segment, AGB or AHB, will generate a spherical segment. To find the volume of a spherical segment; find the volume of the corresponding sector, and also the volume of the cone whose vertex is at the centre, and whose base is the base of the segment. If the greater segment is required, add them together; if the lesser segment is required, take their difference, and the result will be the volume required. The portion of a sphere between any two parallel secant planes, is a segment.

SEM'I-CIR-CLE. [L. *semi*, half; half of a circle]. Every diameter divides the circle in which it is drawn into two equal parts, each of which is called a semicircle. The diameter is called the diameter of the semicircle.

SEM-I-CU'BI-CAL PARABOLA. A

parabola which may be referred to co-ordinate axes such that the squares of the ordinates of its points shall be to each other as the cubes of the abscissas of the same points. In this case the equation of the curve is of the form

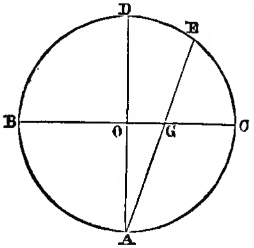
$$y^2 = p^2x^3,$$

the origin being at the vertex and the axis of X coinciding with the axis of the curve. The curve consists of two branches, convex towards the axis of X, and tangent to it at the origin of co-ordinates, or vertex of the curve. See *Parabola*.

SEM-I-DĪ-AM'E-TER. Half a diameter, or radius of a circle or sphere.

SEM-I-OR'DI-NATE. A term used by some of the old writers to designate half of a chord of a curve perpendicular to an axis. It is now called an ordinate.

SEM-I-TAN'GENT. In Spherical Projections, the tangent of half an arc. Let ABDE represent any circle; AD and BC two diameters at right angles, and DE any arc estimated from D as an origin. Draw the chord AE intersecting BC in G; then is OG the semi-tangent of the arc DE, the radius of the circle being 1.



SĒ'RIĒS. In Analysis, an infinite number of terms following one another, each of which is derived from one or more of the preceding ones, by a fixed law, called the *law of the series*. Whenever a sufficient number of terms are given, and the law of the series is known, any number of succeeding terms may be deduced.

Sometimes the law of a series is given by means of a general term, from which any term may be deduced by making proper suppositions upon the arbitrary quantities that enter it. Series are derived in a great number of ways, and by a great variety of analytical processes, generally from the development of some function. They receive names from the nature of the function developed, as the logarithmic series, the exponential series, &c.

A series is said to be *decreasing* when the numerical value of each term is less than that of the preceding.

A series is *increasing* when the numerical value of each term is greater than that of the preceding.

A series is *converging* when the greater the number of terms taken, the nearer will their sum approach in value to a fixed quantity, which is called the *sum of the series*. All other series are *diverging*.

The *summation* of a series is the operation of finding an expression for the sum of any number of terms of the series. When the series is converging, we can often find an expression for the sum of an infinite number of terms, which is the total sum of the series.

We shall indicate some of the most useful series, together with the method of generating and the method of summing them.

1. *Arithmetical Series.* An arithmetical progression is a series in which each term is derived from the preceding, by the addition of a constant quantity called the common difference. The sine of n terms of such a series. is given by the formula

$$S = n\left(\frac{a+l}{2}\right) \cdots (1).$$

in which a denotes the first term, l the n^{th} term, and n the number of terms. See *Arithmetical Progression*.

2. *Geometrical Series.* A geometrical progression is a series in which each term is derived from the preceding one by multiplying it by a constant quantity, called the *ratio* of the progression. The sum of n terms of such a series is given by the formula

$$S = \frac{lr - a}{r - 1} = \frac{ar^n - a}{r - 1} \cdots\cdot (2).$$

in which l denotes the n^{th} term, a the first term, and r the ratio. When $r < 1$, the series is converging, and the sum of an infinite number of terms is given by the formula

$$S = \frac{a}{1 - r} \cdots\cdot (3) \cdots$$

See *Geometrical Progression*.

3. *Recurring Series.* A recurring series

is a series in which each term is equal to the algebraic sum of the products obtained by multiplying one or more of the preceding terms by certain fixed quantities; these quantities, taken in their order, constitute the *scale of the series*. For the method of deducing and summing these series, see *Recurring Series*.

4. *Logarithmic Series*. A series derived by developing the logarithm of $(1 + y)$ according to the ascending powers of y.

The simplest logarithmic series is

$$l(1+y)=M\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}-\text{\&c.}\right)\cdots\cdot(4).$$

This series is not adapted to the computation of tables. By suitable transformations a great variety of converging series have been deduced, some of which we give below.

$$l(y+1)=ly+\left\{\frac{1}{2y+1}+\frac{1}{3}\cdot\frac{1}{(2y+1)^3}+\frac{1}{5}\cdot\frac{1}{(2y+1)^5}+\frac{1}{7}\cdot\frac{1}{(2y+1)^7}+\text{\&c.}\right\}\cdot\cdot(5).$$

A series by means of which we can compute logarithms of numbers when we know those of the preceding ones.

$$l(y+1)=2ly-l(y-1)$$
$$2\left\{\frac{1}{2y^2-1}+\frac{1}{3}\cdot\frac{1}{(2y^2-1)^3}+\frac{1}{5}\cdot\frac{1}{(2y^2-1)^5}+\text{\&c.}\right\}(6)$$

a very rapidly converging series.

$$l(y+2)=2l(y+1)+l(y-2)-2l(y-1)+2\left\{\frac{2}{y^3-3y}+\frac{1}{3}\left(\frac{2}{y^3-3y}\right)^3+\frac{1}{5}\left(\frac{2}{y^3-3y}\right)^5+\text{\&c.}\right\}\ (7).$$

5. *Exponential Series*. Exponential series are derived from the development of exponential functions.

The simplest form of an exponential series is

$$a^x=1+kx+\frac{k^2x^2}{1.2}+\frac{k^3x^3}{1.2.3}\cdot\cdot+\frac{k^nx^n}{1.2..n}+\cdots\cdot(5),$$

in which k is equal to the Naperian logarithm of a.

If $a=e=2.718281828$, $k=1$, and the formula becomes

$$e^x=1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\cdots\cdots\cdots(6).$$

If $x=1$ in this series, we have

$$2.71828\cdot\cdot=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{\&c.}$$

6. *Trigonometrical Series*. Trigonometrical series are derived from developing some of the trigonometrical functions, and when they can be summed approximately, they generally give the value of π, or the ratio of diameter to the circumference of a circle. We give some of the simplest:

$$\sin x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}+\text{\&c.}\ (7).$$

$$\cos x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\text{\&c.}\ldots(8).$$

$$\frac{1}{\sin x}=\operatorname{cosec}x=\frac{1}{x}+\frac{x}{1.2.3}+\frac{14x^3}{1.2.3.4.5.6}+\frac{744x^5}{1.2.3\ldots9}+\frac{100584x^7}{1.2\ldots12}+\text{\&c.}\ (9).$$

$$\frac{1}{\cos x}=\sec x=1+\frac{x^2}{1.2}+\frac{5x^4}{1.2.3.4}+\frac{61x^6}{1.2.3.4.5.6}+\frac{1385x^8}{1.2.3.4.5.6.7.8}+\text{\&c.}(10).$$

$$\tan x=\frac{\sin x}{\cos x}=x+\frac{2x^3}{1.2.3}+\frac{2^3.12x^5}{1.2.3.4.5.6}+\frac{2^6.306x^7}{1.2\ldots9}+\text{\&c.}\ldots(11).$$

$$\cot x=\frac{\cos x}{\sin x}=\frac{1}{x}-\frac{2x}{1.2.3}+\frac{2^4x^3}{1.2.3.4.5.6}+\frac{2^6.12x^5}{1.2\ldots9}+\text{\&c.}\ldots(12).$$

These may be used for computing tables.

The following show the values of arcs in terms of trigonometrical lines:

$$\sin^{-1}x=x+\frac{x^3}{1.2.3}+\frac{3x^5}{1.2.3.4.5}+\text{\&c.}\ (13).$$

$$\cos^{-1}x = \frac{\pi}{2} - x - \frac{1}{2}\cdot\frac{x^3}{1.2.3}$$
$$- \frac{1.3}{2^2}\cdot\frac{x^5}{1.2.3.4.5} - \&c. \ldots\ldots (14).$$
$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$
$$+ \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \&c. \ldots (15).$$

By means of these series, the value of π may be computed by making x equal to any arc whose sine, cosine, or tangent, is known. The 15th formula is best adapted to this computation, making use of one of the following auxiliary formulas:

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\ldots$$
$$\frac{\pi}{4} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}.$$
$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2^2-1} + \tan^{-1}\frac{1}{2^3-1} + \tan^{-1}$$
$$\frac{1}{2^4-1} + \tan^{-1}\frac{1}{2^5-1} + \&c. \&c.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\frac{\pi}{4} = \tan^{-1}x + \tan^{-1}\frac{1-x}{1+x}.$$

By making, in (15), x equal to $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}$, the series is converging, and the sum of the results obtained by using only a few terms, gives a close approximation to the value of $\frac{\pi}{4}$.

A great variety of trigonometric series have been deduced; the foregoing are, however, some of the most important.

7. *Series of Figurate Numbers.* These series are of different orders, and may be derived from the general expression:

$$\frac{n(n+1)(n+2)(n+3)\ldots(n+m)}{1.2.3\ldots(m+1)}:$$

by assigning to m a particular value, and then in the result making n, in succession, equal to 1, 2, 3, 4, &c. The order of the series is denoted by the value of m.

Making $m = 1$, we have the general term of the first order of figurate series,

$$\frac{n(n+1)}{1.2},$$

and making n, successively equal to 1, 2, 3, &c., we have the series,

$$1, 3, 6, 10, \ldots \frac{n(n+1)}{1.2} \ldots (16).$$

In like manner, for $m = 2, 3, 4$, &c., we have the series,

$$1, 4, 10, 20, \ldots \frac{n(n+1)(n+2)}{1.2.3} \ldots\ldots (17)$$
$$1, 5, 15, 35, \ldots \frac{n(n+1)(n+2)(n+3)}{1.2.3.4} \ldots (18).$$
$$\&c. \qquad \&c. \qquad \&c. \qquad \&c.$$

Figurative series may be summed by means of the formula,

$$S = ma + \frac{m(m-1)}{1.2}d_1 + \frac{m(m-1)(m-2)}{1.2.3}d_2$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}d_3 + \&c.,$$

in which a denotes the first term, m the number of terms considered, and d_1, d_2, d_3, &c., the first terms of the successive orders of differences.

In the series of the first order,

$$d_1 = 2, \quad d_2 = 1, \quad d_3 = 0, \quad \&c.;$$

whence,

$$S = m + m(m-1) + \frac{m(m-1)(m-2)}{1.2.3}.$$

In the second order,

$$d_1 = 3, \quad d_2 = 2, \quad d_3 = 1, \quad d_4 = 0, \quad \&c.;$$

whence,

$$S = m + \frac{3m(m-1)}{1.2} + \frac{m(m-1)(m-2)}{1.2.3}$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}:$$

and so on, for the remaining orders.

8. *Series derived from the general expression,*

$$\frac{q}{n(n+p)}.$$

These series bear considerable analogy to figurate series, and are deduced in each case by attributing a fixed value to p, and then giving suitable successive values to q and n. The following are the most useful of the series of this class:

If we make

$p = 1$, $q = 1$, and $n = 1, 2, 3, 4$, &c. we have

$$\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}, \frac{1}{5.6}, \frac{1}{6.7}, \&c. (19).$$

If we make

$p = 0$, $q = 1$, and $n = 1, 2, 3, 4$, &c. we have

$$\frac{1}{1.3}, \frac{1}{3.5}, \frac{1}{5.7}, \frac{1}{7.9}, \frac{1}{9.11}, \&c. \ldots\ldots (20).$$

If we make

$p = 3$, $q = 1$, and $n = 1, 2, 3, 4$, &c. we have

$$\frac{1}{1.4}, \frac{1}{2.5}, \frac{1}{3.6}, \frac{1}{4.7}, \frac{1}{5.8}, \text{ \&c.} \cdots (21).$$

If we make

$p = 4$, $q = 4$, and $n = 1, 5, 9, 13$, &c. we have

$$\frac{4}{1.5}, \frac{4}{5.9}, \frac{4}{9.13}, \frac{4}{13.17}, \frac{4}{17.21}, \text{ \&c.} \cdots (22).$$

If we make

$p = 2$, $q = 2, -3, +4, -5, +6$, &c., and $n = 3, 5, 7, 9$, &c., we have

$$+\frac{2}{3.5}, -\frac{3}{5.7}, +\frac{4}{7.9}, -\frac{5}{9.11}, +\frac{6}{11.13}, \text{\&c.} \cdots (23).$$

If we make

$p = 2$, $q = +1, -1, +1, -1$, &c., $n = 1, 2, 3, 4$, &c., we have

$$\frac{1}{1.3}, -\frac{1}{2.4}, +\frac{1}{3.5}, -\frac{1}{4.6}, +\frac{1}{5.7}, \text{ \&c} \cdot (24).$$

In this way a great number of useful series may be deduced; and since

$$\frac{q}{n(n+p)} = \frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right),$$

it follows that

$$\Sigma\left(\frac{q}{n(n+p)}\right) = \frac{1}{p}\Sigma\left(\frac{q}{n} - \frac{q}{n+p}\right)$$
$$= \frac{1}{p}\left[\Sigma\frac{q}{n} - \Sigma\frac{q}{n+p}\right],$$

that is, the sum of any number of terms of one of these series can be found, when the difference of the sums of the corresponding terms of two auxiliary series deduced, according to the same law, from the expressions, $\frac{q}{n}$ and $\frac{q}{n+p}$, can be found. For example, in series (19), we have

$$\Sigma\left(\frac{q}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n};$$

$$\Sigma\frac{q}{n+p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \frac{1}{n+1};$$

whence,

$$\Sigma\left(\frac{q}{n(n+p)}\right) = \frac{1}{p}\left[\Sigma\left(\frac{q}{n}\right) - \Sigma\left(\frac{q}{n+p}\right)\right]$$
$$= 1 - \frac{1}{n+1}.$$

If $n = \infty$, we have $S = 1$, S denoting the sum of the entire series.

In the series (20),

$$\Sigma\left(\frac{q}{n}\right) = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2n-1};$$

$$\Sigma\left(\frac{q}{n+p}\right) = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2n-1} + \frac{1}{2n+1};$$

whence,

$$\Sigma\left(\frac{q}{n(n+p)}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right).$$

If $n = \infty$, we have

$$S = \frac{1}{2}.$$

In series (21), we find, in like manner,

$$\Sigma\left(\frac{q}{n(n+p)}\right) = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right);$$

whence, if $n = \infty$,

$$S = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}.$$

In series (22), we have

$$\Sigma\left(\frac{q}{n(n+p)}\right) = 1 - \frac{1}{1+4n}, \quad \text{and} \quad S = 1.$$

In series (23), we have, according as n is even or odd,

$$\Sigma\left(\frac{q}{n(n+p)}\right) = \frac{1}{2}\left(-\frac{1}{3} + \frac{n+1}{2n+3}\right); \quad \text{or,}$$
$$= \frac{1}{2}\left(\frac{2}{3} - \frac{n+1}{2n+3}\right). \qquad S = \frac{1}{12}.$$

In series (24),

$$\Sigma\left(\frac{q}{n(n+p)}\right) = \frac{1}{2}\left(\frac{1}{2} \mp \frac{1}{n+1} \pm \frac{1}{n+2}\right) \text{and } S = \frac{1}{4}.$$

The upper signs are used when n is even, and the lower ones when n is odd.

9. *Series of powers of natural numbers.* These series are found by raising the natural numbers to powers; thus,

1 . 4 . 9 . 16 . 25 . 36 . 49 . &c. . (25),
1 . 8 . 27 . 64 . 125 . 216 . 343 . &c. . (26),
1 . 16 . 81 . 256 . 625 . 1296 . 2401 . &c. . (27),
&c. &c. &c.

These may all be summed by the method of differences. See *Summation by differences.*

10. *Series of reciprocals of powers.* These series are formed by taking the reciprocals of the different powers of the natural numbers. Their sums are given in the formulas.

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{\&c.} \quad (28),$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{\&c.} \quad (29),$$

$$\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{\&c.} \quad (30),$$

&c. &c. &c.

Series of reciprocals of powers can be summed in terms of π, whenever the exponent of the power is even.

SES-QUI-DŬ'PLI-CATE RATIO. [L. *sesqui*, one and a-half, and *duplicatus*, double]. A ratio in which the consequent is $2\frac{1}{2}$ times the antecedent. Thus, the ratio of $\frac{5a}{2a}$ is sesquiduplicate.

SEX'A-GEN-A-RY. [L. *sexagenarius*; from *sex*, six, and a word signifying ten, seen in *viginti*, twenty]. Something appertaining to the number sixty.

SEXAGENARY, IN NUMBERS. A scale in which the modulus is 60 : see *Scale of Numbers*. This scale is used in treating of the divisions of the circle. If the circumference be divided into 6 equal parts, the chord of each part is 1 ; each arc contains 60 degrees, each degree contains 60 minutes, each minute contains 60 seconds, each second 60 thirds, and so on. This division of the circle is called *sexagenary*.

SEX-A-GES'I-MALS. Pertaining to the number 60. Sexagesimal fractions are those whose denominators are some power of 60 ; that is, fractions written in a descending sexagenary scale.

SEX'TANT. [L. *sextans*, one-sixth]. A sixth part of the circumference of a circle. An instrument used in measuring angles, founded upon the optical principle that a ray of light twice reflected from plane reflectors, makes, with the ray before reflection, an angle equal to twice the angle of inclination of the reflecting surfaces. The graduated arc is equal to the sextant of the circle, and is divided into 120 equal parts, so that if the image of an object is made to coincide with the image of the second object, after two reflections, the angle read off on the limb of the instrument is the angle subtended by the two objects at the instrument.

SEX'TU-PLE. [L. *sextuplus* ; *sex*, six, and *duplus*, double]. Six fold ; six times as much.

SHĀDES AND SHADOWS. A branch of applied Geometry, having for its object the graphic construction of the shades and shadows of bodies. The shade of a body is that part of the surface of the body from which light is excluded by the body itself The indefinite shadow of a body is that part of space from which light is excluded by the body. The shadow of a body on a body, is that part of the surface of the second body from which the light is excluded by the first. That part of the surface of a body upon which the light falls freely, is called the illuminated part. The line which separates the shade of a body from the illuminated part, is called the *line of shade*. The line which separates the shadow on a body from the illuminated part, is called the *line of shadow*.

In considering the theory of shades and shadows, it is usual to regard the light as emanating from a body at so great a distance that the rays may be regarded as parallel. A ray of light is a straight line parallel to the direction of the light. A *plane of rays* is a plane parallel to a ray of light. A *cylinder of rays* is a cylinder whose elements are parallel to a ray of light.

The lines of shade and shadow are determined by passing a cylinder of rays enveloping the body casting the shadow, and finding its intersections with the surface receiving the shadow. The line of contact of the enveloping cylinder of rays with the first body, is the line of shade on that body ; and the line of intersection of the cylinder with the surface of the second body, is the line of shadow on that body. Hence. we see that the line of shade casts the line of shadow ; the former is always a line of contact, the latter, a line of intersection.

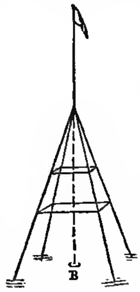
The shadow of a point on a surface is the point in which a ray of light through the point pierces the surface. The shadow of a straight line on a surface, is the line of intersection of the surface, with a plane of rays passing through the line.

The shadow of a curve on a surface is the intersection of the surface, with a cylinder of rays passed through the curve. The bodies

casting shadows, the surfaces receiving them, and the direction of the light, are all given by their projections. The lines of shade and shadow are found, by the application of the rules of Descriptive Geometry.

SĪGN. [L. *signum*]. A symbol employed to denote an operation to be performed, to show the nature of a result of some previous operation, or to indicate the sense in which an indicated quantity is to be considered. See *Notation*.

SIG'NAL. [L. *signum*, a sign]. In Trigonometrical Surveying, an object used to mark the positions of the triangulation points. The simplest form of a signal, is a vertical staff planted at the triangulation point, so that its axis shall pass through the point to be marked. To render the signal visible at a distance, the staff is often painted in checks of black and white, and a flag is attached to its upper end. To distinguish different stations, the flags are formed by sewing together pieces of different colors arranged according to some preconceived system. Another method is to place a frustum of a tin cone, of a few inches radius, concentric with the axis of the staff. This, by reflecting the rays of the sun, in all directions, serves to render the signal visible at a greater distance than the bare pole, or even the pole with a flag, could be seen. When the length of the staff is considerable, it should be braced by strong staves driven into the ground obliquely and nailed to the staff. The signal should be so arranged, that it may be readily removed for the purpose of planting a theodolite exactly over the centre of the station. When a very elevated signal is needed, the staff may be supported by a frame-work of timber firmly braced. One of the simplest examples of such a signal, consists of three or four pieces of scantling firmly bedded in the ground and meeting at a point, with a short staff projecting vertically from the apex. This staff should be so fixed, that its axis shall be exactly over the centre of the station, and it may be marked either by a flag or a tin cone. When, on account of the nature of the soil, the pieces of scantling cannot be bedded in the ground, they may be confined in their proper places by heaping stones around them, to such a height as to render them stable.



If it is necessary to raise the signal still higher, as often happens when the station is on low ground, or when the surrounding country is covered with trees or brush wood, a regular framework must be constructed, of sufficient height to sustain a scaffolding on which the instrument is to be placed when the station becomes one of observation. In this case a staff surmounted by a tin cone, may be set up in the scaffolding and thoroughly braced from the timbers of the platform. The framework of the signal should be strongly braced, so as to resist the action of tempests, and preserve the axis of the signal uniformly in the same position. No detailed rules can be given for the construction of signals, as in nearly every instance some peculiarity of structure is required. It is to be noted that when tin cones are used the brilliant element formed by the reflected rays proceeding to the eye, does not, in general, lie in the plane passing through the station of observation and the axis of the signal observed upon. In this case, a slight correction is used, called the correction for *reduction to the centre of the signal*. This correction is given by the formula

$$C = \pm \frac{r \cos^2 \frac{1}{2} z}{D \sin 1''};$$

in which r denotes the radius of the signal, or the mean radius of the frustum, z the horizontal angle at the point of observation subtended by the sun and the signal, D the distance from the point of observation to the signal.

SIG-NIF'I-CANT [L. *significans*]. Figures standing for numbers are called significant figures. They are 1, 2, 3, 4, 5, 6, 7, 8 and 9.

SIM'I-LAR FIGURES. [L. *similas*, like]. In Geometry, figures made up of the same number of parts, these parts being arranged in the same manner, so that the figures shall be of the same form and differ from each other only in magnitude.

Two polygons are similar when they have the same number of angles, which are equal

each to each, and the sides about these angles taken in the same order. Proportional similar polygons are to each other as the squares of their homologous sides; or, as the squares of any of their homologous lines. All mutually equiangular triangles are similar. All regular polygons having the same number of sides are similar.

Two sectors, arcs, or segments of circles, are similar, when they correspond to equal angles at the centre. Two curves of the same name or kind are similar, when, if any polygon be inscribed in the one a similar polygon can always be inscribed in the other. Two ellipses or two hyperbolas are similar when their axes are respectively proportional to each other. In this case their eccentricities are equal.

Two polyhedrons are similar when they are bounded by the same number of mutually similar faces, similarly placed; their polyhedral angles are then equal, each to each. Two right cylinders are similar when they may be generated by the revolution of similar rectangles about their homologous sides. Two cones are similar when they can be generated by the revolution of similar triangles about their homologous sides. All regular polyhedrons of the same name are similar solids. The volumes of two similar solids are to each other as the cubes of any two homologous lines. Two solids of the same name, bounded by curved surfaces, are similar when any polyhedron, being inscribed in one, a similar polyhedron can always be inscribed in the other.

SIM'I-LAR-LY. In like manner.

SIM'PLE. [L. *simplex*]. Not complicated. A simple quantity is a quantity containing but one term. It is the same as a monomial. A simple equation is one of the first degree. Simple addition is the addition of numbers expressed in a uniform scale. Simple subtraction, multiplication, division, &c., have corresponding significations. In this sense the term is used to distinguish the operations from the corresponding operations upon numbers expressed in varying scales; such operations are called *compound.*

The essential nature of the operation is the same in each case, but the practical application of the rules are more *simple* in the first case than in the last.

SĪ-MUL-TĀ'NE-OUS. [L. *simul*, at the same time]. Two equations are *simultaneous* when the values of the unknown quantities which enter them are the same in both, at the same time. A group of equations is *simultaneous* when the value of the unknown quantities is the same in them all, at the same time. Any single equation containing more than one unknown quantity, is indeterminate, and any two such equations may always be regarded as simultaneous. The very act of combining two equations implies that the values of the unknown quantities which enter them are the same in both, and consequently the act of combination renders them simultaneous. It is impossible to render a greater number of equations simultaneous than there are unknown quantities entering them. If we make the attempt to combine a group containing more equations than there are unknown quantities, we shall arrive at one or more equations independent of the unknown quantities, which express the relations that must exist between the known quantities, in order that the proposed equations may be simultaneous. These equations are equations of condition for *simultaneity*, and they are as many in number as the number of equations exceeds the number of unknown quantities. These relations being satisfied, some of the equations must be dependent upon the others, so that there will only be as many independent equations as there are unknown quantities. When we combine a less number of equations than there are unknown quantities, we arrive, by elimination, at a single equation, which is indeterminate, and by the aid of which we may render one or more additional equations simultaneous.

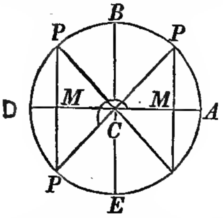
SIMULTANEOUS CHANGES. The corresponding changes resulting from the relation which exists between the function and the variable. If we give any increment to the variable, and substitute the variable thus increased for the variable, the function will receive a corresponding increment, and these two increments are said to be simultaneous. The term increment, as here used, does not necessarily imply *increase*, but is used in the most general sense to cover the case of a decrease of value as well as of an increase. If the increment of the variable is infinitely small, or the differential of the variable, the corresponding in-

crement of the function, is the differential of the function. If the increment of the variable is itself variable, and the corresponding increment of the function be divided by it, the limit of the ratio of their simultaneous increments is the *differential co-efficient of the function.*

SINE OF AN ARC. [L. *sinus*]. In Trigonometry, the distance of an extremity of an arc upon the diameter drawn through the other extremity. If from any point on one side of a plane angle, a perpendicular be let fall upon the other side, thus forming a right angled triangle, the ratio of the hypothenuse of this triangle to the perpendicular is the *sine of the angle.*

The terms sine of an angle, and sine of an arc, are often used as synonymous terms, but they are only so on the supposition that the radius of the arc is taken equal to 1. This supposition is tacitly made in Analysis, and we shall suppose it to obtain for the present discussion. The sine of an angle is designated by the abbreviation *sin*; thus *sin* x, denotes the sine of the angle x, or the sine of an arc of x^0, in a circle whose radius is 1.

Let C be the centre of a circle whose radius is 1, and DA and BE two diameters of the circle at right angles to each other.



All arcs are by convention estimated from A, the right hand extremity of the horizontal diameter DA, and are considered positive when estimated around in a direction contrary to the motion of the hands of a watch. Let AP be an arc thus estimated, which terminates, as shown in the figure, in either of the four quadrants. Then will PM be the sine of the arc or angle. In accordance with an analytical convention, all distances estimated upwards from DA are positive, and as a consequence, all distances estimated downwards must be regarded as negative. By inspecting the figures we see that the sine of the arc AP is positive, when it terminates in the first or second quadrants, and negative when it terminates in the third or fourth quadrants.

The following continued equations show the relations existing between the sine of an arc and the other trigonometrical functions of the arc.

$$\sin x = \cos x \tan x = \frac{\cos x}{\cot x} = \sqrt{1-\cos^2 x}$$

$$= 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x = \frac{\tan x}{\sqrt{1+\tan^2 x}} = \frac{1}{\operatorname{cosec} x}.$$

$$\sin x = \tfrac{1}{2}\sqrt{1-\cos 2x} = \frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}$$

$$= \frac{2}{\cot\frac{1}{2}x+\tan\frac{1}{2}x} = \frac{\sin(30^\circ+x)-\sin(30^\circ-x)}{\sqrt{3}}$$

$$\sin x = 2\sin^2(45^\circ+\tfrac{1}{2}x)-1$$

$$= 1-2\sin^2(45^\circ-\tfrac{1}{2}x) = \frac{1-\tan^2(45^\circ-\frac{1}{2}x)}{1-\tan^2(45^\circ-\frac{1}{2}x)}$$

$$\sin x = \frac{\tan(45^\circ+\frac{1}{2}x)-\tan(45^\circ-\frac{1}{2}x)}{\tan(45^\circ+\frac{1}{2}x)+\tan(45^\circ-\frac{1}{2}x)}$$

$$= \sin(60+x)-\sin(60^\circ-x).$$

The following equations show the value of the sine of an arc in terms of the arc itself; also in terms of an aliquot part of the arc.

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5}$$

$$- \frac{x^7}{1.2.3.4.5.6.7} + \&c. \ldots -$$

The following formulas give the value of the sine of an arc or angle for certain given values of the angle or arc.

$\sin 0^\circ = 0$; $\sin 9^\circ = \frac{1}{4}\left\{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right\}$; $\sin 15^\circ = \frac{1}{4}\left\{\sqrt{6}-\sqrt{2}\right\}$;

$\sin 18^\circ = \frac{1}{4}\left\{\sqrt{5}-1\right\}$; $\sin 27^\circ = \frac{1}{4} \times \left\{\sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right\}$; $\sin 30^\circ = \frac{1}{2}$;

$\sin 36^\circ = \frac{1}{4}\sqrt{10-2\sqrt{5}}$; $\sin 45^\circ = \frac{1}{2}\sqrt{2}$;

$\sin 54^\circ = \frac{1}{4}\sqrt{1+\sqrt{5}}$; $\sin 60^\circ = \frac{1}{2}\sqrt{3}$

$\sin 72^\circ = \frac{1}{4}\sqrt{10+2\sqrt{5}}$;

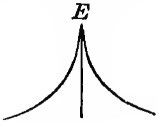
$\sin 75 = \frac{1}{4}\left\{\sqrt{6}+\sqrt{2}\right\}$; $\sin 90^0 = 1$.

For further information in regard to sines, see *Trigonometry*.

SIN″GU-LAR POINT. [L. *singularis*, from *singulus*, single]. A singular point of a curve is a point at which the curve possesses some peculiar properties not possessed by other points of the curve.

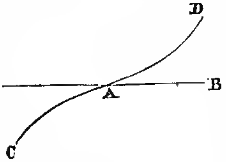
The most important of the singular points are *cusp points*, *multiple points*, *conjugate or isolated points*, *collection of isolated points forming pointed branches*, and *points of abrupt termination*, or, as the French term them, *points d'arrêt*.

CUSP POINTS. A cusp point is a point of a curve at which a curve ceases to proceed in a certain direction, and returns in a contrary direction, the two branches thus formed having a common tangent at that point. Cusps are of two kinds: 1st., when the two branches lie on opposite sides of the common tangent. 2d., when they lie on the same side of the common tangent.



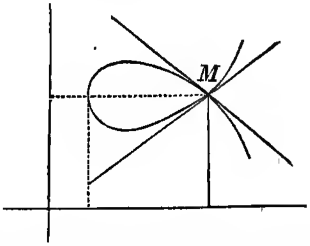
The point E is a cusp of the first kind, and the point A is a cusp of the second kind.

No fixed rules can be given for determining the existence of cusp points; their existence and position must be ascertained from a discussion of the equation of the curve, with respect to the properties given in the definition. See *Cusp*.



POINTS OF INFLEXION. Points at which a curve, from being convex or concave, with respect to a straight line not passing through the point, becomes concave or convex, with respect to the same line. Thus, A, in the last figure, is a point of inflexion. These points are sometimes called points of contrary flexure. The analytical characteristic of a point of inflexion is a change of sign of the second differential co-efficient of the ordinate at the point. See *Inflexion, Point of Inflexion*, &c.

MULTIPLE POINTS. The points at which two points of a curve cross each other, or intersect each other. Thus, M is a multiple point. If two branches intersect each other, it is



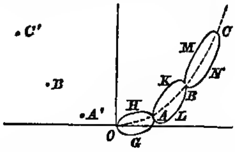
called a double multiple point; if three, a triple multiple point, &c. There is another singular point which closely resembles the multiple point, and is generally known by the name of multiple point, but which differs from the multiple point in having the two branches tangent to each other instead of intersecting. At this point the two branches have a common tangent, but do not intersect each other at that point, nor is that point a cusp. See *Multiple Points*.

CONJUGATE OR ISOLATED POINTS, are points whose co-ordinates satisfy the equation of the curve, but which have no consecutive points. The analytical characteristic of such points is, that the first differential co-efficients of the ordinate of the curve, at those points, is imaginary. See *Conjugate Points*.

POINTED BRANCHES are made up of a collection of separate points which have no consecutive points. The equation,

$$y = ax^2 + \sqrt{x} \sin bx,$$

affords an example of a pointed branch.



If we construct the parabola OAC, whose equation is

$$y = ax^2,$$

it will be a diametral curve; that is, it will bisect a system of chords parallel to the axis of Y; for every positive value of x, there will be two points of the curve, or the curve will be made up of a succession of branches OA, AB, like the links of a chain.

For negative values of x, y is imaginary, except when

$$\sin bx = 0 \, ;$$

the corresponding points lie on the parabola CO at P, P′, &c., and constitute the pointed branch. The points P, P′, P″, &c., are separated by finite intervals; but, in some cases, there are points, making up pointed branches which are separated only by infinitely small intervals; such, for example, is the case in the curve given by the equation $y = a^x$.

Now, when x is a fraction, with an even denominator, there are two real values of y, and equal with contrary signs; consequently, there are two corresponding points of the curve, one on each side of the axis of X; but, when x is a fraction with an odd denominator, there is but one value of x that is real, and this corresponds to but one point. Hence, there is one branch which is continuous, and one pointed branch made up of points, separated by infinitely small intervals. Such a curve cannot be generated by the motion of a point, but only exists as the result of a particular functional relation between y and x.

Points of Abrupt Termination, or Points d'arrêt. These are points at which a curve terminates abruptly. Thus, the curve whose equation is

$$y = x \log x,$$

has but a single branch, terminating abruptly at the origin of co-ordinates. In seeking for points d'arrêt, we have to find the points of a single branch, which have no consecutive points on one side. They always arise from exponential relations between the co-ordinates of the points of the curve.

The curve whose equation is

$$x = b + (y - a) \left[\sqrt{y - a}, + \sqrt{y - c} \right],$$

gives a point of abrupt termination (the radicals having the positive sign only), whose co-ordinates are $y = a$, and $x = b$.

SLĪD'ING-RŪLE or SCALE. A mathematical rule or scale consisting of two parts, one of which slides along the other. Each part has several scales marked upon it, being so numbered and arranged, that, when a given number on one scale is brought to coincide with a given number on the other, the product, or some other function of the two numbers may be found by inspection. The instrument is chiefly used in gauging, and for the mensuration of timber.

SLŌPE. Oblique direction. The slope of a plane is its inclination to the horizon. This slope is generally given by its tangent. Thus, the slope, $\frac{1}{2}$, is equal to an angle whose tangent is $\frac{1}{2}$; or, we generally say, the slope is 1 upon 2; that is, we rise, in ascending such a plane, a vertical distance of 1, in passing over a horizontal distance of 2. The slope of a curved surface, at any point, is the slope of a plane, tangent to the surface at that point.

If, through any point of a curved surface, any number of vertical planes be passed, they will cut out lines of different slopes, the slope of each being the same as the slope of the line which the same plane cuts from the tangent plane. Of all these sections, that has the greatest slope which is cut out by a plane perpendicular to the tangent plane. This may be shown, as follows:

Take any horizontal plane as a plane of reference, and through the point of contact pass any number of vertical planes: these will cut out straight lines intersecting each other at the point of contact, and meeting the horizontal trace at different points. Of all these lines, that one will make the greatest angle with the horizontal plane, which is shortest; for, in a right-angled-triangle with a given perpendicular, the angle at the base increases when the hypothenuse diminishes, and the reverse. But the shortest of these lines is that which is perpendicular to the horizontal trace; but the vertical plane which is perpendicular to the horizontal trace of a plane, is perpendicular to the plane itself; hence we conclude, that the line of greatest slope, cut out of a surface by any number of vertical planes through a point on the surface, is that one cut out by a plane perpendicular to the tangent plane at the point.

SOL'ID. [L. *solidus*, solid]. In Geometry, a magnitude which possesses the attributes of *length*, *breadth* and *thickness*. The term volume would be preferable to solid, as the latter conveys the idea of matter; whereas geometry is only conversant with volumes, or spaces, irrespective of what those spaces may be filled with, or whether they be entirely void.

SO-LID'I-TY. The number of times th t a volume or solid contains another volume or solid, taken as a unit of measure; or, it is the ratio of the unit of volume to the given volume. In general, a volume varies as its length into its breadth into its thickness. If the three dimensions increase proportionally, the solidity varies as the cube of either dimension. If one dimension remains constant, the volume varies as the product of the other two. If two dimensions remain constant, the volume varies as the other dimension. These remarks are perfectly general, and require modification in particular cases; but they serve to give a general view of the nature of a volume. See *Mensuration.*

SOL-STĬ'TIAL POINTS. See *Projections Spherical.*

SO-LŪ'TION. [L. *solutio*; from *solvo*, to loosen]. The solution of an equation, in Analysis, is the operation of finding such values for the unknown quantities that enter it, as will satisfy the equation; that is, when substituted for the unknown quantities, will make the two members equal to each other. See *Equation.*

Solution of a Problem. The operation of finding such values for the unknown parts, as will satisfy the conditions of the problem. Problems may be solved algebraically or geometrically. See *Determinate Geometry*, *Analytical Geometry*, *&c.*

SOUND'ING. A measured depth of water, ascertained by means of a lead and line. Soundings are made in Maritime Surveying, for the purpose of ascertaining the depth of water, the nature of the bottom, the channels, bars, sunken reefs, &c., for the benefit of navigation or some other purpose. The operation of sounding is conducted as follows:

A suitable boat and crew being provided, the surveyor directs the crew to row backwards and forwards in different directions between established signals, and at suitable intervals, he heaves the lead and notes the depth of water by marks attached to the line; these are entered in the note-book of the survey, together with the time or other memoranda, that aid him in fixing the position of the point at which the sounding is made. See *Maritime Surveying.*

In shallow water, soundings may be made by a pole graduated to feet or other suitable units of measure. Before a sounding is used in plotting the contour of the bottom of a harbor, it must be corrected for the height of the tide, which is ascertained by a record of a tide-gauge, which is noted and recorded every 15 or 30 minutes during the time in which sounding operations are being carried on. The surveyor should, in connection with the soundings, keep a record of the direction and strength of the wind, and the nature of the bottom. The former serves to show the relation between different tides, as affected by winds; and the latter, when laid down on the chart, serve as guides to the mariner, in selecting suitable places for anchorage.

Sounding Line. A long line having a heavy piece of lead attached to one extremity, used in sounding. It is usually divided in spaces of 1 fathom in length, the points of division being marked so that the person who heaves the lead can easily determine the depth to which it sinks before reaching bottom. The lead attached to the line is of a conical shape, 5 or 6 inches in length, and hollowed out at the bottom, so as to afford space for inserting some grease, the object of which is to bring up specimens of the bottom.

SOUTH. [Fr. *sud*]. One of the cardinal points of the compass.

SOUTH'ING. When, in Surveying, the second extremity of a course is further south than the first extremity, the course is said to make southing. This is indicated by the meridional letter which is prefixed to the bearing. Thus, a course which is recorded S. 4° E., is said to make southing. The amount of the southing made depends upon the length of the course and upon the bearing. It is equal to the length of the course into the cosine of the bearing. For ordinary purposes its value is taken from a traverse table. The southing is equal to the distance between two east and west lines, one drawn through each extremity of the course. In navigation the term southing has a similar meaning. See *Navigation.*

SPĀCE. [Fr. *espace*; L. *spatium*, from *spatior*, to wander]. That which extends to an infinite distance in all directions, and contains all bodies. Nothing but a negative definition can be given of space, as it is im

material, and destitute of tangible attributes. Although we may not be able to conceive a clear idea of space in general, we may nevertheless form a correct notion of limited portions of space; it is upon such portions that all reasonings of Geometry are based. A limited portion of space, extending in all directions, is called a volume, or sometimes a figure. That which divides this limited portion from the surrounding, indefinite space, is a surface. We may then regard a surface as a portion of space possessing the two attributes of length and breadth. If we take a portion of a surface we may regard the boundary of that portion as a part of space; hence, we consider a line as a portion of space, possessing but the single attribute of length. Volumes, lines, and surfaces are limited portions of space; taken together, they form the subject of all geometrical reasoning.

SPĒ'CIĒS OF LINES. [L. *species*, sort; from *specio*, to see]. A subdivision of an order of lines. Thus, in analysis, all lines are first grouped in two classes, *algebraic* and *transcendental*; these classes are variously subdivided.

Algebraic Lines are classed in orders, according to the degree of their equation. Those whose equations are of the first degree are of the first order, those whose equations are of the second degree, are of the second order, &c. The orders are further divided into *species*, according to some analytical property. For example, lines of the second order, whose equations may always be reduced to the form

$$ay^2 + bxy + cx^2 + dx + ey + f = 0,$$

are subdivided into three species, according as $b^2 - 4ac$ is less than 0, greater than 0, or equal to 0.

When $b^2 - 4ac = 0$, the species is named *parabola*; when $b^2 - 4ac < 0$, it is named *ellipse*; when $b^2 - 4ac > 0$, it is named *hyperbola*. Species of lines are again subdivided into varieties, containing many individuals.

Species of Surface. A subdivision of an order of surface. Surfaces are classed, like lines, into *algebraic* and *transcendental*. The algebraic surfaces are divided into *orders*, according to the degree of their equations; the orders are subdivided into *species*; these are again subdivided into *varieties*, &c. For example, surfaces of the second order are divided into three species, depending upon the nature of their plane sections; these species are respectively named *ellipsoids*, *paraboloids* and *hyperboloids*; each of the species has several varieties; thus, the varieties of the ellipsoid are elliptical ellipsoids, ellipsoids of revolution, including the sphere, the point, and the imaginary surface. The varieties of the paraboloids are *elliptical paraboloids*, *parabolic paraboloids*, and *hyperbolic paraboloids*. The varieties of hyperboloids are elliptical hyperboloids, or hyperboloids of two nappes, and hyperbolic hyperboloids, or hyperboloids of one nappe.

SPHĒRE. [L. *sphæra*. Gr. *σφαιρα*]. A solid or volume bounded by a surface, every point of which is equally distant from a point within, called the *centre*. Or it is a volume that may be generated by revolving a semicircle about its diameter as an axis. The distance from any point of the surface to the centre is called a radius of the sphere.

Every section of a sphere made by a plane is a circle, and all sections made by planes equally distant from the centre, are equal. A circle of the sphere whose plane passes through the centre, is a *great circle*; all other circles are *small circles*. All great circles are equal, and their radii are equal to the radii of the sphere.

If a regular semi-polygon, of any number of sides, be inscribed in a semicircle, and a similar semi-polygon be circumscribed about the semicircle, and then the whole be revolved about their common diameter, the surfaces generated by the perimeters of the polygons may be made to differ from each other and from the given sphere by less than any assignable surface, and their volumes will thus differ from each other, and from that of the sphere by less than any assignable volume. The surface and volume of the sphere may then be regarded as the limits of the surface and volume generated by revolving a regular semi-polygon about its diameter.

The surface of a sphere is equal to the product of the diameter by the circumference of a great circle; or it is equivalent to the area of four great circles. Denoting the radius of the sphere by r, and its diameter by d, we have the following formulas for the surface.

$$s = 4\pi r^2 = \pi d^2.$$

The surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters; or in general, as the squares of any two homologous lines that can be drawn in the sphere.

If a right cylinder be circumscribed about a sphere, the area of the convex surface of the cylinder is equal to the area of the surface of the sphere.

The volume of a sphere is equal to the product of its surface by one-third of its radius. It is also equivalent to two-thirds of the volume of its circumscribing cylinder. The following formula gives the value of the volume of any sphere, whose radius is r, and diameter is d,

$$v = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3.$$

The volumes of any two spheres are to each other as the cubes of their radii, or as the cubes of their diameters; and in general, as the cubes of any two homologous lines that may be drawn in the spheres.

In analysis, the sphere is a surface of the second order, and its equation may always be reduced to the form,

$$x^2 + y^2 + z^2 = r^2,$$

in which the origin of co-ordinates is taken at the centre. If we denote the co-ordinates of any arbitrary point by α, β and γ, and place the centre of the sphere at that point, its equation will become

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2.$$

The quantities, α, β, γ and r, are all arbitrary, and by means of them we can make the sphere pass through any four points arbitrarily assumed.

Oblique Sphere in Spherical Projections. The case in which the projection is made upon the plane of the horizon of any place not on the equator, or at the poles.

Parallel Sphere. The case in which the projection is made upon the plane of the equator, or upon the horizon of the poles.

Right Sphere. The case in which the projection is made upon the plane of a meridian, or upon the horizon of a place on the equator.

SPHER'IC-AL. Relating to the sphere, as spherical angle, spherical triangle, spherical polygon, &c.

Spherical Angle. An angle included between the arcs of two great circles intersecting each other on the surface of a sphere. The measure of a spherical angle is the same as that of the angle included between two tangents, one drawn to each arc at their point of intersection. This angle is the same as that included between the planes of the two arcs. If the radius of the sphere is taken as 1, the angle is also measured by the arc of a great circle intercepted between the two arcs, and which is perpendicular to them both.

Two great circles intersecting form four spherical angles at the same point, whose sum is 360°, and which are equal two and two. The sum of any two adjacent ones is equal to 180°.

Spherical Co-ordinates. The same as trigonometrical co-ordinates. See *Trigonometrical Co-ordinates*.

Spherical Excess. The excess of the sum of the three angles of a spherical triangle over 180°. It is of practical use in a trigonometrical survey conducted on a large scale. See *Geodesy*.

Spherical Lune. A portion of the surface of a sphere included between two great semicircles, having a common diameter. The angle of the lune is the same as the angle of the planes of the circles. See *Lune*.

Spherical Polygon. A portion of the surface of a sphere bounded by the arcs of three or more great circles. Like plane polygons they are named from the number of sides or angles. See *Polygon*.

Spherical Projections. The projections of the circles of a sphere upon a plane. See *Projection*.

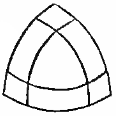
Spherical Pyramid. A portion of a sphere bounded by a spherical polygon, and by three or more sectors of great circles meeting at the centre of the sphere. The centre of the sphere is called the *vertex*, and the polygon is called the *base*. See *Pyramid*.

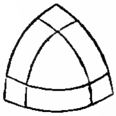
Spherical Sector. A portion of a sphere which may be generated by revolving a sector of a circle about a straight line through its vertex as an axis. See *Sector*.

Spherical Segment. A portion of a

sphere included between a zone of the surface and a secant plane, or between two parallel secant planes. See *Segment*.

Spherical Triangle. A spherical polygon of three sides. It is a portion of the surface of a sphere bounded by the arcs of three great circles. The points where the arcs meet are called *vertices* of the triangle, and the arcs are called sides. If from the three *vertices* of a spherical triangle, as poles, the arcs of three great circles be described, forming a second spherical triangle, this is the polar *triangle* of the given triangle: and conversely, the first triangle is the polar triangle of the second. By following the directions above given, for constructing the polar triangle, four triangles are found: the central one, is the one referred to.





The polar triangles bear such relation to each other, that any angle of either is measured by a semicircumference minus the side of the other triangle lying opposite to the angle. For a further account of spherical triangles, see *Triangles*.

Spherical Trigonometry. That branch of Trigonometry which explains the method of solving spherical triangles, when three of the parts are given; it also treats of the general relations existing between the six parts of which the triangle is composed. A spherical triangle is composed of six parts, three sides, and three angles, any three of which being given, the triangle can be solved; that is, the remaining parts may be found. The only exception to this rule, is in the case of a birectangular triangle. In that case, there are an infinite number of solutions, when the two right angles and a side opposite one of them are given. For the method of solving all cases of spherical triangles, see *Trigonometry*.

Spherical Ungula. A portion of the sphere bounded by a lune and two semicircles meeting in a diameter of the sphere. The lune forms the *base* of the ungula. The angle of the ungula is the same as the angle of the lune. The volume of the ungula is equal to the area of the lune multiplied by one-third of the radius of the sphere.

Spherical Zone. A portion of the surface of a sphere included between two parallel planes. The circles in which these planes intersect the surface of the sphere are called *bases*, and the distance between the planes is called the altitude of the zone. It may happen that one of the planes is tangent to the surface; in that case, the corresponding base becomes a point. If, at the same time, the other plane becomes tangent to the surface, the zone embraces the entire surface of the sphere. The area of a spherical zone is equal to that of a great circle multiplied by the altitude of the zone. Hence, all zones of the same sphere, having the same altitude, are equivalent in area. As a surface of revolution a zone may be generated by revolving any arc of a circle about a diameter of the circle as an axis.

SPHĒ'ROID. [Gr. σφαιρα and ειδος, form]. A solid resembling a sphere in form, and which may be generated by revolving an ellipse about one of its axes. If an ellipse be revolved about its transverse axis, the spheroid generated is called a *prolate spheroid*; if it be revolved about its conjugate axis, the spheroid generated is called an *oblate spheroid*. If we denote the semi-transverse and semi-conjugate axes of the ellipse by a and b, and the volume by V, we shall have for the prolate spheroid,

$$V = \frac{4}{3}\pi b^2 a;$$

and for the oblate spheroid,

$$V = \frac{4}{3}\pi a^2 b;$$

in either case the volume is equivalent to two-thirds of that of the circumscribing cylinder.

If the same ellipse be, in turn, revolved about its two axes, the prolate spheroid is to the oblate spheroid generated, as b is to a.

SPHE-ROID'AL. Appertaining to a spheroid. A *spheroidal angle* is an angle included between two plane sections of the surface made by two normal planes to the surface at the same point. This point is the vertex of the angle, and the measure of the angle is the same as the measure of the angle of the planes.

Spheroidal Excess is the excess of the

sum of the three angles of a spheroidal triangle over 180°.

SPHEROIDAL POLYGON, SECTOR, SEGMENT, &c. Terms analogous to the corresponding ones referred to the sphere, and having corresponding significations.

SPHEROIDAL TRIANGLE. A triangle on the surface of a spheroid, analogous to a spherical triangle. Such a triangle is formed by drawing geodesic lines on the surface of the earth, so as to connect three trigonometrical points of the surface. In most cases of practical geodesy, no error will arise from considering a spheroidal triangle as coinciding with a spherical triangle taken upon the surface of a sphere whose radius is equal to the radius of curvature of the meridian at the mean latitude of the three stations.

SPIN'DLE. A solid generated by revolving a portion of a curve about a chord perpendicular to an axis of the curve. The spindle takes its name from the curve which is revolved, as the *hyperbolic*, the *parabolic*, the *elliptic*, &c., spindles.

SPĪ'RAL. [L. *spira*, a spire]. A curve which may be generated by a point moving along a straight line, in the same direction, according to any law, whilst the straight line revolves uniformly about a fixed point, always continuing in the same plane. The portion generated during one revolution is called a spire. The moving point is the *generatrix* of the curve, the fixed point is the pole of the spiral, and the distance from the pole to any position of the generatrix is the *radius vector* of that point. The law according to which the generatrix moves along the revolving line, is the *law of the spiral*, and determines the nature of the curve. Any position of the revolving line, assumed at pleasure, is called the initial line; the angle through which the revolving line moves is estimated from this line, and is taken positively in a direction contrary to the motion of the hands of a watch. If we denote the angle swept over by the revolving line, from the initial lines, by t, and the corresponding length of the radius vector by u, the general relation

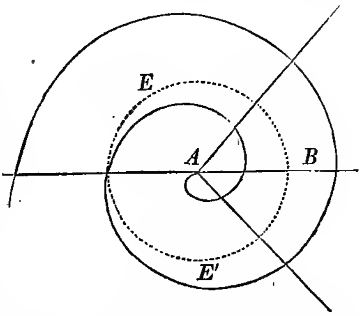
$$u = f(t),$$

will be the equation of the spiral. A great variety of spirals have been investigated, some of the most important of which will be noticed.

1. *The Spiral of Archimedes.* The law of this spiral is, that the generatrix moves uniformly along the revolving line. The equation of this spiral is

$$u = at,$$

in which u is any arbitrary constant. In the spiral of Archimedes, the radii vectors are proportional to the entire angle swept over by the revolving line from the initial line, the initial line being taken so that $u = 0$ when $t = 0$. This property of the curve enables us to construct it by points. If in the equation we make $t = 2\pi$, we find $u = 2a\pi$. If from any point A as a centre, and with a radius AB, equal to $2\pi a$, we describe a circle



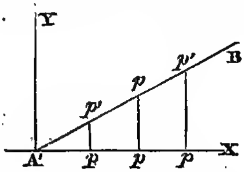
BEE', and divide its circumference into any number of equal parts, as n, by radii, and then lay off on these radii, from A, distances respectively equal to

$$\frac{1}{n}, \quad \frac{2}{n}, \quad \frac{3}{n}, \quad \frac{4}{n}, \ \&c.$$

of AB, the curve drawn through the points thus constructed is a spiral of Archimedes. The accuracy of the construction will be increased by taking a greater number of radii. If the angle between any two radii vectors be bisected, the corresponding radius vector is an arithmetical mean between the then radii vectors. Following this principle, the construction may be continued beyond the first circle drawn, by the method of interpolation of radii.

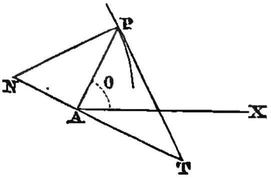
This spiral may also be constructed by means of an auxiliary straight line, as follows: Let A'B be a straight line whose equation, referred to the rectangular axes,

A′X and A′Y, is $y = ax$. With A as a centre, (last figure), and with a radius AB equal



to $2\pi a$, describe a circle. Assume any abscissa, as $A'p$, and lay it off on the circumference of the circle from B around to the left, and from its extremity draw a radius; from A lay off on this radius a distance equal to the corresponding ordinate pp', the extremity of this distance is a point of the spiral. In like manner any number of points may be constructed, and a curve drawn through them. This construction shows the analogy between Archimedes' spiral and the straight line.

If PT is tangent to the curve at P, and PA be drawn through P, and AT be also drawn perpendicular to AP, then is AT the subtangent. The subtangent in Archimedes' spiral



is always equal to m times the circumference described with the radius vector of the point of contact. If PN be drawn perpendicular to PT, then is AN the subnormal, PT is the tangent, and PN the normal.

The locus of the point T, the extremity of the subtangent in this curve, is a spiral whose equation is

$$u = at^2,$$

in which t is reckoned from an initial line, making an angle of 90° with the initial line of the given spiral. This secondary spiral is a parabolic spiral of the first order, and will soon be discussed. If, in like manner, we form the subtangent of this spiral, the locus of its extremity is a spiral whose equation is

$$u = \frac{a}{2} t^3;$$

this is a parabolic spiral of the second order. In like manner, the locus of the extremity of the subtangent, in this curve, is a spiral whose equation is

$$u = \frac{a}{2.3} t^4;$$

and if we continue the process we shall find a system of parabolic spirals, whose equations are, beginning at the spiral of Archimedes,

$$u = at; \quad u = \frac{a}{1} t^2; \quad u = \frac{a}{1.2} t^3; \quad u = \frac{a}{1.2.3} t^4;$$

$$u = \frac{a}{1.2.3.4} t^5 \ldots u = \frac{a}{1.2.3 \ldots (n-1)} t^n \ldots$$

and in each, the angle t is estimated from an initial line which makes, with the preceding initial line, an angle of 90°.

If, from the pole, a perpendicular be let fall on the tangent to the curve, at a point whose radius vector is u, the equation of the spiral of Archimedes may be written,

$$p = \frac{u^2}{\sqrt{a^2 + u^2}}.$$

The arc of the spiral of Archimedes, estimated from the pole to a point whose radius vector is u, is equal to the arc of a parabola whose parameter is $2a$, estimated from the vertex to a point whose ordinate is u. The corresponding area is equal to one-half that of the area of the parabola.

2. The Parabolic Spiral. The law of this spiral is, that the distance from the pole to the generatrix, varies as the square root of the angle swept over by the revolving line. Its equation is

$$u = at^2.$$

This curve may be constructed by means of an auxiliary parabola in the same manner that we constructed Archimedes' spiral, by the aid of a straight line, using the abscissas and ordinates of the parabola,

$$y = ax^2,$$

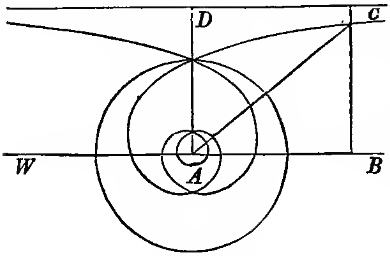
instead of those of the straight line. Parabolic spirals of the higher orders may be constructed in like manner.

3. The Hyperbolic Spiral. The law of this curve is, that the distance from the pole to the generatrix varies inversely as the distance swept over. The equation of this spiral is

$$ut = a,$$

an equation entirely analagous to that of the equilateral hyperbola referred to its centre

and asymptotes. The curve may be constructed by means of an auxiliary equilateral hyperbola, in the same manner as we have heretofore explained. Let A be the pole, AB the initial line, AD a perpendicular to AB.



Lay off on AD, a distance AD $= a$, and through D draw DC parallel to AB; then is DC an asymptote to the spiral. The curve has its origin at an infinite distance, and approaches the pole as t is increased, but only reaches it when t becomes equal to ∞; that is, after an infinite number of turns. If we had considered the line AW as the initial line, and reckoned angles positive in the same direction as the motion of the hands of a watch, we should have found a second spiral entirely symmetrical with the first, having DC′ for its asymptote.

If, from A as a centre, and with any radius whatever, an arc of a circle be described, the part intercepted between the initial line and the curve is constant, and equal to AD. This principle enables us to construct the curve by points.

If we denote the distance from the pole to the tangent by p, as before, we have the equation

$$p = \frac{au}{\sqrt{a^2 + u^2}}.$$

4. Cotes' Spirals. If we take the equation,

$$p = \frac{bu}{\sqrt{a^2 + u^2}},$$

in which p is the same as before, the spirals represented by it will be of three different classes, according as $a = b$, $a < b$, or $a > b$.

If $a = b$, we have the case of the logarithmic spiral. See *Logarithmic Spiral.*

If $a < b$, the polar equation of the curve takes the form,

$$t = \frac{b}{c} \log \left\{ \frac{\sqrt{a^2 + c^2} - c}{u} \right\},$$

in which

$$c = \sqrt{a^2 - b^2}.$$

In this case, if

$$u = 0,\ t = \infty;\ \text{if } t = 0,\ u = \infty.$$

If $a > b$, the polar equation of the curve becomes

$$t = \frac{b}{c} \sec^{-1} \frac{u}{c}, \quad \text{or,} \quad u = c \sec \frac{ct}{b},$$

in which

$$c = \sqrt{b^2 - a^2}.$$

These curves comprehend the simple case

$$p = \frac{bu}{a},$$

which gives for the polar equation,

$$t = \frac{b}{\sqrt{a^2 - b^2}} \cdot \log \frac{u}{a},$$

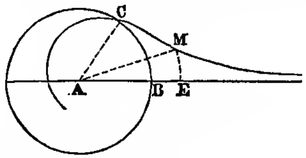
the logarithmic spiral.

These spirals are of scientific interest, from the fact, that they are the trajectories that would be described by material points projected in different directions with different velocities, and these acted upon by a central force, varying in intensity inversely, as the cube of the distance from the point to the centre of attraction.

5. Lituous Spiral. The lituous is a spiral whose equation is

$$u^2 = a^2 t^{-1}, \quad \text{or,} \quad u^2 t = a^2.$$

If, with A as a centre, and a radius AB $= a$,



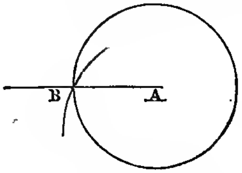
a circle be described cutting the curve in C, the angle t is estimated from the line AB, determined by making the arc CB equal to a. The radius, AB, produced, is an asymptote to the curve. Wherever the point M is taken on the curve, the sector AME is constant, and equal to the sector, ACB.

6. Logarithmic Spiral. The characteristic property of this spiral is, that the angle included between any radius vector, and the tangent at its extremity is constant: from

this circumstance it has been called the *equiangular spiral*. If we denote the tangent of the constant angle by M, then is the equation of the curve

$$t = M \log u.$$

If, from A as a centre, and with a radius



AB = 1, a circle be described. the origin of the spiral is at B, and there are an infinite number of spires both without and within the circle described. The radii vectors making equal angles with each other are in geometrical progression, and when any two are constructed, the curve may be constructed by points from this principle. Although there are an infinite number of spires within the circle before reaching the pole, the length of the arc is finite, and from the pole to a point whose radius vector is u, the length is equal to $u\sqrt{2}$.

The evolute of this curve is a logarithmic spiral, similar to the given one. The involute of it is also a similar logarithmic spiral. The caustics of reflected and refracted rays, under certain restrictions, are logarithmic spirals.

There are a great variety of other spirals, but the ones mentioned are the most important.

Every curve having an infinite branch, when referred to rectangular axes, admits of a corresponding spiral, which may be constructed by points in the same manner as was explained in treating of the spiral of Archimedes.

SPĪRE. That portion of a spiral which is generated during one revolution of the straight line revolving about the pole. Every spiral consists of an infinite number of spires. See *Spiral.*

SQUĀRE. [Fr. *carré*]. In Geometry, an equilateral and equiangular quadrilateral. Each of the angles of a square is equal to 90° or a right angle. The diagonals of a square are equal, and mutually bisect each other at right angles. The ratio of either side of a square to its diagonal is that of 1 to $\sqrt{2}$. The square is employed as a unit of measure in determining the area of surfaces, whence the term *square measure*, in that connection. The area of any square is equal to the product of two adjacent sides.

The law of relation of the different orders of units of square measure is given in the following table :

Square Miles	Acres.	Square Chains	Perches.	Square Yards.	Square Feet.
1	640	6400	102400	3097600	27878400
	1	10	160	4840	43560
		1	16	484	4356
			1	$30\frac{1}{4}$	$272\frac{1}{4}$
				1	9

SQUARE, MAGIC. See *Magic Square.*

SQUARE. In geometrical construction, an instrument for laying off a right angle. It consists of two branches or arms fastened together at right angles, or sometimes it is formed of a single piece of wood or metal, cut so that two of its adjacent edges shall be perpendicular to each other. See *Rule and Triangle.*

SQUARE OF A QUANTITY. In Algebra, the result obtained by taking that quantity twice as a factor. To square any quantity is to multiply that quantity by itself : the resulting product is the square required.

SQUARE NUMBER. In Arithmetic, a number which may be resolved into two equal factors. The following are some of the principal properties of square numbers :

1. Every square number is of the form, $4n$ or $4n + 1$; that is, every square number, when divided by 4, will leave a remainder either equal to 0 or to 1. Here, 4 is called a *modulus*. Square numbers may likewise be expressed in terms of other moduli, as 5, 6, 7, 8, &c.

Modulus.	Forms of Square Numbers.			
4	$4n$,	$4n + 1$,		
5	$5n$,	$5n \pm 1$,		
6	$6n$,	$6n + 1$,	$6n + 3$,	$6n + 4$,
7	$7n$,	$7n + 1$,	$7n + 2$,	$7n + 4$,
8	$8n$,	$8n + 1$,	$8n + 4$,	
9	$9n$,	$9n + 1$,	$9n + 4$,	$9n + 7$,
10	$10n$,	$10n \pm 1$,	$10n \pm 4$,	$10n \pm 5$.

2. The sum of two odd squares cannot be a square.

3. An odd square, taken from an even square, cannot leave a remainder which is a square.

4. If the sum of two squares is a square, one of the three must be divisible by 5, and consequently by 25.

5. Square numbers must terminate with one of the digits, 0, 1, 4, 5, 6, or 9.

6. If we take the series of squares of natural numbers, as $1^2, 2^2, 3^2, 4^2, 5^2$, &c., the mean proportional of any two squares of this series is equal to the less square plus its square root multiplied by the difference of the square roots of the two squares; thus,

$$\sqrt{11^2 \times 7^2} = 7^2 + 7 \times 4 = 28 + 49\text{; also}$$

$$\sqrt{7^2 \times 4^2} = 4^2 + 4 \times 3 = 28.$$

7. The arithmetical mean of any two squares exceeds their geometrical mean by half the square of the difference of their square roots. Thus,

$$\frac{7^2 + 4^2}{2} - \sqrt{7^2 \times 4^2} = \frac{3^2}{2} = \frac{9}{2} = 4\tfrac{1}{2}.$$

8. Of three equi-distant squares in the series, the geometrical mean of the extremes is less than the middle square by the square of the common difference of their square roots. Thus,

$$\sqrt{5^2 \times 1^2} = 3^2 - 2^2 = 5.$$

9. The difference between two consecutive squares is equal to twice the square root of the lesser increased by 1. Thus,

$$5^2 - 4^2 = 2 \times 4 + 1 = 9.$$

10. If the natural numbers be cubed, giving the series of natural cubes, the sum of any number of consecutive terms of this series, beginning at the first, is a perfect square. Thus,

$$1^3 + 2^3 = 3^2, \quad 1^3 + 2^3 + 3^3 = 6^2,$$
$$1^3 + 2^3 + 3^3 + 4^3 = 10^2, \text{ \&c.;}$$

and, in general,

$$1^3 + 2^3 + 3^3 + \cdots + n^3$$
$$= (1 + 2 + 3 + \cdots + n)^2 = \tfrac{1}{2}n(n + 1).$$

Square Root of a Quantity. A quantity which, being taken twice as a factor, will produce the given quantity. Thus, the square root of 25 is 5, because $5 \times 5 = 25$. When the square root of a number can be expressed in exact parts of 1, that number is a perfect square, and the indicated square root is said to be commensurable. All other indicated square roots are incommensurable.

To extract the square root of a whole number:

I. Separate the number into periods of two figures each, beginning at the right hand: the left hand period will often contain but one figure.

II. Find the greatest perfect square in the first period on the left, and place its root at the right, after the manner of a quotient in division. Subtract the square of this root from the first period, and to the remainder bring down the second period for a dividend.

III. Double the root already found, and place it on the left for a divisor. See how many times this is contained in the dividend, exclusive of the right hand figure, and place the quotient in the root, and also at the right of the divisor.

IV. Multiply the divisor thus obtained by the last digit of the root found; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend. If the product just found exceeds the dividend, diminish the last digit of the root found, and proceed as before till the product is less than the dividend.

V. Double the root already found for a new divisor, and proceed as before till all the periods have been brought down. If the final remainder is 0, the root found is exact, and the number is a perfect square. If it is not 0, the root is not exact, but is true to within less than 1.

To extract the square root of a vulgar fraction:

Reduce the fraction to its lowest terms; then extract the square root of the numerator for a numerator, and the square root of the denominator for a denominator: the result will be the square root required.

If, when the fraction is reduced to its lowest terms, both terms are not perfect squares, the fraction is not a perfect square, and its root can only be found by approximation.

To extract the square root of a fraction to within less than its fractional unit:

Multiply the numerator by the denominator, and extract the square root of the product to within less than 1; *divide this result by the denominator of the given fraction: the quotient will be the root required.*

Let it be required to extract the square root of $\frac{5}{8}$ to within less than $\frac{1}{8}$: We have the square root of 40 equal to 6 to within less than 1; hence, $\frac{3}{4}$ or $\frac{6}{8}$ is the approximate root.

To extract the square root of a whole number to within less than any fractional unit:

Multiply the number by the square of the denominator of the fractional unit, and extract the square root of the product to within less than 1; *divide the result by the denominator of the fractional unit: the quotient will be the approximate root.*

To extract the square root of a vulgar fraction to within less than any fractional unit:

Multiply the numerator by the quotient of the square of the denominator of the fractional unit by the denominator of the given fraction; extract the square root of the product to within less than 1, *and divide the result by the denominator of the fractional unit: the quotient will be the approximate root.*

Let it be required to extract the square root of $\frac{2}{5}$ to within less than $\frac{1}{15}$: Multiply 2 by $\frac{225}{5}$, or 45; we find, for the product, 90: the square root of 90 to within less than 1, is 9; hence, the approximate root is $\frac{9}{15} = \frac{3}{5}$.

To extract the square root of a whole number or vulgar fraction to any number of decimal places:

Convert the fraction into a decimal, and, if necessary, annex 0*'s till the whole number of decimal places is equal to twice the number required in the root. Extract the square root of this, regarded as a whole number, to within less than* 1, *and from the result point off the required number of decimal places.*

A very good approximate result may be obtained, in extracting the square root of numbers containing 10 or 12 places of figures, by following the rule, as laid down, till at least half of the number of figures of the root are found; then bring down all the remaining periods, and form a divisor according to the rule, with which continue the division till the requisite number of places of figures are found in the root.

Square Root of Algebraic Quantities.

To extract the square root of a monomial:

Extract the square root of the co-efficient for a new co-efficient, after which write each letter entering the given expression, giving to each an exponent equal to one-half of its original exponent: the result is the root required.

A monomial cannot be a perfect square, unless its co-efficient is a perfect square, and the exponents of all the letters which enter it are even numbers.

To extract the square root of a polynomial.

I. Arrange the polynomial according to one of its letters. Extract the square root of the first term of the arranged polynomial: this will be the first term of the root.

II. Divide the second term of the polynomial by twice the first term of the root: the quotient will be the second term of the root. Subtract the square of the sum of these terms from the given polynomial for the first remainder.

III. Divide the first term of this remainder by twice the first term of the root, and the quotient will be the third term of the root. From the first remainder subtract twice the sum of the first and second terms of the root plus the third term, by the third term, for a second remainder.

IV. Divide the first term of the second remainder by twice the first term of the root, and the quotient will be the fourth term of the root. Subtract from the second remainder twice the sum of the first three terms of the root plus the fourth term, by the fourth term, for a third remainder.

V. Continue in this manner, until a remainder is found equal to 0, or the first term of which is not exactly divisible by twice the first term of the root. In the former case, the root is exact; in the latter, the root cannot be extracted.

If, on inspection, it is seen that any term, which is of the highest or of the lowest degree with respect to any letter, is not a perfect square, it is useless to attempt to apply the rule; for, in this case, the polynomial is not a perfect square. A binomial can never be a perfect square. In order that a trinomial may be a perfect square, it is necessary that two of its terms be perfect squares, and that the third be equal to twice the product of their square roots. When a quantity is not a perfect square, its square root can only be indicated. See *Radicals*.

The square root of an imperfect square number may be found approximately by means of continued fractions. A single ex-

ample will best illustrate the method. Let it be required to extract the square root of 19. The operation is as follows:

$$(1).\ \sqrt{19} = 4 + \frac{\sqrt{19}-4}{1} = 4 + \frac{3}{\sqrt{19}+4}$$
$$= 4 + \frac{1}{\left(\dfrac{\sqrt{19}+4}{3}\right)}.$$

$$(2).\ \frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3} = 2 + \frac{5}{\sqrt{19}+2}$$
$$= 2 + \frac{1}{\left(\dfrac{\sqrt{19}+2}{5}\right)}.$$

$$(3).\ \frac{\sqrt{19}+2}{5} = 1 + \frac{\sqrt{19}-3}{5} = 1 + \frac{2}{\sqrt{19}+3}$$
$$= 1 + \frac{1}{\left(\dfrac{\sqrt{19}+3}{2}\right)}.$$

$$(4).\ \frac{\sqrt{19}+3}{2} = 3 + \frac{\sqrt{19}-3}{2} = 3 + \frac{5}{\sqrt{19}+3}$$
$$= 3 + \frac{1}{\left(\dfrac{\sqrt{19}+3}{5}\right)}.$$

$$(5).\ \frac{\sqrt{19}+3}{5} = 1 + \frac{\sqrt{19}-2}{5} = 1 + \frac{3}{\sqrt{19}+2}$$
$$= 1 + \frac{1}{\left(\dfrac{\sqrt{19}+2}{3}\right)}.$$

$$(6).\ \frac{\sqrt{19}+2}{3} = 2 + \frac{\sqrt{19}-4}{3} = 2 + \frac{}{\sqrt{19}+4}$$
$$= 2 + \frac{1}{\left(\dfrac{\sqrt{19}+4}{1}\right)}.$$

&c, &c. &c.

Substituting in the first equation these results,

$$\sqrt{19} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{8 + \&c.}}}}}$$

Whence we deduce the successive approximating fractions,

$4,\ 4\frac{1}{2},\ 4\frac{1}{3}.\ 4\frac{4}{11},\ 4\frac{5}{14},\ 4\frac{14}{39},\ 4\frac{117}{326}$, &c., each of which is nearer the true value of the square root than the preceding.

The principal advantage of this method lies in its applicability to the solution of indeterminate equations of the second degree.

LEAST SQUARES. In Astronomical and Physical researches it is frequently necessary to determine the values of certain elements by means of several equations which only express the relations existing between the elements approximately. These approximate equations of condition are usually derived from a series of observations, or of experiments, which are necessarily liable to certain errors. It becomes a question in what manner these equations may be combined so that the values of the required quantities may be found affected with the least errors. The theory of probabilities affords one of the best methods, and it is the one that we propose to explain. This method is that of *least squares*. It is shown in the theory of probabilities, that the probable error will be least when the sum of the squares of the errors is a minimum.

In a series of observations or experiments, let us suppose that the errors committed are denoted by e, e', e'', &c., and suppose that by means of the observations we have deduced, the equations of condition

$$\left.\begin{aligned} e &= h + ax + by + cz + \&c. \\ e' &= h' + a'x + b'y + c'z + \&c. \\ e'' &= h'' + a''x + b''y + c''z + \&c. \\ e''' &= h''' + a'''x + b'''y + c'''z + \&c. \end{aligned}\right\} (1)$$
$$\&c. \qquad \&c. \qquad \&c.$$

Let it be required to find such values of x, y, z, &c., that the errors e, e', e'', e''', &c., with reference to all the observations, shall be the least possible.

If we square both members of each equation, in group (1), and add them together, member to member, we shall have

$$e^2 + e'^2 + e''^2 + \&c. = x^2(a^2 + a'^2 + \&c.)$$
$$+ 2x\{(ah + a'h' + \&c.) + a(by + cz + \&c.)$$
$$+ a'(b'y + c'z + \&c.) + \&c.\} + \&c.,$$

an equation which may be written

$$e^2 + e'^2 + e''^2 + \&c. = u = Px^2 + 2Qx + R + \&c.$$

Now, in order that $e^2 + e'^2 + \&c$, or u, may be a minimum, it is necessary that its partial differential co-efficients, taken with respect to each variable. in succession, should be separately equal to 0. Hence,

$$\frac{du}{dx} = Px + Q = 0, \quad \text{or} \quad x(a^2 + a'^2 + \&c.)$$
$$+ ah + ah' + \&c. + a(by + cz + \&c.)$$
$$+ a'(b'y + c'z + \&c.) + \&c. = 0;$$

and similar equations for each of the other variables. Hence, we deduce the principle that in order to form an equation of condition for the minimum, with respect to one of the unknown quantities, as x for example, we have simply to multiply the second members of each of the equations of condition by the co-efficient of the unknown quantity in that equation, then take the sum of the products and place the result equal to 0. Proceed in this manner for each of the unknown quantities, and there will result as many equations as there are unknown quantities, from which the required values of the unknown quantities may be found by the ordinary rules for solving equations.

Let it be required, for example, to find, from the equations

$$\left.\begin{array}{r} 3 - x + y - 2z = 0 \\ 5 - 3x - 2y + 5z = 0 \\ 21 - 4x - y - 4z = 0 \\ 14 + x - 3y - 3z = 0 \end{array}\right\}. \qquad (2)$$

such values of x, y and z, as will most nearly satisfy all of the equations.

Following the rule, and multiplying the first member of each equation by the co-efficient of x in that equation, we get the products

$$\begin{array}{r} -3 + x - y + 2z \\ -15 + 9x + 6y - 15z \\ -84 + 16x + 4y + 16z \\ 14 + x - 3y - 3z \end{array}$$

and placing their algebraic sum equal to 0, we have

$$27x + 6y - 88 = 0 \quad . \quad . \quad . \quad . \quad (3)$$

Proceeding in like manner with respect to the unknown quantities y and z, we obtain the equations

$$6x + 15y + z - 70 = 0 \quad . \quad . \quad . \quad (4)$$
$$y + 54z - 107 = 0 \quad . \quad . \quad . \quad . \quad (5)$$

Combining equations (3), (4), and (5), we find

$$x = 2.4702, \; y = 3.5507, \text{ and } z = 1.9157,$$

which most nearly satisfy all of the equations in group (2).

To show the practical application of this principle, we will suppose that it is required to investigate the values of the constants in an equation, by means of several independent experiments.

It is demonstrated from theory that the length of the pendulum which beats seconds, in any latitude, is given by the formula

$$L = x + y \sin^2 l \quad . \quad . \quad . \quad . \quad (6)$$

in which L denotes the length of the pendulum; l the latitude of the place on the surface of the earth, and x and y are constants to be determined. In consequence of errors incident to observation, the values of x and y cannot be accurately determined by means of a single observation; taking the *metre*, (denoted by m), as the unit of measure, suppose that the length of the seconds pendulum has been measured in 6 different places, whose latitudes are known, and that the following equations have been deduced:

$$\left.\begin{array}{l} e' = x + y \times 0^m.3903417 - 0^m.9929750 \\ e'' = x + y \times 0^m.4972122 - 0^m.9934620 \\ e''' = x + y \times 0^m.5667721 - 0^m.9938784 \\ e^{iv} = x + y \times 0^m.4932370 - 0^m.9934740 \\ e^{v} = x + y \times 0^m.5136117 - 0^m.9935967 \\ e^{vi} = x + y \times 0^m.6045628 - 0^m.9940932 \end{array}\right\} (7)$$

Applying the rule already deduced to these equations, we find the equations,

$$6x + y \times 3^m.0657375 - 5^m.9614793 = 0 \; . \; (8),$$
$$x \times 3^m.0657375 + y \times 1^m.5933894$$
$$-3^m.0461977 = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (9)$$

Combining equations (8) and (9), we obtain

$$x = 0^m.9908755, \quad y = 0^m.0052942.$$

Substituting these in equation (6), it becomes

$$L = 0^m.9908755 + 0^m.0052942 \sin^2 l \; . \; (10).$$

By means of formula (10), the length of the second pendulum may be found, by computation, at any place whose latitude is known. In like manner, the method of least squares may be applied to a multitude of similar cases. This principle is one of the most useful that has been furnished by Mathematics for the advancement of the physical sciences.

STĂFF. An instrument employed in Surveying, consisting of a rod of wood taking different forms, according to the use to be made of it.

CHAINMAN'S STAFF. A staff of 6 feet in length, and an inch and a-half in diameter, carried by the chainmen, for the purpose of

alligning the chain between two stations. It is shod with a pointed piece of iron, that it may be planted firmly in the ground, and near the bottom there is a cross-piece of iron about 4 or 5 inches in length. The object of the cross-piece is, that the ring of the chain which is put over the staff may be prevented from slipping off.

Flagman's Staff. A staff used for marking the positions of stations. It is generally from 8 to 12 feet in height, and an inch and a-half in diameter. At one end, it is shod with a sharp piece of iron, and a flag is attached to the other end, which may be red, or red and white. It is well to have the flag-staff painted of different colors, as black and white, in alternate checks, so that it may be more readily visible in a brush-wood country.

Jacob Staff. A staff sometimes used, instead of a tripod, to support the compass. It is about 4 feet in length, and an inch and a-half or 2 inches in diameter. It is shod with iron at its lower extremity, that it may be easily and firmly planted in the ground; and at its upper extremity, it is prepared so as to fit accurately into a socket on the under side of the compass.

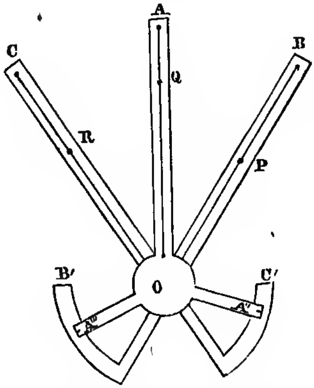
Offset Staff. A staff carried by the surveyor, for the purpose of measuring offsets. It is generally 10 links in length, and divided into 10 equal parts, which are numbered for convenience of reading. See *Offset*.

STA'TION. [L. *statio*; from *sto*, to stand]. Points selected in surveying, from which observations are made with an instrument. In ordinary field surveying, the stations are generally taken at the angular points of the field to be surveyed, and are temporarily marked by flag staves. They may, however, be taken at any prominent part of the field, and then connected with the angular points of the field by suitable observations and measurements.

In surveying, for the purpose of filling in a map of a portion of country, the stations are generally selected at the bends of roads and streams, at the corners of fields and other prominent points, and are temporarily marked by flag staves. In Geodesic surveying, the principal stations are selected with great care, in accordance with the principles laid down under the head of Geodesy, and are marked by permanent signals. See *Signals*. The secondary stations are also selected with the same care, and are marked by signals of an inferior order.

In leveling, the stations selected for the instrument are generally on the line of proposed section, but no particular rule can be laid down, except, that they should be so chosen that the leveling rods may be distinctly seen from them. It should, in general, be an object to place the instrument as nearly midway between the two leveling rods as may be. See *Leveling*.

Station Pointer. An instrument used in plotting the place of an observation made upon three fixed points; that is, for plotting *the problem of three points*.



It consists of three arms, A, B, C, turning about a common axis O, through the centre of which a small opening is made. To the arm A, two branches, A′ and A″, are attached, having a vernier marked upon each. The arm B is attached to a graduated arm B′, which moves with it; the arm C is attached to a graduated arc C′, which also moves with it. The graduation of these arcs is such, that when the instrument is shut up, the verniers both stand at the 0 of the arcs. Through the central part of each arm a slit is cut, along which a fine hair is stretched, whose line of direction passes through the point O. To use the instrument: open the arm B till the reading at the 0 of the vernier A‴ is equal to that of the angle subtended by the central and right hand object at the place of observation. Then open the arm C till the reading

at the 0 of the vernier A″ is equal to that of the angle subtended by the central and left hand object at the point of observation. Then lay the instrument on the paper and move it about, without disturbing the angles, till the central wires pass through the plots of the three points, supposed P, Q and R; then with a needle point mark the point O on the plot, and this will be the required plot of the point.

STE-RE-O-GRAPH'IC PROJECTION. See *Projection*, and *Spherical Projections.*

STE-RE-OM'ETRY. [Gr. στερεος, firm, and μετρεω, to measure]. The art of measuring solid bodies, and determining their solid contents. See *Mensuration.*

STE-RE-OT'O-MY. [Gr. στερεος, firm, and τεμνω, to cut]. The art of cutting solid bodies into specified forms. This art is particularly employed by stone-cutters, in forming stones of suitable shapes for building arches, abutments, piers, &c. The principles of Descriptive Geometry are used in guiding the stereotomist; in fact, this is one of the most useful, as well as most extensive branches of the application of that part of Mathematics.

STRAIGHT LINE. [L. *strictus*; from *stringo*]. In Geometry, a line which does not change direction between any two of its points. When a straight line is designated, its direction alone is intended to be pointed out; it is supposed to extend indefinitely in that direction. In Elementary Geometry, however, only limited portions of straight lines are considered, and these are given by designating their extreme points. Thus, we say the straight line AB, A and B being the two extreme points of the portion considered. It is an axiom of Geometry, that a straight line marks the direction of the shortest distance between two given points. It is also an axiom, that two straight lines cannot include a space or area.

In analysis, a straight line is given by its equation, and is supposed to be infinite in length. The general equation of a straight line is

$$ax + by + c = 0.$$

Any equation of the first degree between two variables, is necessarily the equation of a straight line. If such an equation be solved with reference to either variable, the co-efficient of the other variable will represent the ratio of the sines of the angles which the line makes with the axes, and the absolute term will denote the distance from the origin of co-ordinates to the point in which the line cuts the axis of the variable with reference to which the equation is solved. When the axes are at right angles, as they are usually taken, the ratio of the sines of the angles which the line makes with the axes, becomes the tangent of the angle which it makes with the axis of the variable with reference to which the equation is not solved.

The most ordinary form of the equation of a straight line is

$$y = ax + b,$$

in which a is the tangent of the angle which the line makes with the axis of X, and b the distance from the origin to the point in which it cuts the axis of Y. The equation of a straight line passing through a point whose co-ordinates are y'' and x'', is

$$y - y'' = a(x - x''),$$

a being the same as before, and perfectly arbitrary. This should be so, for an infinite number of lines can be drawn through any given point. The equation of a straight line passing through two points whose co-ordinates are x'', y'' and x', y', is

$$y - y'' = \frac{y'' - y'}{x'' - x'}(x - x'')$$

or $$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

Only one straight line can be drawn through two different points, as these two conditions determine the values of the two constants in the equation.

STURM'S THEOREM. A theorem in Algebra, first demonstrated by Sturm, which has for its object to explain the method of determining the number and places of the real roots of an equation, involving but one unknown quantity.

Let $$X = 0 \ldots\ldots (1),$$

represent an equation containing the single unknown quantity x: X being an entire polynomial of the n^{th} degree with respect to x, and the co-efficients of the different powers

of x being all real. If this equation has any equal roots, they may be found, and the equation freed of them by the method of equal roots. We may therefore regard the equation $X = 0$, as having no equal roots.

Let us denote the first derived polynomial of X by X_1, and then apply to X and X_1 a process similar to that for finding their greatest common divisor, differing, however, in this respect, that instead of using the successive remainders, as at first obtained, we change their signs, and take care also in preparing for the division, neither to introduce or reject any factor, except a positive one.

If we denote the several remainders, after their signs have been changed, by

$$X_2, X_3, X_4, \&c. \ldots . X_r$$

and the corresponding quotients by

$$Q_1, Q_2, Q_3, \ldots . Q_{r-1},$$

we shall have the following relations:

$$X = X_1Q_1 - X_2 \cdots\cdots\cdots\cdots (2).$$

$$\left.\begin{array}{l} X_1 = X_2Q_2 - X_3 \\ \cdots\cdots \\ X_{n-1} = X_n\, Q_n - X_{n+1} \\ \cdots\cdots\cdots \\ X_{r-2} = X_{r-1}\, Q_{r-1} - X_r \end{array}\right\} \cdots (3).$$

Since by hypothesis $X = 0$ has no equal roots, no common divisor can exist between X and X_1; hence the last remainder $-X_r$, will be different from 0, and will not contain x.

Now suppose that the number p has been substituted for x in the expression X, X_1, X_2, X_3, &c., and that the signs of the results, together with the sign of X^r, have been arranged in a line in their order; also that another member, q, greater than p, has been substituted for x, and the signs of the results arranged in like manner.

Then will the number of variations of the signs found in passing along the first line, diminished by the number of variations found in passing along the second line, be exactly equal to the number of real roots included between p *and* q; the latter inclusive, should it be a root.

The demonstration of this Theorem depends upon the following lemmas, which may be easily proved.

1. *If any number* p *be substituted for* x *in the expression* X, X_1, X_2, *&c., it is impossible that any two consecutive ones can become* 0 *at the same time.*

2. *If any one of these expressions reduces to* 0, *by the substitution of a particular value for* x, *the preceding and following ones will have contrary signs for the same value.*

3. *If any number, insensibly less than one of the real roots of the equation* X = 0, *be substituted for* x *in* X *and* X_1, *the results will have contrary signs, and if a number insensibly greater than this root be substituted, the results will have the same sign.*

4. *From the nature of the expression,* X, X_1, X_2, *&c., it follows that no one of them can change its sign from plus to minus or from minus to plus,* x *being regarded as variable, without first becoming equal to* 0.

The demonstration of the Theorem proceeds as follows:

Let any number, k, less, that is, nearer $-\infty$ than any of the real roots of the several equations

$$X = 0 \quad X_1 = 0 \ldots X_{r-1} = 0 \ldots (4)$$

be substituted for x in the expressions X, X_1, X_2, &c., and the signs of the several results arranged in order; then let the value of x be increased by insensible degrees, or continuously, till it becomes equal to h, the least of all the roots of equations (4). As there is no root of any of the equations between k and h, none of the signs can change, (Lemma 4), whilst x is less than h, and the number of variations of signs in the several sets of results will remain the same as in the result first obtained.

When x becomes equal to h, one or more of the expressions X, X_1, X_2, &c., will reduce to 0: suppose X_n to become 0 for this value; then (lemmas 1 and 2), since both X_{n-1} and X_{n+1} cannot become 0 for the same value, but must have contrary signs in passfrom one to the other (omitting $X_n = 0$), there will be *one*, and *only one* variation; and since their signs have not changed, one must be *the same as*, and the other *contrary to*, that of X_n, both before and after it becomes 0; hence, in passing over the three, either just before or just after X_n becomes 0, there is *one*, and *only one* variation. Hence, where the value of x passes a root of any of equations (4) except the first, there will neither be a gain nor a loss in the number of variations of signs.

If $x = a$ reduces X to 0, then h is the least real root of the proposed equation. Now (lemma 3) just before x becomes equal to X and X_1 have contrary signs, and there will be a variation in passing over them, just after x becomes equal to h, they will have the same sign, and there will be no variation in passing them: hence, when the value of x passes a real root of the given equation, there is always a loss of one variation of sign, and since the same thing happens in passing each real root of the given equation, it follows that there will be as many variations of sign lost, whilst x varies up to q, as there are real roots, and no more; hence, the truth of the proposition ennunciated.

A consideration of the principles involved in the preceding demonstration, will show that the loss of variation takes place in passing to the root from the preceding value, and not in passing from the root to the succeeding value; hence, if p is a real root of the given equation it will not be included in the number given by the rule, but if q is a root of it, it will be included in the number given by the rule.

If in the application of the preceding principles it is found that any one of the expressions X_1, X_2, X_2, &c, has the same sign for all values of x between p and q, it will be unnecessary to use the succeeding expressions, or even to deduce them. For, if X_n preserves the same sign for all values of x between p and q, it is plain (lemmas 1 and 2) that no variation of signs can be lost amongst the expressions following X_n, between the given values of x. Whenever, therefore, in the division a remainder is found, which when placed equal to 0, gives an equation having only imaginary roots, it will be useless to obtain any of the succeeding remainders.

If now we suppose $p = -\infty$, and $q = +\infty$, the application of the rule will make known the whole number of real roots, and by taking this number from that indicating the degree of the equation, we shall have the number of imaginary roots.

Having obtained the number of real roots, we may ascertain their places by substituting for x, in succession, the values 0, 1, 2, 3, &c., until a number is found that will give the same number of variations as $+\infty$; this will be the smallest superior limit of the positive roots in whole numbers. Then substitute -1, -2, -3, &c., until a number is found that will give the same number of variations as $-\infty$; this will be *numerically* the smallest superior limit of the negative roots in whole numbers.

Now, by commencing with this limit, and observing the number of variations lost from each number to the next in order, we shall discover how many roots are included between each two consecutive numbers, and thus of course, know the entire part of each root.

The decimal part is then found by other methods.

Example: Take the equation

$$8x^3 - 6x - 1 = 0,$$

Hence, $\quad X = 8x^3 - 6x - 1,$

and $\quad X_1 = 24x^2 - 6,$

or omitting the factor $+6$, $X_1 = 4x - 1$.

Applying the rule above deduced to these expressions, we have, by collecting the results,

$$X = 8x^3 - 6x - 1$$
$$X_1 = 4x^2 - 1$$
$$X_2 = 4x + 1$$
$$X_3 = +3.$$

Substituting for x, $-\infty$, we have, after arranging the signs of the results in a line,

$-, +, -, + \cdots\cdots$ three variations.

Substituting for x, $+\infty$, we have

$+, +, +, +$, no variations.

Hence, the equation has three real roots. To find their places,

For	$x = 1$,	$+ + + +$	0 variations.
"	$x = 0$,	$- - + +$	1 variation.
"	$x = -1$,	$- + - +$	3 variations.

Hence, $+1$ is the superior limit of positive roots, -1 the superior limit of the negative roots (*numerically*). One of the roots lies between 0 and $+1$, and two of them between -1 and 0. In the same manner we may proceed with other equations.

This principle enables us to deduce the relations that must exist in order that the roots of a cubic equation may all be real.

Suppose the cubic equation reduced to the form

$$x^3 + px + q = 0.$$

We shall have

$$X = x^3 + px + q.$$
$$X_1 = 3x^2 + p.$$
$$X_2 = -2px - 3q.$$
$$X_3 = -4p^3 - 27q^2.$$

Now, in order that all the roots may be real, there must be three variations of signs lost in passing from $-\infty$ to plus ∞; this can only happen when, for $x=-\infty$, the expressions are, alternately, plus and minus; and when, for $x=+\infty$, they are all plus. For, $x=-\infty$, X is negative, and X_1 positive; in order that X_2 may be negative, p must be negative; and in order that X_3 may be positive, $4p^3 > 27q^2$. If p is essentially negative, all the expressions will be positive for $x=+\infty$.

Hence, the two conditions which render the roots of the cubic equations all real, are

$$p<0, \quad \text{and} \quad \frac{p^3}{27} > \frac{q^2}{4}.$$

STȲLE. [L. *stylus*; Gr. *στυλος*; a pier; a column]. In Dialing, the line whose shadow determines the hour. The gnomon. See *Gnomon*, and *Dialing*.

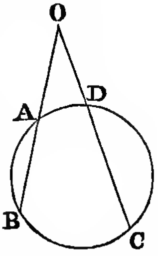
SUB-CON′TRA-RY. [L. *sub* and *contrarius*, from *contra*, against]. In a contrary order.

Sub-contrary Section. In any surface of the second order, if two planes be passed perpendicular to the same principal plane, but not parallel to each other, and so that the sections are similar, both the planes and the sections are *sub-contrary* with respect to each other.

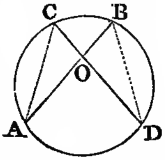
In general, every section of a surface of the second order has a sub-contrary section. Every plane passed parallel to that of a sub-contrary section, also cuts a sub-contrary section; hence, every section has an infinite number of sub-contrary sections.

The use of the term is almost entirely confined to the sections of a scalene cone. If, in any scalene cone with a circular base, a plane be passed through the axis, and perpendicular to the plane of the base, it is a principal plane. If a second plane be then passed perpendicular to it, making with one of the elements cut out, an angle equal to that which the base makes with the other element, the section, thus formed, is sub-contrary, and is a circle. If the base of the cone is an ellipse, whose plane is perpendicular to the principal plane of the cone, its sub-contrary section is a similar ellipse. If the cone is a right cone with a circular base, the sub-contrary sections become parallel sections.

Sub-contrary Straight Lines. If two straight lines, AB and CD, be intersected by two other straight lines, BD and CA, making the angle ACD equal to the angle ABD, the last two lines are sub-contrary with respect to the first two. The first two lines are also sub-contrary with respect to the last two.



If the point O is at an infinite distance, the lines AB and CD will be parallel to each other, and AC, BD, are anti-parallel.



SUB-DI-VĪDE′. [L. *sub*, and *divido*, to divide]. To divide into smaller parts.

SUB-DI-VĬ″SION. A part obtained by subdividing anything.

SUB-DŬ′PLI-CATE RATIO. [L. *sub*, and *duplus*, double]. The ratio of the square roots, or square root, of a ratio. The subduplicate ratio of a to b, is the ratio of

$$\sqrt{a} \text{ to } \sqrt{b}; \quad \text{or,} \quad \sqrt{\frac{b}{a}}.$$

SUB-MUL′TI-PLE. [L. *sub*, and *multus*, many]. A quantity which is contained in another an exact number of times. Thus, 7 is a submultiple of 42.

SUB-NOR′MAL. [L. *sub*, and *norma*, a rule]. That part of the axis on which the normal is taken, contained between the foot of the ordinate through the point of normalcy of the curve, and the point in which the normal intersects the axis.

In the Conic Sections, the subnormal is often taken on a diameter: in that case, the ordinate through the point of contact, is drawn parallel to the chords which the diameter bisects. Unless the contrary is mentioned, the subnormal is regarded, as taken on the principal axis of the curve; we shall henceforth so regard it. In other curves, the subnormal is generally taken on the axis of X, and the ordinate through the point of contact, is drawn perpendicular to the axis of X. In this case, the analytical formula for a sub-

normal to any curve, at a point whose co-ordinates are y'' and x'', is

$$S\text{-}N = y'' \frac{dy''}{dx''}.$$

This, for the conic sections referred to the principal axis, and tangent at its vertex, becomes

$$S\text{-}N = p + r^2 x''.$$

In the parabola, $r^2 = 0$, and $S\text{-}N = p$; that is, the subnormal is constant, and equal to one-half of the parameter.

In the ellipse and hyperbola, the subnormal varies from point to point of the curve, being in both, equal to p, or half of the parameter, at the principal vertex. See *Subtangent*.

SUB-QUAD′RU-PLE. [L. *sub*, and *quadruplus*]. Containing one part out of four.

SUB-QUIN′TU-PLE. [L. *sub*, and *quintuplus*]. Containing one part out of five.

SUB-SEX′TU-PLE. [L. *sub*, and *sextuplus*]. Containing one part out of six.

SUB-SID′I-A-RY. [L. *subsidiarius*, auxiliary]. Something introduced to facilitate the solution of a problem, or to aid in a demonstration. The term is particularly applied to auxiliary angles. Since the trigonometrical tables give great power in their management, they are frequently introduced, even into problems which involve no question of angular quantity. For example, suppose it were required to compute a great number of results from the formula

$$y = ax + b\sqrt{1 - x^2}.$$

Assume $x = \cos\theta$, $a = r\cos\phi$, $b = r\sin\phi$: substituting these in the formula, we have

$$y = r\cos\phi\cos\theta + r\sin\phi\sin\theta = r\cos(\phi - \theta)$$
$$= \frac{a\cos(\phi - \theta)}{\cos\phi}.$$

By the aid of tables and the preceding formula, the computation can be made much more rapidly than by the aid of the given formula.

In the solution of trigonometrical problems, subsidiary angles are continually employed. No rule can be given for their use, but each case must be treated as it arises.

SUB-SIST′. [L. *sub*, and *sisto*, to be fixed]. To be; to have an existence: thus, two inequalities are said to subsist in the same sense, when the first members are the greatest in both, or least in both; they subsist in a contrary sense, when the first member is greatest in one, and least in the other.

SUB-STI-TŪ′TION. [L. *sub*, and *statuo*, to set; to put in place of another]. The operation, in Algebra, of replacing one quantity by another, in an algebraic expression.

There is a process of approximation much used, which has received the name of *successive substitution*.

Suppose an equation to have been reduced to the form,

$$x = a + bf(x),$$

in which a is less than 1.

To find the value of x by successive substitution: If we assume $x = a$, the error will be less than $f(x)_{x=a}$. Take this value of x, and substitute it in the second member of the equation; we shall have for x the approximate value

$$x = a + bf(x)_{x=a};$$

which value is, in general, nearer the true value than the former value found. Denote this value by a', and again substituting, we get

$$x = a' + bf(x)_{x=a'};$$

which will, in general, be still nearer the true value. Denoting this value by a'', and proceeding as before, we shall continually approach, nearer and nearer, the true value of x.

SUB′STȲLE. [L. *sub*, under, and *stylus*, style]. In dialing, the orthographic projection of the style upon the plane of the dial.

SUB-TAN′GENT. [L. *sub*, under, and *tangens*, touching]. That part of an axis included between the points in which a tangent cuts it and the foot of the ordinate through the point of contact. The subtangent and subnormal are projections of the tangent and normal upon the axis on which they are taken, or to which they are referred.

We shall first consider the case in which the subtangent is taken on the axis of X, the curve being referred to rectangular axes.

The general formula, in this case, for the sub-tangent drawn at a point whose co-ordinates are x'' and y'', is

$$S\text{-}T = y'' \frac{dx''}{dy''}.$$

For the conic sections referred to the principal vertex, the equation being

$$y^2 = 2px + r^2x^2,$$

this formula gives

$$S\text{-}T = \frac{(2px'' + r^2x''^2)}{p + r^2x''}.$$

The subtangent together with the subnormal, form the hypothenuse of a right-angled triangle, whose other sides are the tangent and the normal; hence, the square of the ordinate of the point of contact is always equal to the product of the subtangent and subnormal. This remark is readily verified by multiplying the formulas already given, for the subtangent and subnormal, member by member. This multiplication gives

$$S\text{-}T \times S\text{-}N = y''^2.$$

When a curve is given by means of polar co-ordinates, the formulas for the subnormal and subtangent deduced, either from those given, or by direct reasoning, are

$$S\text{-}N = \frac{du}{dt}, \quad \text{and} \quad S\text{-}T = u^2\frac{dt}{du},$$

in which u and t are the polar co-ordinates of the point of contact. In this case, as before, we have

$$S\text{-}N \times S\text{-}T = u^2.$$

The subtangent and subnormal are both taken on an axis perpendicular to the radius vector of the point of contact.

SUB-TENSE′. [L. *sub* and *tensus*, stretched out]. The same as chord. See *Chord*.

SUB-TRACT′. [L. *subtraho*, *subtractus*; from *sub* and *traho*, to draw]. To withdraw, or take a part from a whole.

SUB-TRAC′TION. L. *subtraho*, to take away from]. The operation, in Arithmetic, of finding the difference between two numbers; or it is the operation of finding a number that being added to the lesser of two numbers, will produce the greater. The greater number is called the *minuend*; the lesser, the *subtrahend*, and their difference, the *remainder*. If the minuend and subtrahend are equal, the remainder is 0.

In Algebra, the definition of subtraction requires extension to correspond with the more general language of that science. In Algebra it is by no means necessary that the minuend should be greater than the subtrahend, on the contrary, it is often less. We may then define subtraction in Algebra, to be the operation of finding a quantity which being added to the second of two given quantities, will produce the first. The first is the minuend, the second is the subtrahend, and the third, or quantity sought, is the *algebraic difference*. If the minuend exceeds the subtrahend, the *algebraic difference* is the same as the arithmetical difference or remainder: when they are equal, the difference is 0; and when the minuend is less than the subtrahend, the *algebraic difference is negative*, but *numerically* equal to the arithmetical remainder. This definition and explanation does away with all discussion as to the nature of subtraction, when the subtrahend exceeds the minuend, a discussion which can only be founded upon a partial understanding of the general term subtraction. We shall consider arithmetical and algebraic subtraction in their order.

1. *Arithmetical Subtraction.*

The first condition required is, that both minuend and subtrahend be expressed in the same scale of numbers; this being premised, let us illustrate by two simple cases in which the numbers are written in the scale of tens.

Let it be required to subtract 52 from 76.

Operation:

$$\begin{array}{r} 76 \\ \underline{52} \\ 24. \end{array}$$

Here we see that 6 exceeds 2 by 4, and 7 tens exceed 5 tens by 2 tens; hence, the remainder is 24. Again, let it be required to subtract 59 from 94.

Operation:

$$\begin{array}{r} 94 \\ \underline{59} \\ 35. \end{array}$$

Commencing with the units, we see that 9 is greater than 4, and consequently, cannot be taken from it in the arithmetical sense, but by adding 1 ten to 4, gives 14, which exceeds 9 by 5: having added 1 ten to the minuend, it becomes necessary to add 1 ten to the subtrahend, that the difference may remain unchanged; this we do by increasing 5 tens by 1 ten, giving 6 tens; now 9 tens exceeds 6 tens by 3 tens; hence, in this case, 35 is the true remainder. This course of reasoning may be extended to any extent; hence, we have the rule for arithmetical subtraction.

Set down the less number under the greater, so that units of the same order shall fall in the same column; then beginning with the unit or

the lowest order, subtract each from the one above it. When the number of units of any order in the subtrahend exceeds that in the minuend, suppose as many units of that order to be added in the minuend as make one unit of the next higher order; after which, add one unit of the next higher order in the subtrahend, and proceed as before till all the units have been subtracted: the result is the remainder required.

The principle employed here is, that if the same number be added to both minuend and subtrahend, the difference will remain unchanged.

2. *Algebraic Subtraction.*

In Algebra, if from one quantity we wish to subtract another, the operation may be indicated by inclosing the second within a parenthesis, prefixing the minus sign, and then writing it after the first. To deduce a rule for performing the operation thus indicated, let us represent the minuend by a, the sum of all the additive terms in the subtrahend by b, and the sum of all the subtractive terms by $-c$; then will the operation be indicated thus,

$$a - (b - c),$$

where it is required to subtract the difference of b and c from a.

If now, we diminish the quantity a by the number of units in b, the result, $a - b$, will be too small by the number of units in c, since c should have been subtracted from b, before taking b from a. Hence, to obtain the true remainder, we must increase the first result by c, giving the expression

$$a - b + c,$$

which is the true algebraic difference required. By comparing it with the given minuend and subtrahend, we see that we have changed the signs of all the terms of the latter, and added the result to the former. To facilitate the operation, the similar terms are to be written in the same column. Hence, for the subtraction of algebraic quantities,

1. *Write the quantity to be subtracted under that from which it is to be taken, placing similar terms, if there are any, in the same column.*

2. *Change the signs of all the terms to be subtracted, or conceive them to be changed, and then add the result to the other quantity.*

The sign of subtraction is $-$, called minus, and though used in Arithmetic, is always to be understood as algebraic in its nature, indicating a subtraction in the most general sense of the term.

Subtractive Quantity. A quantity, in Algebra, preceded by the sign $-$. If the sign of the quantity is itself minus, the combination of the sign of the quantity with the sign of operation, conspire to render the essential sign of the expression positive.

SUB-TRA-HEND'. In Arithmetic and Algebra, the quantity to be subtracted. See *Subtraction*.

SUB-TRIP'LE. [L. *sub* and *triplex*, triple]. One part out of three.

SUB-TRIP'LI-CATE RATIO. [L. *sub* and *triplex*, threefold]. Of two quantities, the ratio of their cube roots. Thus, the subtriplicate ratio of a to b, is

$$\frac{\sqrt[3]{b}}{\sqrt[3]{a}} = \sqrt[3]{\frac{b}{a}}$$

The subquadruplicate ratio is

$$\frac{\sqrt[4]{b}}{\sqrt[4]{a}};$$

the subquintuplicate ratio is,

$$\frac{\sqrt[5]{b}}{\sqrt[5]{a}};$$

the subsextupilicate ratio is

$$\frac{\sqrt[6]{b}}{\sqrt[6]{a}}, \text{ \&c.}$$

SUM. [L. *summa*, a sum]. In addition, the aggregate of two or more quantities. In Arithmetic, the sum of several numbers is a number which contains as many units as are contained in all the given numbers taken together. Hence, the sum is greater than any of its parts.

In Algebra, the term *sum* does not necessarily imply increase; for, if we aggregate several quantities, some of which are positive, and some negative, it may happen that the sum is numerically less than any one of the parts, it may even be 0. This sum is therefore distinguished, as the *algebraic sum*.

SU-PER-FĬ''CIAL. Appertaining to a surface, as superficial contents, &c.

SU-PER-FĬ''CIĒS. [L. *super*, upon, and

facies, the face]. The area of a surface. The difference between this term and the term surface, is simply this. The term surface is abstract, and simply implies that magnitude which has length and breadth without thickness, whilst the term superficies does not refer to the nature of the magnitude, but simply refers to the number of units of surface which the given surface contains.

SU-PĒ'RI-OR. [L. *super*, above]. Lying above, or having a higher place.

SUPERIOR LIMIT OF A QUANTITY. A limit towards which the quantity may approach to within less than any assignable quantity of the same kind; it is always greater than the quantity.

A SUPERIOR LIMIT of the roots of a numerical equation of the form

$$x^m + Px^{m-1} + Qx^{m-2} + \ .\ .\ + Tx + U = 0,$$

is any number greater than the greatest positive root of the equation. From the definition, it is plain that such an equation has an infinite number of superior limits. In the solution of numerical equations, it becomes an object to find as small a superior limit as possible, and for this purpose a variety of rules have been given. The simplest method of finding a superior limit is this:

A superior limit of the positive roots of a numerical equation is always found by taking the numerical value of the greatest co-efficient of any term, and adding 1 to it. This will, in general, be much greater than it is necessary to use. A simpler limit is named the *ordinary limit*. To find this, extract that root of the numerical value of the greatest negative co-efficient of any term, whose index is the number of terms which precede the first negative one, and to the result add 1. If there are any terms wanting they must be supplied by inserting $+ 0$ in their places. This will, in general, be smaller than the one before considered, but there is still another limit. which is usually smaller than the ordinary limit, which is called *Newton's limit*. To find it, form from the first member of the given equation its successive derived polynomials; then determine by trial the least number which will render the first member and all its successive derived polynomials positive; and such that all greater numbers will render them positive; then will the first number found be a superior limit. The method of finding the least superior limit in whole numbers depends upon Sturm's theorem. Find the first derived polynomial of the first member, and to this and the first member apply the process of finding their greatest common divisor, with this exception, that instead of using the remainder as found, change their signs, and take care not to introduce or reject any factors except positive ones. Continue the process until a remainder is found which is independent of the unknown quantity. Denote the first member of the given equation by X, its first derived polynomial by X_1, and the several remainders with their signs changed by X_2, X_3, &c., X_r; then write the expressions

$$X,\ X_1,\ X_2,\ X_3\ .\ .\ .\ X_r$$

in a row, and substitute in them $+ \infty$ for the unknown quantity, and write the signs of the results in a row. Find by trial the smallest positive number, which, when substituted for the unknown quantity, will give the same number of variations of signs in passing along the row; this will be the smallest superior limit in whole numbers.

The superior limit of the negative roots, (numerically considered), may be found from the same expression, as follows:

Substitute in them $- \infty$ for the unknown quantity, and write the signs of the results in a row; then find by trial the smallest negative number (numerically considered), which, being substituted for the unknown quantity, will give the same number of variations of signs in passing along the row. This number will be the superior limit of the negative roots (numerically considered).

SUP'PLĒ-MENT. [L. *supplementum*; from *sub* and *pleo*, to fill]. In Trigonometry, the supplement of an angle is the remainder obtained by subtracting the angle from 180°, or two right angles. If the angle exceeds 180° the supplement will be negative. The trigonometrical functions of the supplement of an angle are given by the following equations:

$$\sin(180^\circ - A) = \sin A.$$
$$\cos(180^\circ - A) = -\cos A.$$
$$\tan(180^\circ - A) = -\tan A.$$
$$\cot(180^\circ - A) = -\cot A.$$
$$\sec(180^\circ - A) = -\sec A.$$

$$\text{cosec}\ (180° - A) = -\ \text{cosec}\ A.$$
$$\text{versin}\ (180° - A) = 2 - \text{ver-sin}\ A.$$

The radius being taken equal to 1.

SUP-PLĒ-MENT'ARY CHORDS, in an ellipse or hyperbola, any two chords drawn through the extremities of a diameter, and intersecting on the curve. If we refer the curve and the supplementary chords to the centre, as an origin, the axis of X coinciding with the diameter through whose extremities the chords are drawn, and the axis of Y coinciding with its conjugate, the equation of condition for supplementary chords in the ellipse, is

$$cc' = -\frac{b'^2}{a'^2};$$

in the hyperbola it is

$$cc' = \frac{b'^2}{a'^2};$$

c and c' denoting the ratio of the sines of the angles which the chords make with the conjugate diameters, a' the semi-diameter through the extremities of which the chords are drawn, and b' its semi-conjugate.

If the chords are drawn through the extremities of the transverse axis, the equations of condition become, for the ellipse,

$$cc' = -\frac{b^2}{a^2};$$

for the hyperbola,

$$cc' = \frac{b^2}{a^2};$$

c and c' are the tangents of the angles which the chords make with the transverse axis, and a and b the semi-axes of the curve. Either chord is called a supplement of the other.

It is a property of supplementary chords in either of the curves, that if any chord is parallel to a diameter, its supplement is parallel to the conjugate of that diameter, and also to the tangent to the curve through the vertex of that diameter. This affords a method of constructing the conjugate of any diameter, and also a method of drawing a tangent line to either curve at a given point, parallel to any given straight line. See *Ellipse* and *Hyperbola*.

The supplementary chords drawn through the extremities of the transverse axis of an ellipse, make an obtuse angle with each other; these drawn through the extremities of the conjugate axis make an acute angle with each other. In the circle they are always at right angles to each other. Supplementary chords drawn through the extremities of the transverse axis in the hyperbola, always make an acute angle with each other.

SURD. [L. *surdus*, deaf]. An indicated root of an imperfect power of the degree indicated. It is the same as a radical. Any expression involving a surd is called a surd expression, a surdal quantity, or a radical quantity. See *Radical*.

SUR-SOL'ID. A fifth power; thus, a^5 is the sursolid of a.

SUR-VEY'ING. [L. *sur*, and *video*, to see]. In its most general signification, embraces all the operations for finding, 1st, the area or superficial contents of any portion of the earth's surface; 2d, the lengths and directions of the bounding lines; 3d, the contour or shape of the surface; and, 4th, the accurate delineation of the whole upon paper.

Surveying is divided into three branches: *Topographical*, *Plane*, and *Geodesic Surveying*.

TOPOGRAPHICAL SURVEYING embraces all the operations incident to finding the contour of a portion of the earth's surface, and the various methods of representing it upon a plane surface. For an account of Topographical Surveying, see *Topography*.

PLANE SURVEYING embraces all the operations of surveying, carried on under the supposition that the surface of the earth is a plane. The radius of the earth being very great. (nearly 4000 miles), if only a limited portion of the surface is considered, as a few miles in extent, it may, without error, be regarded as a plane, disregarding the minor inequalities, or conceiving the whole to be projected on a plane.

GEODESIC SURVEYING comprises all the operations of surveying carried on under the supposition that the earth is spheroidal. An outline of the subject of Geodesic Surveying has been given under the head of *Geodesy*, which see.

Geodesic Surveying embraces what is generally denominated Maritime Surveying.

PLANE SURVEYING. The operations of plane surveying may be classed under three heads: 1st, *Field Operations*; 2d. *Computations*; and,

3d. *Plotting*; each of which will be noticed in turn.

I. *Field Operations* comprise all measurements made in the field, the results of which are recorded in a book for the purpose, and constitute what are called *field notes*.

The measurements incident to a field survey are of two kinds: *measurement of angles* and *measurement of distances*.

Angles are measured by means of the *theodolite*, *compass*, *sextant*, *plane table*, *circumferentor*, or some other instrument, contrived for the purpose. For the method of performing the measurements, and for an account of the different instruments employed, see the articles *Theodolite*, *Compass*, *Plane Table*, &c. A sufficient number of angles are measured at the different stations to afford the means of determining the relative positions of all the points which it is desired to locate by the survey.

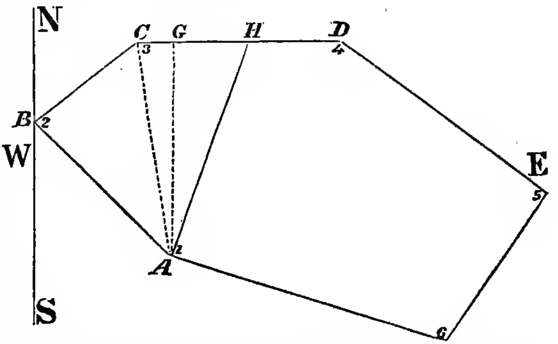
Distances are measured by means of a *chain*, *tape*, *rod*, or any *scale of equal parts*, by continually applying them along the direction of the required distance. The number of distances measured will depend upon the nature of the survey, and also upon the number of angles measured. The measured distances are sometimes called *courses*; this is particularly the case in surveying with the compass.

In field surveying, undertaken for the purpose of determining the area of a piece of land, or for dividing or laying off any piece of land, distances are generally measured with Gunter's chain, or a tape of the same length as Gunter's chain, and divided into the same number of equal parts. In connection with Gunter's chain, ten marking pins are used, consisting of a piece of iron one-eighth of an inch in thickness, and twelve or fourteen inches in length, sharp at one end, with a ring at the other. There are also required two chainman's staves, for straightening and aligning the chain, and two flag staves to mark the stations. For a description of Gunter's chain, and the manner of using it, see *Gunter's Chain*.

In surveying for the purpose of filling in a geodesic survey, or for making a map of a limited territory, as a village or town, a chain of 50 or 100 feet in length, or a tape of the same length divided into feet, is generally used, and the same additional instruments are required as in using Gunter's chain. When irregular lines are to be surveyed, an offset staff is also employed. See *Offset Staff*.

We shall illustrate the method of obtaining and recording the field notes of a field survey, and, for the purpose of filling in a trigonometrical survey, or for mapping a tract of country, by giving a single example of each.

To obtain and record the field notes of a field survey:



Let BCDEA represent the piece of ground to be surveyed, NS being a meridian line through B. Rule the pages of the field book into three spaces by vertical lines, and head the columns thus formed, respectively, *Stations*, *Bearings*, and *Distances*.

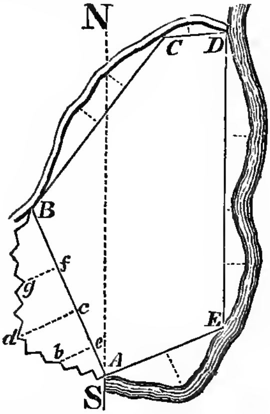
Select some convenien angle of the field, as A for example, and call that station 1. Number the remaining angles around in order, keeping the field on the right, and enter these numbers in the column marked *Stations*. At station 1 take the bearing of the line to station 2, and enter it in column headed *Bearings*, and measure the distance to station 2, and enter it opposite station 1 in the column headed *Distances*. Carry the compass to station 2 and take a reverse bearing to station 1, and see if it agrees with the direct bearing; if not, both must be repeated;

if it does, take the bearing of station 3, and measure the distance to it, entering the bearing and distance as before, in the proper columns. Continue this operation of measuring bearings and distances till all of the courses have been measured.

FIELD BOOK.

STATIONS.	BEARINGS.	DISTANCES.
1	N. $46\frac{1}{2}$° W.	20 ch.
2	N. $51\frac{3}{4}$° E.	13.80 ch.
3	E.	21.25 ch.
4	S. 56° E.	27.60 ch.
5	S. $33\frac{1}{4}$° W.	18.80
6	N. $74\frac{1}{2}$° W.	30.95

When the field to be surveyed is bounded by a broken and irregular line, the survey is best made by means of offsets.



Let ABD represent such a field. In this case, select several of the prominent points in the boundary of the field, as A, B, C, D and E, for stations. Take with the compass the bearings from A to B, from B to C, from C to D, &c., as before directed. At convenient points of the course AB, as *e*, *c*, *f*, measure the offsets *ab*, *cd*, and *fg*; measure also the distances A*e*, A*c*, A*f*, and record both the offsets and these distances, either in two columns ruled in the field notes, and headed *offsets to the right*, and *offsets to the left*, or the record may be kept as in the method of surveying for the purpose of mapping, as will now be explained.

In this kind of surveying, compass lines are run along the principal lines of the country to be mapped, and offsets are measured on each side to the principal objects. The field notes comprise not only a record of the bearings and lengths of the courses, together with the lengths and positions of the offsets, but a rough sketch of the country along the lines run. (See figure, next page.)

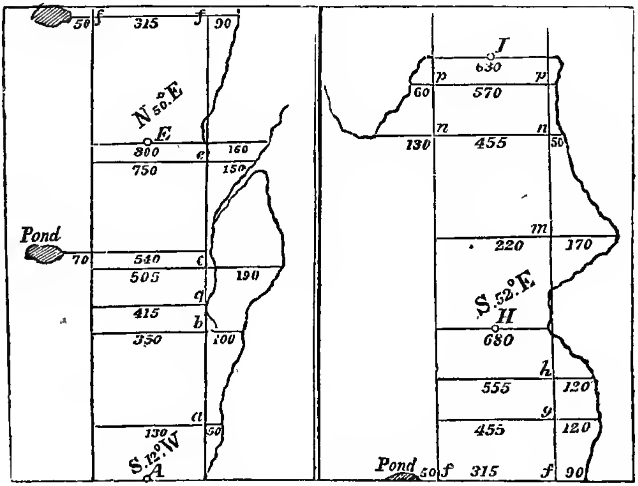
Each page of the field book is ruled into three columns, the middle space being much narrower, usually, than the lateral spaces, and the records are made from the bottom of each page upwards for the purpose of having the ground and the sketch before the eye in the same relative positions.

Starting at some principal station as A, suppose the bearing of the first course is S. 12° W.; this is entered at the bottom of the page. Measure along this course, say 130 feet; this distance is entered in the middle column, at a distance from the bottom, corresponding to 130, taken from a rough scale of equal parts, or if the paper is ruled, this distance may be estimated, allowing a certain number of feet for the distance between two consecutive ruled lines. Draw a horizontal line, and suppose that an offset is measured to the right, 50 feet, to a stream; produce the horizontal line to the right, a distance of 50 feet to the scale adopted, and enter the length of the offset under this line in the right hand column. Measure again along the line to a distance say of 350 feet from the station. Draw a horizontal line in the middle column at a distance from the bottom, equal to 350 feet from the scale. and suppose that we measure another offset to the same stream of 100 feet to the right. Enter and plot this offset as before. and then with the eye sketch in the general courses of the stream between the two points found on the plot.

Continue, in this manner, measuring along the main line, taking offsets and plotting them roughly till the end of the course is reached. In the case considered, the length of the first course is supposed to be 800 feet. Whenever an offset is measured to the left, as in the case of the pond, in the diagram, the offset must be plotted on the left hand side of the middle column. When a new course is to be commenced, a circle is made in the central column to indicate the fact, and the bearing of the course is entered just above it;

then the operations are continued as before. When the top of one page is reached in the field book, the work is transferred to the bottom of the next, and the work is continued in the same manner till the survey is complete. The note book, with the rough sketches, will then afford all the data necessary for plotting in the principal features of the country surveyed. It may be well to confine the notes to the left hand page, leaving the right hand page for any remarks that may be desirable to enter with respect to the nature of the country, &c.

This method is often used in surveying



large estates, running compass lines through the estate in different directions, and making offsets to angular points of fields, &c.

There is still another method of surveying a field, which consists in selecting two prominent stations visible from each other, and from which the angular points of the field are also visible. The distance between them is carefully measured; then their bearing from each other is measured with the compass. The bearings of each of the angular points of the field are measured from each station, and all these measurements are entered in the note book.

II. We come next to the computations.

The computations incident to a geodesic survey are explained under the head of Geodesy, and as far as they are applicable, the same computations are to be made in extensive plane surveys. The computations necessary to determine the distances of objects, (accessible or inaccessible) trigonometrically, are explained under the title of *Distances*, which see. We shall therefore introduce, in this place, only the computations necessary to make a field survey.

In the first place, if the length or bearing of any course be lost or suspected of being in error, it may be thrown out and the distance or bearing, or both, and the course required may be computed from the remaining notes. To do this, find from the traverse table the latitudes and departures of all the other courses, and enter them under the proper headings of N. S. E. and W. Take the sum of the northings and the sum of the southings; their difference will be the northing or southing of the required course. If the sum of the northings exceeds the sum of the southing, the required course will make southings; if the reverse, the required course will make northing.

Take the sum of the eastings and the sum of the westings; their difference will be the easting or westing of the required course. If the sum of the eastings exceeds the sum of the westings, the required course makes westing. If the reverse is the case, the re-

quired course makes easting. The square root of the sum of the squares of the latitude and departure thus found, is equal to the length of the required course. Divide the departure by the length of the course, and the quotient will be the natural sine of the bearing, which may then be found from a table of natural sines.

If it is found that some error has been committed in taking the field notes, and it is not known on which course the mistake has been made, each course may be thrown out in turn, and computed by the rule just given; the computed result compared with that in the notes; and in this manner the error may be reached. If it is suspected that an error has been committed in reading an angle, as is often the case, compute the interior angles of the field, from the notes, by the following rules, and take their sum; this ought to be equal to two right angles taken as many times as the field has sides, less two.

Rules for finding the interior angles from the field notes, the courses being taken around the field.

1. If the meridional letters are *unlike*, and those of departure *unlike*, the interior angle is equal to the *difference of the bearings*.

2. If the meridional letters are *unlike*, and those of departure *alike*, the interior angle is equal to the *sum of the bearings*.

3. If the meridional letters are *alike*, and those of departure are *unlike*, the interior angle is equal to 180° *minus the sum of the bearings*.

4. If the meridional letters are *alike*, and those of departure *alike*, the interior angle is equal to 180° *minus the difference of the bearings.*

Having tested the accuracy of the field notes, rule a sheet of paper into 12 columns, and head them as in the annexed example. Write the field notes in the first three columns. We have taken the same example as was used in explaining the field operations. Then find from the traverse table the latitude and departure of each course, and enter them under the proper headings, observing to write the latitude under the head N. when the meridional letter of the bearing is N., and under S. when it is S. and also to enter the departure under E. when the letter of departure of the course is E, and under W. when it is W. See *Traverse Table.*

Next, balance the work; that is, apportion the errors in latitude and departure according to the lengths of the courses, and enter the balanced latitudes and departures, with their proper signs, under the heading "Balanced." See *Balancing*. Next form the double meridian distance of each course according to the following rule:

The double meridian distance of the first course is equal to the departure of the course. The double meridian distance of any other course is equal to the double meridian distance of the previous course, plus the departure of that course, plus the departure of the course itself. In using the rule, the proper signs of the latitudes and departures must

CALCULATION.

Stations.	Bearings.	Dist.	Diff. Lat. N. +	Lat. S. −	Departure. E. +	Departure. W. −	Balanced. Lat.	Balanced. Dep.	D.M.D. +	Area. +	Area. −
1	N. 46½° W	20 ch	13.77			14.51	+13.88	−14.56	14.56	202.0928	
2	N. 51¾° E.	13.80	8.54		10.84		+8.61	+10.81	10.81	93.0741	
3	E.	21.25			21.25		. . .	+21.20	42.82	. . . .	
4	S. 56° E.	27.60		15.44	22.88		−15.29	+22.82	86.84		1327.7836
5	S. 33¼° W.	18.80		15.72		10.31	−15.63	−10.36	99.30		1552.0590
6	N. 74½° W.	30.95	8.27			29.83	+8.43	−29.91	59.03	497.6229	
Sum of courses,		132.40	30.58	31.16	54.97	54.65				792.7898	2879.8426
				30.58	54.65						792.7898
Error in Northing, .				0 58	0.32						2)2087.0528
Ans. 104*A*. 1*R*. 16*P*.											1043.5264

be observed. Enter these under the head of D. M. D. In the example, the course marked with a star * is taken as the first course.

Then multiply the double meridian distance of each course by its northing or southing, observing the rule for signs, and enter the positive products under the head of plus areas, and the negative ones under the head of negative areas. Take the sum of the positive areas and of the negative areas separately, and subtract the less from the greater; the remainder will be double the area of the field, expressed in square chains and decimals of a square chain.

III. Plotting the work. The various methods of plotting have already been described under the head of plotting. See *Plotting*, *Plotting Scale, &c.*

SURVEYING THE PUBLIC LANDS The public lands consist of those large tracts that belonged to the United States after the Revolution, together with all that was ceded by the States soon after the formation of the Constitution, with all the additions which have since been acquired by treaty and purchase, embracing many millions of acres. In 1802, Colonel Jared Mansfield, the Surveyor General of the North Western Territory, devised a systematic method of surveying and recording such portions as were to be offered for sale, which method is still adhered to.

The entire public domain is divided into land districts, to each of which a Surveyor-General is assigned, who is charged with the general supervision of all the surveys within his particular district.

The method of making the surveys is, in outline, as follows:

A meridian line is run, with great care, through some prominent point of the district, and through the same point, a line at right angles to it is also run, both reaching through the entire district. These lines, determined astronomically, serve as a system of co-ordinate axes to which the subdivisions are easily referred. Parallel to these lines, and on each side of them, other lines are run six miles distant from each other, dividing the district into squares containing 36 square miles, or 23,040 acres, each. These squares are called townships. Each township is subdivided by lines parallel to the meridians, and east and west lines, into 36 equal squares. These squares are called sections, and each one contains 640 acres. When the land is valuable, these are again divided into quarter sections, and sometimes into eighths of a section.

All the townships lying between two consecutive north and south lines, are called a *range*, and the ranges are numbered from the principal meridian in both directions, 1st., 2d., 3d., &c., to the extreme limits of the land district. The townships in each range are numbered from the principal east and west lines, 1st., 2d., and 3d., &c., in both directions, to the extreme limits of the land district. The sections in each township are numbered from 1 to 36, as shown in the diagram.

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

In describing a section of land, we say, Section No. 2, township 5, N., range 4, W., from the 5th principal meridian, Lawrence County district.

SURVEYING, MARITIME. See *Maritime Surveying*.

SURVEYOR'S CROSS. See *Cross, Surveying*.

SURVEYOR GENERAL. An officer of the United States government, having charge of the survey of the public lands of a land district.

SYM'BOL. [L. *symbolum*]. Any character used in Analysis, to represent a quantity, an operation, a relation, or an abbreviation. See *Notation*.

SYM'ME-TRY. [Gr. *συμμετρια*; from *συν*, with, and *μετρον*, measure]. Regularity of parts with respect to each other

SYM-MET'RIC-AL. Possessing the attribute of symmetry. In Geometry, two points

are symmetrically disposed, with respect to a straight line, when they are on opposite sides of the line and equally distant from it, so that a straight line joining them, intersects the given line, and is at right angles to it.

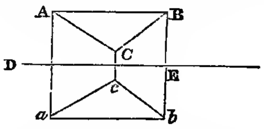
A curve is symmetrical, with respect to a straight line, when its points, taken in pairs, are symmetrically disposed with respect to it; that is, for each point on one side of the line there is a corresponding point on the other side, and equally distant from it. The line is called *an axis of symmetry*.

Thus, in the ellipse, for every point on one side of the transverse axis, there is a point on the other side equally distant from it; the chord joining them is perpendicular to the axis, and bisected by it. Hence, in this case, the axis of the curve is an axis of symmetry.

There is a species of *oblique symmetry*, differing from right symmetry, in the fact, that the chords joining the opposite corresponding points are oblique to the axis.

In an ellipse, any diameter bisects all chords drawn parallel to its conjugate: in this case, the symmetry is oblique. In general, a curve is obliquely symmetrical with respect to any one of its diameters.

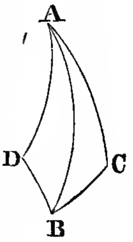
In the conic sections, the axes are the only true axes of symmetry. Two plane figures are symmetrically situated, with respect to a straight line, when each point of one has a corresponding point in the other on the opposite side of the axis, and equally distant from



it: thus the triangles ABC and *abc*, are symmetrically situated, with respect to the line DE. In all such cases, if either figure be revolved about the axis of symmetry through 180°, the two figures may be made to coincide.

A line or surface is symmetrical, with respect to a plane, when for each point on one side of the plane there is a second point on the other side, equally distant from it. The plane is called the plane of symmetry, and is, in the conic sections, a principal plane. Symmetrical lines and surfaces in space cannot, in general, be made to coincide with each other.

Spherical triangles are symmetrical, when their sides and angles are equal each to each, but not similarly situated: thus, the triangles ABC and ABD are symmetrical, but they cannot, by any possibility, be made to coincide with each other; they are equal in area only.



In Analysis, an expression is symmetrical, with respect to two letters, when the places of these letters may be changed without changing the expression. Thus, the expression

$$x^2 + a^2x + ab + b^2x,$$

is symmetrical with respect to a and b: for, if we change the places of a and b, we have

$$x^2 + b^2x + ba + a^2x,$$

the same expression; but it is not symmetrical with respect to x any y. An expression is symmetrical, with respect to several letters, when any two of them may change places without affecting the expression: thus, the expression,

$$a^2b + ba^2 + a^2c + c^2a + b^2c + bc^2,$$

is symmetrical with respect to the three letters, a, b and c. It is not sufficient that certain contemporaneous changes may be made, without affecting the expression, but any two must be interchangeable; thus,

$$a^2b + b^2c + c^2a,$$

remains unaltered, if a is changed to b, b to c, and c to a; but it is not symmetrical, with respect to a, b, and c; for, if a and b only be interchanged, it becomes

$$b^2a + a^2c + c^2b,$$

a different expression from the given one. See *Functions, Symmetrical.*

SYN'THE-SIS. [Gr. συνθεσις, from σον; and τιθημι, to set]. The method by composition, in opposition to the method of resolution or analysis. In synthesis, we reason from axioms, definitions, and already known principles, until we arrive at a desired conclusion. Of this nature are most of the processes of geometrical reasoning. In synthesis, we ascend from particular cases to general ones; in analysis, we descend from general cases to particulars.

SYN-THET'IC-AL METHOD. The method of reasoning by synthesis. This method is purely deductive. See *Synthesis*.

SYS'TEM. [L. *systema*; Gr. *συστημα*]. A regular method or order. A system of co-ordinates comprises the objects to which points are referred together with the method of reference. Thus, we speak of the rectilineal system, the polar system, &c. The rectilineal system is that in which points are referred to straight lines by means of their distances from these lines or from their planes measured on parallel lines. A polar system is one in which points are referred to a fixed line or lines, and a fixed point, by means of a variable angle or angles, and a variable distance.

T. The twentieth letter of the English alphabet. As a numeral it has been used to denote 160; with a dash over it, $\overline{\text{T}}$, signifies 160,000. In Arithmetic, it is an abbreviation for *Ton*.

T SQUARE. An instrument used for drawing parallel straight lines instead of the triangular ruler. It consists of two arms: one of which, called the *blade*, is fastened to the other, called the *stock*, at its middle, and secured by a clamp screw. The stock projects considerably below the blade, forming a shoulder, which, when used, is pressed firmly against the drawing board. The blade may be set at any angle with the stock and clamped. Then, if the stock is pressed against the edge of the drawing board and moved along, the blade will move continually parallel to its first position. Its use is obvious.

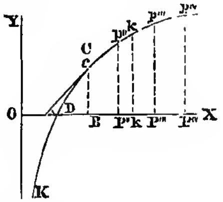
TĀ'BLES. [L. *tabula*]. In Mathematics, tables are of two kinds. The first are simply a collection of particulars, in a small space, for reference and ready application. Such are the tables of weights, measures, currency, &c. The second kind are series of numbers obtained from a general formula, expressing the law of a function, by attributing particular and equidistant values to the independent variable. Such are the tables of logarithms, sines, cosines, &c.

It would exceed our proposed limits to enter into an account of the general method of constructing the different kinds of mathematical tables, but some idea may be acquired of their nature from the following outline:

If we take the equation,

$$y = \log x,$$

and suppose x to have every possible value attributed to it from 0 to ∞, that is, if we suppose it to vary continuously between those limits, and the corresponding values of y to be deduced, they will vary continuously from $-\infty$ to $+\infty$.



Now, if we draw two axes OY and OX, at right angles, and suppose the values of x to be laid off on the axis OX, and the corresponding values of y to be laid off in perpendiculars to this line, the extremities of these perpendiculars or ordinates will make up a continuous curve, which is called the *logarithmic curve*. This curve is of the form represented in the figure, and is the geometrical representation of the *law of relation* between numbers and their logarithms. The curve cuts the axis OX at D, a distance 1 from the origin. Now, if we take from O the distances OD, OB, OP'', OP''', &c., respectively equal to 1, 2, 3. 4, &c., and write them in one column, and then write down in a second column, opposite them, the corresponding values 0, Bc, P''p'', P'''p''', &c., to any limit, the results will constitute a table of logarithms; and, by means of it and known principles, we can find the value of any intermediate ordinate or logarithm, as kk', which shall conform to the law of the function.

This operation of finding an intermediate logarithm is called *interpolation*, and is effected by means of the series of differences. See *Interpolation*.

Instead of taking values of x, corresponding to the series of natural numbers, we might have taken arbitrary values of x, and we should then have had a table of logarithms, but not of so convenient a form as the one before described. If we take the equations,

$$y = \sin x, \quad y = \cos x, \text{ \&c.},$$

and construct the curves expressing the law of the functions, we can, in like manner,

tabulate the values of the function for equidistant values of the variable : these will constitute tables of *sines*, *cosines*, &c., from which, by interpolation, the values of the sines, cosines, &c., may be found for all values of the variable.

The tables of most frequent use in mathematics, are the table of logarithms, sines, cosines, &c., both logarithmic and natural. These tables are generally appended to every treatise on trigonometry and surveying, together with an account of the method of using them.

TAN'GENT. [L. *tangens*, touching]. A tangent line to a plane curve at any point, is the limit of all secant lines through that point. To conceive the idea of a tangent at a point, draw any secant through the point, and revolve it about the given point as an axis ; the other point of secancy will, after a time, approach the given point, and finally coincide with it. At this stage of the revolution, the secant reaches its limit, and becomes a tangent. If the revolution be continued, the second point of secancy passes the first, and the line becomes a secant again, cutting the curve on the other side of the given point. This illustration holds equally true in the case of lines of double curvature, except that the motion ceases to be one of revolution. The motion must be made so that the second point shall always remain on the curve. From this exposition, we deduce the fact that, in general, there can be but one tangent to any curve at the same point. The only exception to this principle is in the case of a multiple point, at which there will be as many tangents as there are branches intersecting at this point, one to each branch.

We have, in the illustration, supposed that only two points of secancy unite to produce a point of contact ; but there is one remarkable exception to this supposition, in the case where the point of contact happens to be a point of inflexion. Let there be a curve having a point of inflexion. If a straight line be drawn through this point, cutting the curve in a point near the assumed point, it will also cut it in another point on the opposite side. Now, if the line be revolved about the assumed point, towards the tangent, these two points of secancy both approach the assumed point together, and ultimately coincide with it, and the tangent *cuts* the curve at the point of contact. If the revolution be continued in the same direction as before, in the next position, the secant only cuts the curve in the single point assumed. When the points of secancy have been made to unite, they are called coincident or consecutive points. Hence, we may define a tangent to a curve to be a straight line passing through *two* (or *three*) consecutive or coincident points.

The first case is the general one, and in that the tangent lies wholly without the curve, in the neighborhood of the point of contact ; that is, all of its points are on the convex side of the curve. The second case is the exception and rarely occurs ; when it does arise, the tangent cuts the curve, but as before, all of its points in the neighborhood of the point of contact lie on the convex side of the curve, the curve changing its curvature at the point of contact. Hence, we may again define a tangent to a curve to be a straight line having but one point in common with the curve, and all of its points in the neighborhood lying on the convex side of the curve. This rule has no exception. We say all the points lying in the neighborhood of the point of contact ; this restriction is necessary, for there is nothing to prevent a tangent, on being produced, from intersecting the curve at some other point.

In the graphic constructions of Descriptive Geometry, we regard the curve as coincident with an inscribed polygon whose sides are so small that they may, without sensible error, be considered as coinciding with the arc. In this case we call the extremities of any side of this polygon consecutive points of the curve, and the prolongation of such a side is called a tangent. Hence, in Descriptive Geometry a tangent is defined to be a straight line passing through two consecutive points of the curve. This definition is only approximately correct ; the approximation approaches nearer to the truth as the length of the side becomes nearer equal to 0. If the sides are infinitely small, and so taken that their projections on the axis of X are equal, each being the differential of x, we have the basis of that system of differential calculus which may be considered as the geometrical method. If we consider the fact, that according to the view just advanced the tangent passes through

two points, it may be asked which is the point of contact; the answer to this question is plain.

The theory of curves supposes the curve to be generated by a point moving continuously and passing over each side of the polygon in succession, and the point of contact is the one first reached in following the motion; and strictly speaking, the tangent ought to be considered as having its origin at this point and proceeding indefinitely from it, in the direction of the element. If we suppose the generating point to reverse its motion and retrace its path in an inverse direction, the direction of the tangents will also be reversed. These views serve to reconcile some apparent contradictions in the language of science, and also to throw much valuable light upon the logic of the science of the differential calculus. In this point of view, a tangent line to a curve at a point of inflexion, becomes simply two tangents, one to each branch, and lying in opposite directions; or in other words, at this point two elements of the curve coincide in direction; and in the language of descriptive geometry we should say that the tangent was a straight line passing through three consecutive points. This view, while it serves to illustrate the subject of tangency of curves, also illustrates that of osculation, as we shall soon see.

Two curves are tangent to each other at a common point, when they have a common rectilinear tangent at this point. In the view of the subject just taken, they both contain a common element at the point. Let us consider the case of a circle tangent to a curve at any point. It is evident that through the two consecutive points that limit an element of the given curve, an infinite number of circles can be passed, all of which will be tangent to each other and to the given curve, because the common element produced is a common rectilineal tangent to them all at the first point of the element. If now we consider the next element in order, of the given curve, we shall have three consecutive points, and through these it is impossible to pass more than one circle. This circle is the osculatory circle, so called, because it has a more intimate contact with the given curve, at the given point, than any other circle. It is also plain that every smaller circle passing through the first two points, will pass within the third point, and every greater one will pass without the third point; whence we see that the osculatory circle separates those tangent circles which pass without the given curve from those which lie wholly within it, in the neighborhood of the point of contact or of osculation. Similar views may be advanced in relation to other osculatory curves to a given curve at a given point. We shall only consider the additional case of the conic sections in general. It is always possible, as may easily be shown, to draw an infinite number of conic sections tangent to any given curve at a given point. Furthermore, since some one of the conic sections may always be made to pass through five points in a plane, it follows that some conic section may be drawn to include four consecutive elements of the curve. This is the osculatory conic section at the first point of the five, and its nature will be dependent upon the nature of the curve in question. Through any four consecutive points, an infinite number of different conic sections may always be made to pass, each of which will have a more intimate contact with each other and with the given curve than any of the infinite number of conic sections which are passed through three of these points.

A tangent plane to a curved surface is the limit of all secant planes to the surface through the point. The point is called the *point of contact*. To conceive the idea of a tangent plane, pass any secant plane cutting the surface in a line; now if this plane be properly turned about this point, the section will, after a time, approximate to a point, that point being the point of contact; or it will approach a single straight line through the point; or to two straight lines intersecting at the point; or to a straight line and curve intersecting at the point, and will always finally reduce to one of these cases; when it does thus reduce, the secant plane has reached its limit and becomes a tangent plane.

We have an instance of the first case of tangency in the sphere or ellipsoid; of the second in the cone or cylinder; of the third in the hyperbolic paraboloid, or the hyperboloid of one nappe; and of the fourth in a warped surface having three curvilinear directrices. A discussion analogous to that just

given, in relation to the tangency of lines, (only much more complicated), might be given in relation to curved surfaces, but as the whole subject may be reduced down to a consideration of the subject of tangency of lines, we shall not enter upon it.

The ordinary definition of a tangent plane to any curved surface is this. A plane is tangent to a surface when the plane and surface have at least one point in common through which, if any number of secant planes be passed, the sections cut out of the plane are tangent to the sections cut out of the surface. This point is called the point of contact. The definition suggests the method of passing a plane which shall be tangent to a given surface at a given point, viz: Draw any two lines on the surface, through the given point, and draw tangents to these lines at the given point; the plane of these lines will be a tangent plane to the surface at the point. In single curved surfaces, as the cylinder, cone, &c., the tangent plane at any point passes through two consecutive elements, and is tangent all along one element; conversely, if a plane passes through two consecutive elements it is tangent to the surface at every point of the first element. In general, only one tangent plane can be passed at the same point of the surface, but there are exceptions to the rule. An instance of this occurs in the case of the cone, for an infinite number of tangent planes may pass to the cone, through the vertex, all tangent to it at that point. In this case the rule for passing a tangent plane evidently fails. This rule also fails in the case of a conoid, where the point of contact falls upon the right line directrix; for, if two straight lined elements be drawn through the point, the plane of their tangents will not be a tangent plane.

Two surfaces are tangent to each other, when they have, at least, one point in common; through which if any number of planes be passed, the sections cut out by each plane will be tangent to each other at the point. This point is called the point of contact. Another definition is this: Two surfaces are tangent to each other, when they have a common tangent plane at a common point. This point is the point of contact.

In Analysis, the equation of a tangent line to any plane curve, at a point whose co-ordinates are x'' and y'', is

$$y - y'' = \frac{dy''}{dx''}(x - x'').$$

The equations of a tangent to any curve in space, at a point whose co-ordinates are x'', y'' and z'', are

$$x - x'' = \frac{dz''}{dx''}(z - z''), \quad \text{and}$$

$$y - y'' = \frac{dz'}{dy''}(z - z'').$$

The equation of a tangent plane is

$$\frac{dz''}{dx''}(x - x'') + \frac{dz''}{dy''}(y - y'') - (z - z'') = 0.$$

When the length of a tangent is spoken of in analysis, it includes that portion lying between the point of contact and the point in which the tangent cuts the axis upon which the subtangent is taken. The formula for the length of a tangent is

$$T = y''\sqrt{1 + \frac{dx''^2}{dy''^2}};$$

in which x'' and y'' are the co ordinates of the point of contact.

In all these expressions, $\frac{dy''}{dx''}$, &c., denote what the differential co-efficients become, when, for x, y, and z, the co-ordinates of the point of contact are substituted.

Tangent in Trigonometry. That portion of a tangent drawn at the first extremity of an arc, and limited by a secant drawn through the second extremity. The tangent is always drawn through the initial extremity of the arc, and is reckoned positive upwards, and consequently, negative downwards.

The following equations express the relations existing between the tangent and the other trigonometrical functions of an arc:

$$\tan a = \frac{\sin a}{\cos a}, \qquad \tan a = \frac{1}{\cot a},$$

$$\tan a = \sqrt{\frac{1}{\cos^2 a} - 1}, \quad \tan a = \frac{\sin a}{\sqrt{1 - \sin^2 a}},$$

$$\tan a = \frac{\sqrt{1 - \cos^2 a}}{\cos a}, \quad \tan a = \frac{2\tan\frac{1}{2}a}{1 - \tan^2\frac{1}{2}a},$$

$$\tan a = \frac{2\cot\frac{1}{2}a}{\cot^2\frac{1}{2}a - 1}, \quad \tan a = \frac{2}{\cot\frac{1}{2}a - \tan\frac{1}{2}a}$$

$$\tan a = \cot a - 2\cot 2a, \quad \cot a = \frac{1 - \cos 2a}{\sin 2a}$$

$$\tan a = \frac{\sin 2a}{1+\cos 2a}, \quad \tan a = \sqrt{\frac{1-\cos 2a}{1+\cos 2a}}$$

$$\tan a = \frac{\tan(45° + \frac{1}{2}a) - \tan(45° - \frac{1}{2}a)}{2}.$$

In which the radius is 1. For a further account of this subject,—see *Trigonometry*.

TĀRE AND TRET. Allowances made in selling goods by weight. Tare is an allowance made on goods sold in boxes, barrels, &c., for the weight of the boxes, &c. Tret is an allowance made for dust, refuse matter, &c. What remains, after these are deducted, is called net weight.

TAU'TO-CHRO-NOUS CURVE. [Gr. ταύτος, the same, and χρονος, time]. A curve such, that a heavy body rolling down it, under the influence of gravity, will always reach the same point at the same time, from whatever point it may start. The inverted cycloid, in a vertical plane, having its base horizontal, is a tautochronous curve.

TAYLOR'S THEOREM. The object of Taylor's theorem is, to show how to develop a function of the algebraic sum of two variables into a series arranged according to the ascending powers of one of the variables, with co-efficients which are functions of the other.

TAYLOR'S FORMULA. The formula for making this development is

$$f(x+y) = u + \frac{du}{dx}y + \frac{d^2u}{dx^2}\frac{y^2}{1\,.\,2} + \frac{d^3u}{dx^3}\frac{y^3}{1.2.3} + \frac{d^nu}{dx^n}\frac{y^n}{1\,.\,2\,.\,.\,n} + \cdots$$

In which the first member is any function of the sum of two variables, and u is what that function becomes when the leading variable y is made equal to 0. This formula is called Taylor's Formula. It fails to develop a function in the particular case in which u, or any of its successive differential co-efficients, becomes infinite for any particular value of the variable which enters them. It only fails for the particular value, holding good for all other values.

TECH'NI-CAL TERM. [L. *technicus*; Gr. τεχνικος, from τεχνη, artifice]. A term belonging to a particular art or science. A word having a certain meaning in common language, has often a very different meaning when used as a term in treating of a science.

TERM. [Gr. τερμα; L. *terminus*, a limit or boundary]. In Algebra, a single expression not connected with any other by the signs plus or minus, equality or inequality. Thus $3a^2b$, $2a \times 3c$, $\sqrt{a^2c}$ are terms. A word having a technical meaning.

TERM'IN-AL. [From L. *terminus*]. Forming an edge or extremity. Thus, we speak of the terminal edge of a polyhedron. Sometimes we speak of the terminal faces of a solid. Terminal is nearly synonymous with limiting.

TER-RES'TRI-AL. [L. *terrestris*; from *terra*, the earth]. Appertaining to the earth. As Terrestrial Magnetism, Terrestrial Equator, &c.

TET'RA-GON. [Gr. τετρα, four, and γωνια, angle]. In Geometry, a polygon having four angles, and consequently four sides See *Quadrilateral.*

TE-TRAG'ON-AL. Having four angles.

TET-RA-HĒ'DRAL ANGLE. A polyhedral angle, having four faces. See *Polyhedral Angle.*

TET-RA-HĒ'DRON, or TET-RA-E'-DRON. [Gr. τετρα, four, and εδρα, face]. A polyhedron bounded by four triangles. If the middle points of the faces be properly joined, two and two, the lines joining them are the edges of a second tetrahedron. A regular tetrahedron is one in which the faces are equal and equilateral triangles. If the middle points of the faces be joined two and two, the lines joining them form the edges of a regular tetrahedron. All regular tetrahedrons, are similar solids. See *Regular Polyhedron.*

TET-RA-HEX-A-HĒ'DRAL. Having four ranges of faces, each range containing six faces.

TET-RA-HEX-A-HĒ-DRON. [From Gr. τετρα, four, and *hexahedron*]. A polyhedron bounded by 24 faces.

THE-OD'O-LĪTE. [Gr. θεαομαι, to view, and δολος, stratagem]. An instrument used in surveying for measuring horizontal and vertical angles. It consists essentially of a telescope, with cross wires, mounted so as to

have a free angular motion in a vertical plane, and also a free motion about a vertical axis. Suitable clamps are attached to restrain either of these motions, and graduated circles are so mounted as to show the amount of angular motion about either axis. A great variety of methods have been adopted for the arrangement of the details of this instrument, and consequently a great variety of instruments have been produced, differing in external appearance, but all agreeing in the same general principle, and all called theodolites. The following outline description of that form of the instrument in most general use, will convey an idea of the theodolite used in surveying.

The horizontal limb is made up of two circular plates of brass, having a common axis, the upper one turning freely upon the lower one. By means of a clamp, the two plates may be firmly attached to each other, so that both must turn together. When thus clamped a slight motion may be imparted to the upper one, by means of a tangent screw. The lower plate bears a graduated circle on its periphery, and the upper plate carries two verniers, situated so that their 0 points lie in a plane passed through the axis of the upper telescope, and the common axis of the two plates. The graduation on the limb is carried to half degrees, or sometimes to 20-minute spaces, and the vernier is so graduated that arcs of 20 seconds may be read with accuracy, and in some instruments arcs of 10 seconds are read. The lower circular plate is firmly joined to a central hollow spindle, at right angles to the plate, through which a similar spindle, joined to the upper plate, passes. A washer of greater diameter than the hollow of the outer spindle, is screwed against the lower end of the solid inner spindle, which whilst permitting the upper plate to revolve freely, prevents it from being raised up from its contact with the lower plate.

The outer spindle passes through a closely-fitting cylinder, concentric with it, and spreading out into a flat plate, against which the leveling screws work. The lower end of the outer spindle terminates in a socket, which grasps a ball, forming a ball and socket, or universal joint. The ball part is firmly joined to the lower plate of the instrument, through which the leveling screws work. The lower plate, when the instrument is in use, is firmly screwed upon a tripod. The lower plate being fixed in position, the motion of the leveling screws enables the observer to bring the horizontal limb parallel to the horizon. The horizontality of the limb is shown by two spirit levels at right angles to each other, resting upon the upper face of the vernier plate. The outer spindle has free motion in the upper leveling plate, but this motion may be restrained by a clamp attached to the upper leveling plate, and then small motions in azimuth may be communicated by a tangent screw. This clamp is called the clamp of the limb; the other one described being the clamp of the vernier. Resting upon the upper face of the vernier plate, and firmly attached to it by screws, is the framework which supports the upper telescope and the vertical limb. The upper, or vertical limb, consists of a graduated circle, turning in supports about an axis at right angles to that of the horizontal limb. It bears in Y's a telescope, whose axis, or line of collimation, moves in a plane perpendicular to the axis of the vertical limb, or in a plane parallel to the axis of the horizontal limb. This is also provided with a clamp and tangent screw. The telescope rests in Y's, from which it may be removed and reversed, by raising two small loops. The telescope bears a level, whose axis should be parallel to the line of collimation, and has in the focus of the eye-glass two fine hairs, intersecting each other, to indicate the line of collimation.

For ordinary purposes of surveying, the diameter of the horizontal limb is about 6 inches, but for accurate work much larger ones are used. The large theodolite of the United States Coast Survey, has a horizontal limb of 30 inches in diameter. Its construction is in many respects different from the surveyor's theodolite above described.

Before using a theodolite, it should be properly adjusted; that is, the different parts should be brought to their proper relative positions. The theodolite is in adjustment when the following conditions are fulfilled: 1st, When the intersection of the cross hairs is in the axis of the telescope; that is, in the line which remains fast when the telescope is turned in the Y's: 2d, When the axis of the attached level is parallel to the axis of the telescope: 3d, When the axes of the levels

on the horizontal limb are perpendicular to the axis of ihe horizontal limb: and, 4th, When the axis of the vertical limb is perpendicular to the axis of the horizontal limb. The adjustments are made as follows:

1st Adjustment. *To bring the intersection of the cross wires into the axis of the telescope.*

Set up the instrument and turn the telescope of the upper limb in its Y's till one of the cross hairs is horizontal, and then direct the telescope to some distant and well defined object, so that the horizontal wire shall appear to pass through some particular point of the object; then, without disturbing the instrument, turn the telescope about in its Y's 180° till the same wire is again horizontal. If in the new position the wire does not pass through the same point that it did before, move it by means of two antagonistic screws, which work against the diaphragm, bearing the cross wires, one half of the space, and again direct the telescope so that the wire shall pass through the same point. Repeat the operation till, by continued trial, the wire passes through the point in both positions of the telescope; then the wire will be adjusted. Turn the telescope in the Y's till the other wire becomes horizontal, and adjust it in the same manner. Then examine the first wire lest the adjustment of the second wire should have disturbed its position. When both wires are adjusted, turn the telescope slowly about in its Y's, and the intersection of the Y's will appear to rest upon the same point during the entire revolution. When this is found to be the case, the first adjustment is complete.

2d Adjustment. *To make the axis of the level attached to the upper telescope, parallel to the axis of the telescope.*

Unclamp the vernier plate, set the vertical limb to the 0 of the vernier, and make the horizontal limb approximately horizontal. Then bring the vertical telescope over two of the leveling screws, and by means of them bring the bubble of the level to the middle. Unloop the telescope and lifting it from the Y's, turn it end for end; if the bubble does not remain in the middle of the tube make half the correction by means of an adjusting screw at one extremity of the level, and the other half by the leveling screws. Repeat this operation continually, till the bubble stands in the middle of the tube in both positions of the telescope; then revolve the telescope slowly in the Y's. and see if the bubble remains in the middle; if it does not, make the correction by means of two lateral antagonistic screws at one end of the level; continue the trial until the bubble is found to remain in the middle of the tube when the telescope is revolved in the Y's; when this is the case the second adjustment is complete.

3d Adjustment. *To make the axes of the levels on the limb, perpendicular to the axis of the instrument.*

Having set the 0 of the vertical limb at the 0 of the vernier, make the horizontal limb approximately horizontal. Turn the vernier plate, in azimuth till the telescope comes over two of the leveling screws; then by means of three screws bring the bubble to the middle of the tube. Turn the vernier plate in azimuth 180°, and if the bubble is not in the middle make half of the correction by the tangent screws of the vertical limb, and half by the leveling screws. Then turn the vernier plate 180°, and correct as before. Continue this operation till the bubble remains in the middle, in both positions. Then will one line of the limb be horizontal. Turn the telescope over the other leveling screws, and by the same process make a second line of the limb horizontal. Turn the vernier plate, in azimuth, through 360°, and if the bubble remains in the middle during the entire revolution, the limb is truly level and its axis vertical. Bring the bubbles of the levels, on the limb, to the middle of their tubes. by means of adjusting screws at their extremities, and they will then have their axes at right angles to the axis of the theodolite, and the third adjustment will be complete.

4th Adjustment. *To make the axis of the vertical limb perpendicular to the axis of the instrument.*

The horizontal limb being leveled, direct the cross hairs of the telescope upon a corner of a house, a plumb line, or other vertical object, and turn the vertical limb about its axis: if the cross wires remain upon the line, the adjustment is complete; if not, and the line described by the cross wires inclines to the left, the right hand end of the axis is too high and must be lowered by

means of adjusting screws at the foot of the framework. If the line inclines to the right the reverse is the case, and the right hand end of the axis must be raised. Continue the trial till the line described by the cross wires is vertical; the adjustment is then complete. After all of the adjustments are made they should be examined in succession, lest some of them should have been disturbed in making the others.

To measure a horizontal angle with the theodolite. Plant the instrument exactly over the angular point by means of a plumb line attached to a hook immediately under the centre of the horizontal limb, and by the aid of the leveling screws bring the bubble of the levels on the limb to the middle of their tubes; the limb will then be horizontal. Clamp the vernier plate and unclamp the horizontal and vertical limbs; direct the telescope upon the left hand signal or object as nearly as possible with the hand, then clamp the horizontal limb, and bring the intersection of the cross wires exactly to coincide with the object by means of the tangent screw; take the reading of the instrument and record it. Unclamp the vernier plate and turn the telescope upon the right hand signal or object, as nearly as possible, and clamp the vernier plate; bring the cross hairs into exact coincidence with the object by means of the tangent screw of the vernier plate, and again take the reading. If the 0 of the vernier has not passed over the 0 of the limb, the difference of the readings is the angle subtended by the signals or objects; if the 0 of the vernier has passed the 0 of the limb, subtract the greater reading from 360°, and add to the result the lesser reading; this will be the angle required. In taking the reading, the minutes and seconds should be read at both verniers and a mean of the results taken. This corrects for a want of exact centering of the graduated circle.

To measure an angle by repetition. Direct he telescope on the left hand object, as before, and take the reading; unclamp the vernier plate and direct the telescope on the right hand object, as before: then leaving the vernier plate clamped, unclamp the limb and direct it again on the left hand object; then, leaving the limb clamped, unclamp the vernier plate and direct the telescope again upon the right hand object. Continue this operation till the telescope has been directed upon the right hand object any number of times, say six; then take the reading of the instrument, add to this reading as many times 360° as the 0 of the vernier has passed the 0 of the limb during the repetition, and from the result subtract the first reading; divide this difference by the number of repetitions, and the quotient will be the angle required. This operation of repetition, if carefully performed, has a tendency to correct errors in graduation, and also in reading the instrument. The errors, in this case, are divided by the number of repetitions. If, now, the telescope be reversed in its bearings, and a second set of six repetitions be taken, a second value of the angle will be obtained; the mean of this and the first result will probably be more nearly accurate than either of them.

The principle of repetition, is now generally adopted in all measurements where great accuracy is required.

To measure a vertical angle, either of elevation or depression. Having planted the instrument and leveled it as before, direct the cross hairs on the object whose elevation or depression is to be measured; take the reading of the instrument and record it: then reverse the telescope in the Y's, turn the vernier plate in azimuth 180°, and again direct the telescope upon the same object as before, and take the reading; a mean of this and the one already found, will be the angle included between a horizontal line through the axis of the vertical limb and a line drawn from that axis to the point observed. The difference of the readings forms what is called the *index error* of the instrument, and when this is known, a single reading of the instrument will suffice to give the angle of elevation. In order to determine the index error, a great number of readings should be taken directly and in reverse, and a mean of all the index errors taken for the true index error.

THE′O-REM. [Gr. θεωρημα; from θεωρεω, to contemplate]. A statement of a principle to be demonstrated; that is, the truth of which is required to be made evident by a course of reasoning, called a demonstration. In the synthetical method of investigation, which is that for the most part employed in

Geometry, it is usual to state the principle to be proved before commencing the demonstration; the demonstration proceeds by a regular course of argumentation to the final conclusion, which is confirmatory of the principle enunciated. The principle being proved, it may properly be employed as a premiss in the deduction of new truths. The principle, as enunciated before the demonstration, is the *theorem*; its statement after demonstration, constitutes a *rule* or *formula*, according as the statement is made in ordinary or in algebraic language. When the principle enunciated is of such a nature, that its statement is not properly a rule, it constitutes a demonstrable truth. In Analysis, it is not customary to enunciate the principle to be proved; but by a proper combination of suitable formulas or equations, an equation is reached, which by interpretation, shows the truth of the principle to be proved. In this case, where the principle is not stated till proved, its statement is not properly called a theorem.

In Analysis, it is sometimes the case, that the object to be attained is indicated before commencing the demonstration, by simply stating what that object is, leaving the complete statement of the principle to be given as a conclusion of the reasoning employed. This statement, with the implied conclusion, constitutes a theorem. Thus, we say that the object of the binomial theorem is to deduce a formula for forming any power of a binomial, without going through the process of continued multiplication, and without stating the nature of the formula, the demonstration is entered upon, and as a result, the equation expressing the law of formation is deduced. Of this nature are most of the theorems of analysis, such as Taylor's theorem, McLaurin's theorem, Lagrange's theorem, &c.

The common definition of a theorem given in Geometry is, that it is a proposition to be proved. A theorem is distinguished from a problem in this; the latter being a statement of something to be done. Thus, the statement of the fact, that a circle is to be inscribed in a triangle, is a problem, or perhaps more correctly, the statement of a problem.

THĒ′O-RY. [L. *theoria*; Gr. *θεωρια*; from *θεωρεω*, to see, to contemplate]. A technical term in Mathematics, denoting an exposition of the principles of any science, or particular branch of science. Thus, the theory of equations embraces a complete discussion of all the principles employed in the transformation and solution of equations, together with an investigation of the various properties of the equations, their derivations, roots, &c. The theory of negative exponents includes a complete discussion of the nature, properties, and uses of negative exponents; it shows the transformations that may be made upon quantities affected with such exponents, and develops the relation between these transformations, and analagous ones as made upon those affected with positive exponents.

The term, *theory*, is sometimes used in contradistinction to *practice*. Thus, the theory of Arithmetic embraces the investigation of the properties and relations of numbers, whilst practice shows the best methods of applying these principles to meet the ordinary wants of life.

Theory is not to be confounded with hypothesis. Theory, as considered in Mathematics, is founded on principles deduced; hypothesis is a mere assumption which may, on investigation, be found either true or false. See *Demonstration*, and *Reductio ad absurdum*.

THICK′NESS. A name given to one of the dimensions of a solid, usually the lesser dimension. Thus, the thickness of a board is its dimension from face to face.

THREE′-SĪD-ED. Having three sides; thus, a triangle is a three-sided figure.

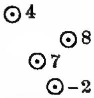
TĪDE-GĀUGE. A contrivance for measuring the height of the tide at any instant.

TO-POG′RA-PHY. [Gr. *τοπος*, place, and *γραφω*, to describe]. A description of the form of the surface of a limited portion of the earth's surface, whether the description be made verbally or by a graphic delineation. It also embraces a description of the natural objects that are found upon the surface, such as rocks, trees, &c., together with all constructions, as roads, streams, bridges, towns, &c.

TOP-O-GRAPH′IC-AL. Appertaining to Topography.

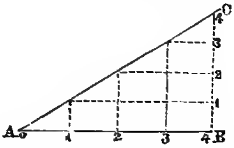
TO-POG'RA-PHER. One who makes a topographical survey.

TOPOGRAPHICAL PROJECTION. A species of projection chiefly employed in representing the contour of ground. In it but one plane of projection is used; it is called the *plane of reference*. We shall consider the plane of the paper as the plane of reference. Points are given by their orthographic projections and numbers expressing their distances from the plane of reference in terms of some assumed unit. For the sake of explanation, we shall take the foot as the unit of the scale; then will the figure represent the points, which are respectively 4, 8, and 7, &c., feet, above the plane of reference. The number written by the side of the projection are called references of the points.

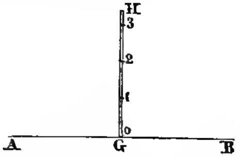


If the references are negative the point is situated below the plane of reference. Thus, the reference -2 indicates that the point is 2 feet below the plane of reference.

Straight lines are given by their orthographic projections and by a scale. To understand the nature of the scale, let AC represent a line in space, AB its projection on the

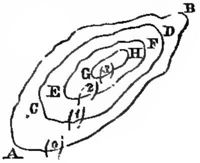


plane of reference, and CB a perpendicular to AB equal, say to 4 feet, on the scale of the drawing. Divide CB into 4 equal parts, and through the points of division draw lines parallel to the line AB, and from the points in which these lines intersect AC, let fall perpendiculars upon AB; mark the points 1, 2, 3, 4, &c., and the line thus divided is called the scale of the line AC. and after one division is found, the scale may be continued to any extent in both directions. The 0 point is where the line pierces the plane of reference. Each division of the scale is equal to the unit of the scale divided by the tangent of the inclination of the line to the plane of reference. It is often necessary, in this kind of projection, to use two different scales; one for horizontal distances, and the other for vertical distances. Suppose, for example, that the vertical scale is ten times as great as the horizontal scale, then will one division on the scale of a line be equal to one-tenth of a unit on the vertical scale divided by the tangent of the inclination of the line; and, in like manner, for any other vertical scale We shall discuss only the cases in which the two scales are equal, leaving the modifications necessary, on account of difference of scale, to be made in accordance with the principle just laid down. Planes are given by their traces and scales of inclination. The scale of inclination of a plane is the same as the scale of a straight line in the plane perpendicular to the trace.



In the figure the plane AB is given, because its trace AB is given, and its scale GH is known.

Curve lines, which are parallel to the plane of reference, are given by their projections and the reference of one point. Thus, the lines AB, CD, EF, and GH, are respectively the representations of lines of the same form as the projections at distances of 0, 1, 2 and 3 feet from the plane of reference.



Horizontal curves are the principal ones considered in this kind of projection. When other curves are given, their projections are given, and the references of a sufficient number of points. Surfaces are given in this kind of projection by the projections of horizontal sections of the surfaces made by planes at equal distances from each other, the reference of each being given; thus, the surface of a hillock is shown in the figure above given.

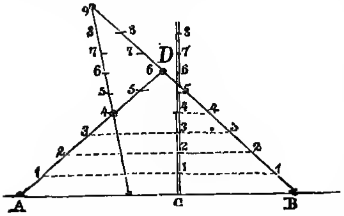
We shall illustrate the method of using this kind of projection by giving a few graphical solutions of problems that may arise in its use.

1. Having given two points, to draw a straight line through them, and find its scale.

Let the given points be 4 and 2. Draw the straight line 4 2, and divide it into two equal parts; set off one of these parts twice to the

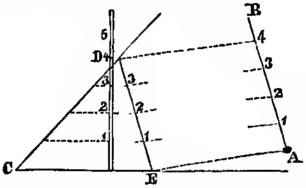
left of 2, and mark the points of division 1, 0; then is 0 the point in which the line pierces the plane of reference, and 0, 1, 2, 3, &c., is the scale of the line.

2. To pass a plane through three points:



Let 9, 6, and 4, be the given points. Draw straight lines through 9, 6, 9 and 4, respectively, by the last problem, and find the points in which they pierce the plane of reference. Join these points by a straight line AB; this will be the trace of the plane on the plane of reference. Draw any line CD, perpendicular to AB, and join the points 1, 1; 2, 2; 3, 3, &c., of the scales of the two lines: their intersection with the line CD will form the scale of the plane.

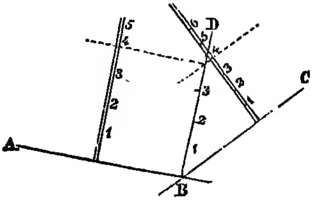
3. To pass a plane through one straight line parallel to another:



Let AB and CD be the two lines: through 4, on the line CD, draw a line DE parallel to and equal to 4A, and join CE: this will be the trace of the required plane, and its scale may be constructed as in the last problem.

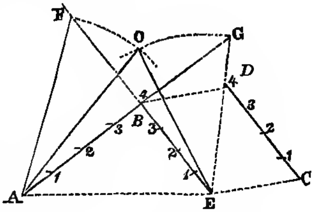
4. To find the intersection of two given planes:

Let AB and BC be the given planes; through the points 4, on each scale, draw lines parallel to the traces of the respective planes, and join their point of intersection with the point, B, in which these traces intersect; this will be the projection of the intersection: to find its scale, draw through 1, 2, 3, &c., on the scale of the plane BC



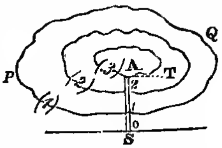
lines parallel to its trace, and their intersections with BD will determine the divisions of the scale.

5. To find the angle included between two straight lines. This angle is equal to the angle included between either line, and a straight line drawn through any point of it parallel to the other.



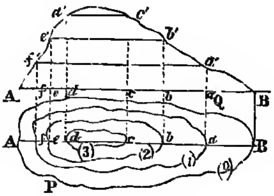
Let AB and CD be the two given lines. Draw through the point 4, on the line AB, a straight line parallel and equal to 4C; join AE. At 4 erect the perpendiculars BF and BG, and make each equal to 4 units of the scale; join GE and FA: with E and A as centres, and EG and AF as radii, describe arcs intersecting at O, and draw OA and OA; then is AOE the angle required.

5. To pass a plane tangent to a given surface at a given point. Let PQ be the topographical representation of the surface, and A the point whose reference is 3.



Draw AT tangent to the curve (3), and AS perpendicular to it at A; take the distance, A2, for the unit of the scale, and set it off three times towards S, and through the extremity 0 draw a line parallel to AT; this is the trace of the plane, and its scale has already been constructed.

6. To find a section of a surface given topographically. Let PQ represent the surface, and AB the trace of the vertical plane



along which the section is to be made. Draw an indefinite line AB, and lay off on it the distances, Ba, ab, bc, &c., as in the figure; at these points, erect perpendiculars and make aa', ff', each equal to 1, bb', ee'' each equal to 2, and cc', dd', each equal to 3; draw the curved line A$f'e'd'c'b'a'$B, and it will represent the required section. These examples serve to show the facility with which problems may be solved by this method of projection. See *Topography*.

Topographical Surveying. That branch of surveying which is undertaken for the purpose of making a topographical description of a part of the earth's surface. It includes the conventional methods of making the representation on paper, or topographical drawing. We shall only touch upon those points which have not been treated of in other articles, omitting all those operations which have been sufficiently described under the heads of *Geodesy*, *Surveying*, *&c.*

The representation to be made, is the horizontal projection of the objects upon the surface, together with the projections of a system of lines cut out of the surface by a system of horizontal planes passed at equal distances apart, usually, 3 or 6 feet.

The method of finding and projecting these *contour lines*, as they are called, is sufficiently explained under leveling. See *Leveling*. After the horizontal projections of the lines of contour are plotted, the drawing is sometimes shaded by lines drawn at right angles to them, according to a conventional rule; where the surface is steep, the shading lines are drawn close together: where the declivity is gentle, they are drawn further apart.

The method of finding and plotting the contour lines, as described in *leveling*, is only employed when considerable accuracy is required, as in surveying a site for a fortification, or for the construction of a building or other improvement. When only a general topographical map of a country is wanted, it will, in general, be found sufficient to survey the country with reference to its fields, roads, rivers, &c. These are to be plotted in the manner indicated for filling in a geodesic survey. Levels are run along the principal lines, as fences, roads, &c., and the highest of the most prominent points of the country are determined with respect to some plane of reference. Then the general outlines of the topography are sketched in by the eye; after the general outline is finished, the principal objects worthy of note are represented by a system of conventional signs.

TŌ'TAL. [L. *totus*, the whole]. The whole sum or amount. The aggregate of several particulars.

TOUCH. To meet, or be in contact with. See *Contact*, *Tangent*.

TRĀCE [L. *traho*, to draw,] of a Plane. In Descriptive Geometry, the intersection of the plane with one of the planes of projection. The trace on the vertical plane is called the *vertical trace*; that on the horizontal plane, the *horizontal trace*. If the plane is not parallel to the ground line it will intersect it at some point, which point will be common to both traces. Since two lines of a plane fix its position, if the traces of a plane are known, the plane is said to be known, that is, a plane is given by its traces. If the plane is parallel to the ground line, its traces are both parallel to the ground line; conversely, if both traces are parallel to the ground line the plane itself is parallel to the ground line. If the plane is parallel to the horizontal plane of projection, it will have no trace on that plane, and its vertical trace will be parallel to the ground line. If the plane is parallel to the vertical plane of projection, it will have no trace on that plane, and its trace on the horizontal plane will be parallel to the ground line. If a plane is perpendicular to the vertical plane of projection, its horizontal trace will be perpendicular to the ground line. If a plane is perpendicular to the horizontal plane of projection, its vertical trace will be perpendicular to the ground line. If

a plane is perpendicular to the ground line, both of its traces will be perpendicular to the ground line.

In Analysis, there are three planes of projection, called co-ordinate planes, and a plane will in general have three traces. To find the equation of the trace of a given plane upon the plane XY, make $z = 0$, in its equation; the resulting equation will be the equation required. To find the equation of the trace upon the plane XZ, make $y = 0$, in the equation of the plane; the resulting equation will be the equation required. To find the equation of the trace upon the plane YZ, make $x = 0$, in the equation of the plane; the resulting equation will be the equation required. Two planes are parallel when their traces on the three co-ordinate planes are respectively parallel.

TRAM′MEL. An instrument used by carpenters for constructing an ellipse, mechanically, by a continuous movement. For an account of the construction of the instrument and the method of using it, see *Ellipse*.

TRANS-CEND-ENT′AL. [L. *transcendo*, to exceed, to go beyond]. A transcendental quantity, is one which cannot be expressed by a finite number of algebraic terms: that is, by the ordinary operation of algebra, viz.: addition, subtraction, multiplication, division, raising to powers denoted by constant exponents, and extraction of roots indicated by constant indices.

TRANSCENDENTAL QUANTITIES are of three kinds, *logarithmic*, *exponential*, and *trigonometrical*. The first are expressed in terms of logarithms, as

$$\log \sqrt{1 - x}, \quad a \log x, \text{ \&c.};$$

the second are expressed by means of variable exponents, as

$$a^x, \quad e^{ax}, \quad ba^xc^y, \text{ \&c.};$$

the third are expressed by means of some of the trigonometrical functions, as

$$\sin x, \tan \sqrt{2 - x^2}, \quad \text{ver-sin } (ax - b), \text{ \&c.}$$

TRANSCENDENTAL EQUATION. An equation expressing a relation between transcendental quantities.

TRANSCENDENTAL FUNCTION. A function in which the relation between the function and variable is expressed by means of a transcendental equation.

TRANSCENDENTAL LINE. A line whose equation is transcendental.

TRANS-FORM′. [L. *trans* and *forma*] To change the form.

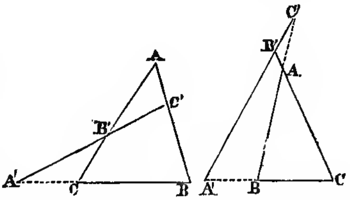
TRANSFORMATION OF AN EQUATION. The operation of changing the form of an equation without destroying the equality of its members. All the operations performed upon equations, in order to simplify them or to solve them, are transformations. See *Equation*.

TRANSFORMATION OF A FRACTION. The operation of changing its form without changing its value. The operations of reducing to simplest terms, of changing the fractional unit, &c., are transformations. See *Fraction*.

TRANS-PĀR′ENT. [L. *trans*, through, and *pareo*, to appear]. A body which will permit light to pass through it freely; used in Shades and Shadows.

TRANS-PŌSE′. [L. *trans*, over, and *pono*, to put]. To transpose a quantity from one member of an equation to another, is to pass the quantity from one member to the other without destroying the equality of the two members. This is done by simply changing its sign.

TRANS-VERS′AL. [L. From *trans* and *versus*, lying across]. A straight line which cuts several other straight lines, is said to be transversal with respect to them. Thus, if a straight line be drawn cutting the three sides of a triangle, or the sides produced, it is a transversal. The following are some of the properties arising from the relation between transversals and the lines which they intersect.



Let ABC be a plane triangle, and B′A′ a transversal cutting the three sides, or sides produced, in the three points A′, B′ and C′; then of the six segments formed, the product

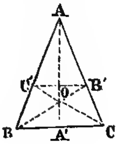
of any three alternate ones is equal to the product of the remaining three; that is,

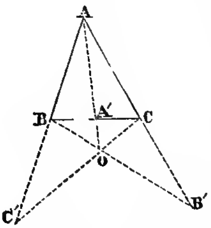
$$AC' \times BA' \times CB' = C'B \times A'C \times B'A.$$

Conversely, if three points A', B' and C' are taken upon the three sides of a triangle, or upon the sides produced, so that the above relation shall exist, the three points are in one and the same straight line.

If from any point O in the plane of a triangle ABC, straight lines be drawn to the three vertices, cutting the sides, or sides produced in the points A', B' and C', then will

$$AC' \times CB' \times BA' = C'B \times B'A \times A'C,$$

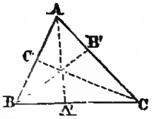




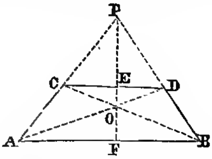
Whatever may be the position of the point O, there is always an odd number of points of intersection upon the sides of the triangle, and an even number upon the sides produced. Conversely, if these points A', B', C', are situated so that an odd number of them are on the sides themselves, and an even number on the sides produced, (0 being considered an even number,) and the above relation exists, then will the straight lines joining them with the opposite vertices of the given triangle pass through the same point. Hence, the straight lines drawn from the vertices of a triangle to the middle of the opposite sides, intersect in the same point. The perpendicular let fall from the vertices upon the opposite sides, intersect in the same point. The lines which bisect the three angles of a triangle, intersect in the same point. If two points, C' and B' be taken upon the sides AB and AC of a triangle, so that

$$AB' : AC' :: AC : AB,$$

and if straight lines BB', CC', be drawn from the vertices of the angles B and C, they will intersect each other on the line AA' drawn from A to the middle point A' of the opposite side; and conversely, if two lines BB' and CC', drawn from the vertices B and C, intersect each other on the line AA', then AB' : AC' : : AC : AB, that is, a straight line joining C' and B' is parallel to BC.



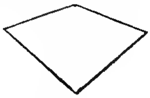
In a trapezoid ABDC, the straight line PO,



drawn through the point P of intersection of the lateral sides, and the point O in which the diagonals intersect, passes through the middle points E and F of the parallel bases CD and AB.

TRANS-VERSE'. [From L. *trans* and *versus*]. The transverse axis of an ellipse or hyperbola, is the axis which passes through the foci. When the length of the transverse axis is referred to, the portion included between the vertices is meant. The real length of the axis is infinite. See *Ellipse*, and *Hyperbola*.

TRA-PĒ'ZI-UM. From Gr. τραπεζιον, a little table]. A quadrilateral, no two of whose sides are parallel to each other. See *Quadrilateral*.



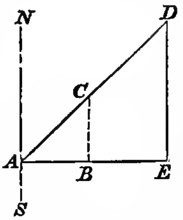
TRAP'E-ZOID. [Gr. τραπεζιον, and ειδος, shape]. A quadrilateral, two of whose sides only are parallel to each other. See *Quadrilateral*.

TRAP-E-ZOID'AL. Having the form of a trapezoid.

TRAV'ERSE. [From L. *trans*, and *versus*]. A line lying in a direction across something else, as a line or figure.

TRAV'ERSE-TĀ'BLE. In Surveying, a table by means of which the latitude and departure of any course can be found by inspection. The table consists of nothing more than a table of sines and cosines of arcs, computed to each quarter of a degree, from 0° to 90°, and for every radius, fom 1 to 100 In the ordinary table the latitude and departure are only carried to two decimal places.

A traverse-table may be computed by means of certain formulas, easily deduced.



Let AD represent any course, NS a meridian through one extremity, DE a meridian through the other extremity and AE an east and west line. The angle NAD equal to ADE, is the bearing. Then, in the right-angled triangle ADE, AE is the departure, and DE the latitude of the course. If we denote the length of the course by c, the latitude of it by l, the departure by d, and the bearing by ϕ, we have

$$l = c \cos \phi \quad \text{and} \quad d = c \sin \phi.$$

If in these formulas we give, in succession, to ϕ values corresponding to every quarter degree from 0 to 45°, and for each value of ϕ give to l every value from 1 to 100 inclusive, and tabulate the results, the table thus formed will be a traverse-table, such as is in common use. It is to be observed that computations need only be carried to 45°, for the latitude of a course corresponding to a given bearing is equal to the departure of the same course corresponding to the complement of the bearing, and the reverse.

Another form of traverse table is sometimes used which is more accurate and but little more difficult to use. The latitudes and departures are computed only to quarter degrees, but are carried to 5 places of decimals. In order to compensate for the increased space occupied, they are only computed for distances from 1 to 9.

We subjoin a specimen of such a table to illustrate its use. The table is constructed as already described.

TRAVERSE TABLE.

Distances.	Bearings.							
	33°		33¼°		33½°		33¾°	
	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.
1	0.83867	0.54463	0.83628	0.54829	0.83388	0.55193	0.83147	0.55557
2	1.67734	1.08927	1.67257	1.09658	1.66777	1.10387	1,66294	1.11114
3	2.51601	1.63391	2.50885	1.64487	2.50165	1.65581	2.49441	1.66671
4	3.35468	2.17855	3.34514	2.19317	3.33554	2.20774	3 32588	2.22228
5	4.19335	2.72319	4.18143	2.74146	4.16942	2.75968	4.15735	2.77785
6	5.03202	3.26783	5.01771	3.28975	5.00331	3.31162	4.98882	3.33342
7	5.87069	3.81247	5.85400	3.83805	5.83720	3·86355	5.82029	3.88899
8	6.70936	4.35711	6.69028	4.38634	6.67108	4.41549	6.65176	4.44456
9	7.54803	4·90175	7·52657	4.93463	7.50497	4.96743	7.48323	5.00013
	57°		56¾°		56½°		56¼°	

To find the latitude and departure of the course N. 33° W., dist. 743 chains:

	Lat.	*Dep.*
700	587.069	381.247
40	33.5468	21.7855
3	2.51601	1.63391
743	623.13181	404.66641

The application is obvious.

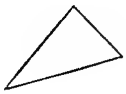
TRĪ′AN″GLE. [L. *triangulum*; *tres*, three, and *angulus*, angle]. A portion of a surface bounded by three lines, and consequently having three angles.

Triangles are either plane, spherical, or curvilinear

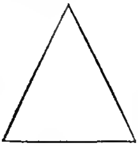
A Plane Triangle is a portion of a plane bounded by three straight lines; these lines are called *sides*, and their points of intersection are the *vertices* of the triangle.

Plane triangles may be classified either with reference to their sides or their angles. When classified with reference to their sides, there are two classes.

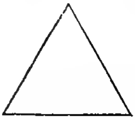
1st. *Scalene triangles*, which have no two sides equal.



2d. *Isosceles triangles*, which have two sides equal. The isosceles triangle has a particular case, called the

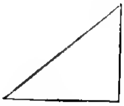


Equilateral triangle, all of whose sides are equal.



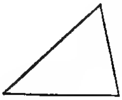
When classified with reference to their angles, there are two classes.

1. *Right angled triangles*, which have one right angle; and

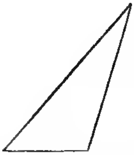


2. *Oblique angled triangles*, all of whose angles are oblique. The latter class is subdivided into

Acute angled triangles, which have all their angles acute; and



Obtuse angled triangles, which have one obtuse angle.



The sides and angles of a triangle are called its *elements*; the side on which it is supposed to stand is called the *base*, and the vertex of the opposite angle is called the *vertex of the triangle*; the distance from the vertex to the base is the *altitude*. Any side of a triangle may be regarded as a base, though in the right angled triangle one of the sides about the right angle is usually taken.

Two plane triangles are *equal to each other*, when they have three elements of the one equal to three elements of the other, each to each, one of the elements being a side. When the three angles only are equal, each to each, the triangles are similar.

The following are some of the most important properties of plane triangles:

1. The greatest side is opposite the great eat angle; the least side is opposite the least angle; and the mean side opposite the mean angle. If two sides are equal, the opposite angles are equal; if the three sides are equal, the triangle is equiangular.

2. Any side is less than the sum of the other two, and greater than their difference.

3. The sum of the angles is equal to two right angles; and if any side be produced, the exterior angle is equal to the sum of the opposite interior angles.

4. If a line be drawn parallel to the base, it will divide the other two sides proportionally, and the triangle cut off will be similar to the whole triangle. The converse property is also true.

5. The line which bisects the vertical angle of a triangle divides the base into two segments, which are proportional to the adjacent sides. If it bisects the exterior angle, the distances from the point in which it cuts the base, produced to the extremities of the base, are proportional to the adjacent sides. These distances are segments.

6. The straight lines which bisect the three angles, intersect in a common point, which is the centre of the inscribed circle. The straight lines which bisect the three sides, and are perpendicular to them, intersect in a common point, which is the centre of the circumscribed circle; hence, two circles can always be constructed, one inscribed within, the other circumscribed about any given triangle.

7. The perpendiculars let fall from the vertices of the three angles upon the opposite sides, intersect in a common point; the straight lines drawn from the vertices to the middle points of the opposite sides, also intersect in a common point; this last point is the centre of gravity of the triangle.

8. If we denote the three angles of a triangle by A, B and C, and their opposite sides respectively by a, b and c, we shall have

when A is a right angle, $a^2 = b^2 + c^2$;
when A is equal to 120°, $a^2 = b^2 + bc + c^2$;
when A is equal to 60°, $a^2 = b^2 - bc + c^2$.

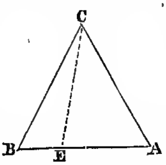
9. If we let a perpendicular fall from the vertex of the angle C, upon the side opposite,

and denote the distance from its foot to the vertex of the angle A, by d, we shall have

when A is obtuse, $a^2 = b^2 + c^2 + 2cd$;
when A is a right angle, $a^2 = b^2 + c^2$; and
when A is acute, $a = b^2 + c^2 - 2cd$.

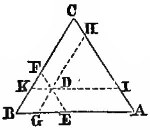
10. If a straight line be drawn from the vertex to the middle of the base, the sum of the squares of the other two sides will be equivalent to twice the square of the line drawn, *plus* twice the square of half the base.

11. If in an isosceles triangle a straight line be drawn from the vertex to any point E of the base, we have



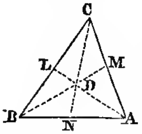
$AC^2 = CE^2 + AE . EB$.

12. If through any point D, in the plane of a triangle, three lines EF, GH and IK be drawn respectively parallel to the three sides; then



$DE \times DK \times DH = DG \times DF \times DI$.

13. If through any point D in the plane of a triangle, three lines CN, BM and AL be drawn to the vertices of the triangle; then



$AN \times BL \times CM = AM \times CL \times BN$.

14. In a right angled triangle, if a perpendicular is drawn from the vertex of the right angle to the hypothenuse,

1st. The triangles formed are similar to each other and to the whole triangle.

2d. Either side about the right angle is a mean proportional between the hypothenuse and the adjacent segment.

3d. The perpendicular is a mean proportional between the two segments.

15. In any triangle the rectangle of the two sides about the vertex is equivalent to the rectangle of the diameter of the circumscribed circle and the altitude.

16. The area of a triangle is equal to the product of its base by half its altitude; hence, triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases; and in general they are to each other as the product of their bases and altitudes. Similar triangles are to each other as the squares of their homologous sides.

17. Two triangles which have an angle equal in each, are to each other as the products of the including sides.

The area of a plane triangle is equal to one half the product of its base and altitude. If we denote the base by b, the altitude by h, and the area by S, the sides and angles being denoted as before, we have the following formulas for finding the area:

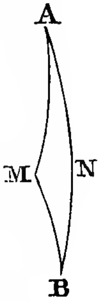
1. $S = \frac{1}{2} ph$;

2. $S = \frac{1}{2} bc \sin A$;

and by making $a + b + c = s$,

3. $S = \sqrt{\frac{1}{2}s\,(\frac{1}{2}s - a)\,(\frac{1}{2}s - b)\,(\frac{1}{2}s - c)}$.

A SPHERICAL TRIANGLE is a portion of the surface of a sphere, bounded by the arcs of three great circles. The arcs are *sides*, their points of intersection *vertices*, and the angles which they form with each other, *angles* of the triangle.



We shall regard the radius of the sphere as 1; in this case, the sides of the triangle will measure the angles at the centre of the sphere which they subtend. The angles of the triangle are the same as those made by the planes of its sides, and we shall also regard them as measured by the arcs of circles having the radius 1. Hence, the sides and angles may be readily compared with each other, being referred to a common unit, and may be referred to indifferently, as arcs or angles.

The angles and sides are called elements, and for the sake of clearness and simplicity, each element will be regarded as less than two right angles, or 180°.

This limitation is necessary to enable us to distinguish between the two portions of the surface of the sphere, both of which are bounded by the same three arcs. By confining the value of the elements within the limits of 180° each, the lesser triangle will always be the one considered. When no limitation is placed upon the elements, they may have any value up to 360°, and the triangle is then called the *general spherical triangle*

Two elements are of the *same species*, when both are less, or both greater than 90°, and of *different species*, when one is greater, and the other less than 90°.

Spherical triangles take the names, *right-angled, obtuse-angled, acute-angled, scalene, isosceles,* and *equilateral,* in the same cases as plane triangles.

A spherical triangle is *birectangular*, when it has two right angles, and *trirectangular*, when it has three right angles. A trirectangular triangle is one-eighth of the surface of the sphere, and is taken as the unit of measure for polyhedral angles.

Two spherical triangles are *polar*, when the angles of the one are supplements of the sides of the other, taken in the same order.

A spherical triangle is *quadrantal,* when one of its sides is equal to 90°.

The following are some of the most important properties of spherical triangles:

1. The greatest side is opposite the greatest angle; the least side opposite the least angle; the mean side opposite the mean angle; and equal sides opposite equal angles. If the three angles are equal, the sides are also equal, and the triangle is equilateral. The converse is also true.

2. Any side is less than the sum of the other two, and greater than their difference.

3. The sum of the angles is always greater than two right angles, and less than six right angles.

4. The sum of any two angles is greater than the supplement of the third.

5. The difference of any two sides is less than two right angles, and the sum of the three sides is less than four right angles.

6. If the sum of any two sides is equal to two right angles, the sum of their opposite angles is also equal to two right angles, and conversely.

7. If the angles are all acute, all obtuse, or all right angles, the sides will be less than, greater than, or equal to, a right angle.

8. Spherical triangles are equal, when they have three elements of the one equal to three elements of the other, each to each, or in the same order.

When the elements of one triangle are respectively equal to the elements of another, but not taken in the same order, the triangles are *symmetrical*, and their areas are equivalent, though incapable of superposition.

The area of a spherical triangle is equal to the area of a trirectangular triangle, multiplied by the ratio of a right angle to the excess of the sum of the three angles over two right angles.

Denoting the angles by A, B and C, expressed in degrees, the area of the trirectangular triangle by T, and that of the given triangle by S, we shall have the following formula,

$$S = T \times \frac{(A + B + C - 180°)}{90°} \; \ldots\ldots (1).$$

If the angles are expressed in terms of the right angle, as a unit, the formula becomes

$$S = T \times (A + B + C - 2) \; \ldots\ldots (2).$$

Formula (1) is more practically useful.

A CURVILINEAR TRIANGLE is one whose sides are curved lines of any kind whatever; as, a spheroidal triangle, lying on the surface of an ellipsoid, &c.

TRĪ-AN″GLED. Having three angles.

TRĪ-AN″GU-LAR. Having three angles. *Triangular Numbers.* See *Figurate Numbers, Numbers, &c.*

TRĪ-AN″GU-LĀ′TION, in Surveying. The operation of measuring the elements necessary to determine the triangles into which the country to be surveyed is supposed divided. The term is principally used in geodesic surveying. See *Geodesy.*

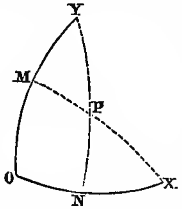
TRĪ′GON. [Gr. τρεις, three, and γωνια angle]. A polygon of three sides. See *Triangle.*

TRIG-O-NO-MET′RIC, TRIG-O-NO-MET′RIC-AL, TRIG-O-NO-MET′RIC-ALLY. Pertaining to trigonometry, or according to the principles of trigonometry. See *Trigonometry.*

TRIGONOMETRICAL CO-ORDINATES, OR SPHERICAL CO-ORDINATES. Elements of reference by means of which the position of a point on the surface of a sphere may be determined with respect to two great circles of the sphere.

Let O be any point on the surface of the sphere, assumed as an origin; OX and OY, arcs of two great circles, assumed as co-ordi-

nate axes. Take OX and OY, each equal to a quadrant, and suppose P to be any point whose position is to be fixed; then, the trigonometrical co-ordinates of P are the tangents of OM and ON: these tangents are designated by the letters x and y, as in rectilineal co-ordinates. It is plain, that if the whole system be projected upon a plane, tangent to the surface of the sphere at O, by lines drawn from the centre of the sphere, OM and ON will be projected into the rectilineal co-ordinates of the projection of P, the projections of the spherical co-ordinate axes being the rectilineal axes.



The equation of a great circle of the sphere, referred to this system, is

$$ax + by + c = 0;$$

in which a, b and c, are constants, depending upon the angles made by its plane with the planes of the axes.

Trigonometrical Series. In addition to the series already considered, under the head of *Series*, there are certain other series of the form,

$$a \sin x + b \sin 2x + c \sin 3x + \&c.,$$

and

$$a \cos x + b \cos 2x + c \cos 3x + d \cos 4x + e \cos 5x + \&c., \&c.,$$

which seem to merit attention. These series are of use in the higher branches of mathematics. The summation of the series, when $a + bz + cz^3 + \&c.$ can be summed, is easy.

For example, let it be required to find the sum of the series

$$1 + a \cos x + a^2 \cos 2x + a^3 \cos 3x + \&c:$$

Assume

$$2 \cos x = z + z^{-1};$$

then,

$$2 \cos nx = z^n + z^{-n};$$

and by substitution and reduction, we find the sum of the series equal to

$$\frac{1}{2(1-a^2)} + \frac{1}{2(1-az^{-1})}; \quad \text{or,}$$

$$\frac{2 - a(z + z^{-1})}{2(1 - a(z + z^{-1}) + a^2)};$$

and by resubstitution, we have for the sum of the given series,

$$\frac{1 - a \cos x}{1 - 2a \cos x + a^2}.$$

The most remarkable property of these series is, that they are capable of representing the ordinates of points of discontinuous lines. This property is of use in the higher branches of Physics.

TRIG-O-NOM'E-TRY. [Gr. τριγωνος, a triangle, and μετρεω, to measure]. That branch of mathematics which has for its object, to show the method of determining the remaining parts of a triangle, when a sufficient number is given or known. It treats also of the general relations which exist between the trigonometrical functions of angles or arcs.

Trigonometry is divided into three branches, *Plane*, *Spherical* and *Analytical*.

Plane Trigonometry treats of the relations existing between the sides and angles of plane triangles. The principal object of plane trigonometry is to show the methods of solving plane triangles; that is, the method of finding the remaining parts of a plane triangle, when three are given, one of the three being a side.

Spherical Trigonometry treats of the relations existing between the sides and angles of spherical triangles. The principal object of this branch is to show the method of solving spherical triangles; that is, the method of finding the remaining parts of a spherical triangle, when any three are given.

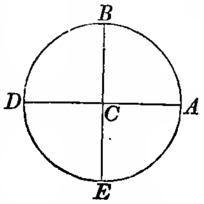
Analytical Trigonometry treats of the general relations and properties of angles, and trigonometrical functions of angles.

We shall consider each of these branches separately, having first explained the meaning of the terms employed, and the conventional principles adopted, in the discussion of the subject.

Definitions and Conventional Principles.

We shall consider the radius of the trigonometrical circle as 1, in which case, the sines, cosines, tangents, &c., of angles and of arcs, may be regarded as identical. For the purposes of trigonometry, the circumference of the circle is divided into four equal parts, by two diameters perpendicular to each

other; each part is called a quadrant. One of these diameters, BA, is taken horizontal, and the other, DE, is vertical, the plane of the circle being regarded as vertical. The right hand extremity of the horizontal diameter, A, is taken as the origin of arcs; or the radius CA is taken as the origin of angles, whose vertices are at C. This line, CA, is called the *initial diameter*, and the diameter, BE, is the *secondary diameter*. Arcs will be regarded as positive when estimated around to the left from the origin, that is, in a direction contrary to the motion of the hands of a watch, and consequently, they must be regarded as negative, when estimated in a contrary direction. AB is called the *first* quadrant; BD, the *second*; DE, the *third;* and EA, the *fourth*. When an angle or arc terminates in either of these, it is said to fall in that quadrant, or to lie in that quadrant.



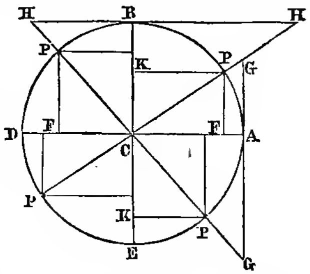
For the purposes of trigonometrical computation, each quadrant or right angle is supposed to be divided into 90 equal parts, called *degrees;* each degree into 60 equal parts, called *minutes;* each minute into 60 equal parts, called *seconds;* so that a right angle contains 32400 seconds.

The *complement* of an angle, or arc, is the result obtained by subtracting it from 90°. If the angle or arc is greater than 90°, the complement is *negative;* if the arc is negative, its complement is greater than 90°.

The *supplement* of an angle, or arc, is the result obtained by subtracting it from 180°. If the angle or arc is greater than 180°, the supplement is *negative;* if the arc is negative, the supplement is greater than 180°.

In what follows, we shall always suppose the tangent to be drawn through the origin A, and the co-tangent to be drawn through the point B, 90° distant from A.

The *sine* of an arc is the distance from the initial line to the second extremity of the arc; thus, FP is the sine of the arc AP. The *cosine* of an arc is the distance from the secondary diameter to the second extremity of the arc; thus, KP, or CF, is the cosine of the arc, AP. The *versed sine* is the distance from the foot of the sine to the origin of arcs;



thus, FA is the versed sine of the arc AP. The *co-versed sine* is the distance from the foot of the cosine to the upper extremity of the secondary diameter. Thus, KB is the co-versed sine of the arc AP.

The *tangent* of an arc is that portion of the tangent included between the origin of arcs and a diameter drawn through the second extremity of the arc; thus, AG is the tangent of the arc AP. The *co-tangent* of an arc is that portion of the co-tangent included between the upper extremity of the secondary diameter, and a diameter drawn through the second extremity of the arc; thus, BH is the co-tangent of the arc AP. The *secant* of an arc is the distance from the centre of the circle to the extremity of the tangent of the arc; thus, CG is the secant of the arc AP. The *co-secant* of an arc is the distance from the centre of the circle to the extremity of the co-tangent; thus, CH is the co-secant of the arc AP. The lines or distances thus defined, are called the trigonometrical functions of the arc AP.

It has been agreed to consider all distances from the initial line, estimated upwards, as *positive;* consequently, all distances from this line downwards, must be regarded as *negative*. It has been agreed to consider all distances estimated from the secondary diameter to the right, as *positive;* consequently, all distances estimated from this diameter to the left, must be regarded as *negative*. It has been agreed to consider all radial distances estimated from the centre towards the second extremity of the arc, *positive;* consequently, those estimated from the centre, in an opposite direction, must be regarded as *negative*.

From a consideration of these conventional

principles, we deduce the following table, showing the signs of the trigonometric functions in the several quadrants.

	1st. Quadrant.	2d. Quadrant.	3d. Quadrant.	4th. Quadrant.
Sine, . . .	+	+	—	—
Cosine,	+	—	—	+
Versed sine, .	+	+	+	+
Co-versed sine,	+	+	+	+
Tangent, . .	+	—	+	—
Co-tangent,	+	—	+	—
Secant, . .	+	—	—	+
Co-secant,. .	+	+	—	—

In order that the above principles may be applicable to an angle, we regard one side of the angle as an initial line, and with the vertex as a centre, and a radius equal to 1, we suppose a circle to be described; that portion intercepted between the two sides is taken as the measure of the angle, and the trigonometrical functions of this arc are the corresponding functions of the angle. The co-functions of an angle, that is, cosine, co-versed sine, co-tangent, co-secant. are respectively equal to the sine, versed sine, tangent, and secant of the complement of the angle. The term *co*, being an abbreviation for the phrase "*of the complement.*"

The following equations show the relations existing between the trigonometric functions of a negative arc, and the corresponding functions of a positive arc, numerically equal to it. Denoting the positive arc by a, we have

$\sin(-a) = -\sin a$; $\cos(-a) = \cos a$;
$\text{vers-sin}(-a) = \text{ver-sin}\, a$; $\tan(-a) = -\tan a$;
$\cot(-a) = -\cot a$; $\sec(-a) = \sec a$;
$\text{co-sec}(-a) = -\text{co-sec}\, a$.

If we suppose the arc to increase continuously from 0° to 90°, we shall have the following results.

The sine. tangent, and secant, increase algebraically with the arc, and the remaining functions decrease algebraically as the arc increases. The versed sine increases numerically for 0° to 180°, and decreases numerically from 180° to 360°. The co-versed sine decreases to 90°, increases to 270°, and decreases to 360°.

The remarks that have been made, apply to plane angles, but they are also equally applicable to spherical angles, if we adopt the principle, that a spherical angle is the same as the plane angle included between two tangents drawn to its sides at the vertex.

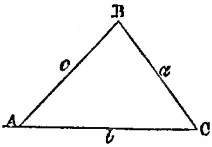
The following table shows some of the particular values of the trigonometrical functions:

NO. DEG.	SINE.	COS.	TAN.	COT.	SEC.	COSEC.
0°	0	+1	0	+∞	+1	+∞
90	+1	0	+∞	0	+∞	+1
180	0	−1	0	−∞	−1	−∞
270	−1	0	−∞	0	−∞	−1
360	0	+1	0	+∞	+1	+∞

The functions of any of these arcs, increased by any number of times 360°, remains the same.

Plane Trigonometry.

In every plane triangle, there are six parts or elements—three angles and three sides. We shall designate the angles by the letters A, B and C, and the sides opposite these angles by the letters a, b and c, respectively. It is to be observed that the letters may be changed, so that either of the capital letters used may represent either angle of the triangle.



When any three parts of a plane triangle are given, one of which is a side, the remaining parts may be found, and the operation of finding them is called *solving the triangle.*

By the aid of the following formulas, all plane triangles may be solved. In them the sum of the three sides is denoted by s, or

$$s = a + b + c;$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \cdots\cdots\cdots (1),$$

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)} \cdot (2);$$

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(\frac{1}{2}s-b)\,(\frac{1}{2}s-c)}{bc}};$$

$$\cos\tfrac{1}{2}A = \sqrt{\frac{\frac{1}{2}s(\frac{1}{2}s-a)}{bc}} \cdots\cdots\cdots (3).$$

If the triangle is right angled at A, the

formulas used in the solution are the following:

$$\sin B = \frac{b}{a};\ \cos B = \frac{c}{a};\ \tan B = \frac{b}{c} \cdot\cdot (4);$$

$$b = a \sin B = c \tan B = \sqrt{(a^2 - c^2)}$$
$$= \sqrt{(a - c)\ (a + c)} \ldots (5).$$

The following cases may arise in solving oblique triangles:

1. *When two angles and a side are given.* In this case, the third angle is found by subtracting the sum of the two given angles from 180°. The sides may then be found by formula (1).

2. *When two sides and an angle opposite one of them are given.* In this case, the angle opposite the other given side may be found by formula (1), after which the solution of the problem may be completed as in the last case.

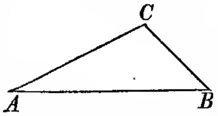
3. *When two sides and their included angle are given.* In this case, the included angle subtracted from 180°, gives the sum of the angles opposite the given sides; then, from formula (2), half the difference of these two angles may be determined. Having half the sum, and half the difference, add the two together, and the result will be the greater angle; subtract the lesser from the greater, and the difference will be the lesser angle. The greater angle lies opposite the greater side, and the lesser angle opposite the lesser side. The angles being determined, the remaining side may be found by formula (1).

4. *When the three sides are given.* In this case, each angle may be determined in succession by formulas (3). Their sum should be equal to 180°.

In solving right angled triangles, select the proper formula form groups (4) and (5). The application is obvious.

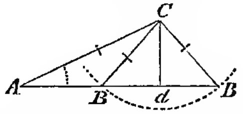
All of these cases may be solved approximately by geometrical construction:

1st. *Case.* Given A, B, and c. Draw an indefinite straight line; from a scale of equal parts, lay off the distance AB = c.



At A and B construct angles respectively equal to the given angles: the intersection of the sides of these angles determines the vertex of the third angle C. Measure the distances AC and BC by a scale of equal parts, and the angle C, with a protractor; the results will be the values of the required parts.

2d. *Case.* Given A, a and b. Draw an indefinite right line AB′, and at any point of



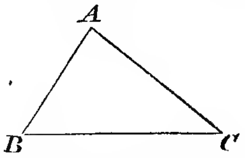
it, as A, construct the given angle A; on one side lay off a distance AC equal to b, and from C as a centre, with a radius equal to a, describe an arc of a circle, cutting the line first drawn. Join the point of intersection with C, and the triangle will be constructed Measure the distance CB with a scale of equal parts, and the angles B and C with a protractor; the results will be the required parts. In this case, there may be three cases:

1. The auxiliary circle may cut the line AB′, as in the figure, in two points, both on the same side of A: in this case, there are two solutions; if it is tangent to the line, the two triangles become one, and that is right angled.

2. It may intersect it in two points on opposite sides of A; in this case, there will be but one solution, the point on the side of the constructed angle being the one required

3. It may neither cut the line nor be tangent to it; in this case, the solution is impossible.

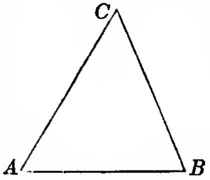
3d. *Case.* Given B, a and c. Construct an angle equal to B, lay off on its sides distances



equal to a and c respectively, and join the extremities of these distances: the figure thus formed will be the required triangle. Measure b, A and C, as before; the results will be the values of the parts required.

4th. *Case.* Given a, b, and c. Draw an indefinite line, and take on it AB equal to c: with A as a centre, and b as a radius, describe an arc; with B as a centre, and a as aradius, describe a second arc cutting the

first in C; join CA and CB. Measure with the protractor the angles A, B, C, of this triangle, and the results will be the values of the required parts.



The same principles will serve to construct all cases of right angled triangles. In most cases, the construction of right angled triangles is much simpler than that of oblique angled triangles.

Spherical Trigonometry.

In spherical, as in plane trigonometry, there are six parts in every triangle—three sides and three angles. When any three are given, the other three may be found, except in the particular case of the birectangular triangle. In that case, if two right angles and a side opposite one be given, each given part will be 90°, and the solution is indeterminate. As in Plane Trigonometry, triangles are solved by means of formulas. The following are sufficient to solve all cases of spherical triangles.

In these formulas the large letters stand for the angles, and the small ones for the sides opposite them. We suppose also that

$$S = A + B + C \quad \text{and} \quad s = a + b + c.$$

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad \dots\dots\dots\dots \quad (1).$$

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin(\frac{1}{2}s - c)\sin(\frac{1}{2}s - b)}{\sin b \sin c}};$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2}s \sin(\frac{1}{2}s - a)}{\sin b \sin c}} \quad \dots \quad (2).$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos(\frac{1}{2}S - C)\cos(\frac{1}{2}S - B)}{\sin B \sin C}} \quad (3).$$

$$\tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2} C \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)};$$

$$\tan \tfrac{1}{2}(A - B) = \cot \tfrac{1}{2} C \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \quad \dots \quad (4).$$

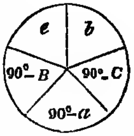
$$\tan \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2} c \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)};$$

$$\tan \tfrac{1}{2}(a - b) = \tan \tfrac{1}{2} c \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \quad \dots \quad (5).$$

For right angled spherical triangles, Napier's formulas for circular parts are used. These formulas require explanation.

In every right angled triangle right angled at A, rejecting the right angle, which is always known, there are five parts. The two sides about the right angle, and the complements of the remaining parts, make up what Napier called *circular parts*.

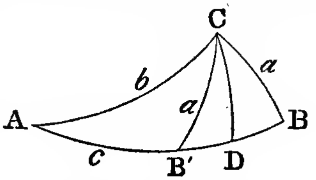
If these be arranged circularly, as in the diagram, and in the same order in which they occur in the triangle, we see that for each part there are two *adjacent* parts and two opposite parts, or parts which are not adjacent. The first part, with respect to these, is called the middle part. If we assume any part as the middle part, and denote it by m, and if we denote the adjacent parts by a and a', and the opposite parts by o and o', we have the following relations:



$$\sin m = \tan a \tan a' = \cos o \cos o' \quad \dots\dots \quad (6).$$

The following cases may arise in the solution of spherical triangles.

1st Case. *Given two sides and an angle opposite one of them*, as A, a and b; B, may be found from formula (1). The formula determines the sine of B, and since there are two



angles, complements of each other, having the same sine, it is necessary to ascertain which is to be taken.

When the sine of the side opposite the required angle is less than the sine of the other given side, that one must be taken which is of the same species as b. In this case there is but one solution.

When the sine of the side opposite the required angle is greater than the sine of the other given side, both angles must be used, and there will be two solutions.

Having determined B, we draw the arc of a great circle, CD, through C, and perpendicular to AB. There will thus be formed two right angled triangles.

There may be two cases; 1st, where CD meets the base; or, 2d, the when it meets the

base produced. Take the first case. Find, from the rightangled triangle BCD, the angle DCB, by means of formula (6). In like manner, find from the triangle ACD the angle ACD by means of the same formula. The sum of these angles will be the value of the angle C, and then the side c may be found from formula (1). In the second case find in like manner the angles ACD and B′CD, and take their difference; this will be angle C, and the side c may be found as before.

2d Case. *Having given two angles, and a side opposite one of them,* as A, B and a. The side b may be found by formula (1). Since there are two arcs corresponding to the same sine, there may be, as before, two solutions, or there may be but one.

When the sine of the angle opposite the required side is greater than the sine of the other given angle, both angles must be used, and there will be two solutions.

When the sine of the angle opposite the required side is less than the sine of the other given angle, that one must be used which is of the same species as B, and there will be but one solution.

Two arcs or angles are of the same species when both are greater or both less than 90°.

Having determined the side b, the remaining side and angle may be found as in the first case.

3d Case. *Having given the three sides of a spherical triangle, a, b* and c. One angle, or all of them, may be found by formula (2). When but one angle is found by formula (2), the remaining ones may be found by means of formula (1).

4th Case. *Having given the three angles A, B and C.* Any side, or all of them, may be found by means of formula (3). When only one side is found by this formula, the remaining ones may be found by formula (1).

5th Case. *Having given two sides, and their included angle, a, b and C.* The half sum of the remaining angles, and their half difference, may be found by formulas (4); then the sum of these results is equal to the greater angle, and their difference is equal to the lesser angle. The greater angle lies opposite the greater side, and the lesser angle opposite the lesser side. The remaining side may be found by formula (1).

6th Case. *Having given two angles and their included side, A, B and c.*

The half sum and the half difference of the remaining sides, may be found by means of formula (5), and then the sum of these results is equal to the greater side, and their difference to the lesser side. The greater side lies opposite the greater angle, and the lesser side opposite the lesser angle. The remaining angle may then be found by formula (1).

To solve any case of a right angled spherical triangle, two parts must be given besides the right angle. Find from them the corresponding circular parts. If they adjoin each other, then the part opposite to them may be found by the formula

$$\sin m = \cos o \cos o'.$$

If they are separated from each other, then the part adjacent to them both may be found by the formula

$$\sin m = \tan a \tan a'.$$

Having found one part, the remaining parts may be found by formula (6), or by means of formula (1).

Quadrantal spherical triangles may be solved by means of right angled spherical triangles. See *Quadrantal Triangles.*

The solution of spherical triangles is some times facilitated by means of auxiliary formulas. The following indicate the method of using these auxiliary formulas:

$$\cos a = \frac{\cos b \sin(c+\phi)}{\sin \phi}; \quad \cot \phi = \tan b \cos A \quad (1)$$

$$\cos A = \frac{\cos B \sin(C-\phi)}{\sin \phi}; \quad \cot \phi = \tan B \cos a \quad (2)$$

$$\cot a \sin b = \frac{\sin(C+\phi)}{\sin \phi}; \quad \cot \phi = \frac{\cot A}{\cos b}. \quad . \quad (3)$$

Analytical Trigonometry.

Analytical trigonometry treats of the relations and properties of the trigonometrical functions.

We shall give some of the most important formulas, embracing those which are most commonly used in Analysis. The radius is equal to 1.

Fundamental Formulas.

$$\left.\begin{array}{l}\sin a = \sqrt{1-\cos^2 a}; \quad \cos a = \sqrt{1-\sin^2 a}; \\ \sin^2 a + \cos^2 a = 1.\end{array}\right\} (1)$$

$$\tan a = \frac{\sin a}{\cos a}; \quad \cot a = \frac{\cos a}{\sin a}; \quad \tan a \cot a = 1 \quad (2)$$

$$\left.\begin{array}{l}\sec a = \dfrac{1}{\cos a};\ \operatorname{cosec} a = \dfrac{1}{\sin a} \\ \sec^2 a - \tan^2 a = \operatorname{cosec} a^2 - \cot^2 a = 1\end{array}\right\} \cdot (3)$$

$$\text{ver-sin } a = 1 - \cos a = \frac{\sec a - \cos a}{\sec a} \cdot \cdot (4)$$

General Formulas.

$$\left.\begin{array}{l}\sin 2a = 2 \sin a \cos a;\ \sin (n+1) a \\ \quad = 2 \sin na \cos a - \sin (n-1) a\end{array}\right\} \cdot (5)$$

$$\left.\begin{array}{l}\cos 2a = 2 \cos^2 a - 1 \ .\ .\ \cos (n+1) a \\ \quad = 2 \cos na \cos a - \cos (n-1) a\end{array}\right\} \cdot (6)$$

$$\left.\begin{array}{l}\tan 2a = \dfrac{2 \tan a}{1 - \tan^2 a}; \\ \tan 3a = \dfrac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}; \\ \tan 4a = \dfrac{4 \tan a - 4 \tan^3 a}{1 - 6 \tan^2 a + \tan^4 a}.\end{array}\right\} \cdot \cdot (7)$$

$$\sin \frac{1}{2} a = \sqrt{\frac{1 - \cos a}{2}};\ \cos \frac{1}{2} a = \sqrt{\frac{1 + \cos a}{a}} (8)$$

$$\left.\begin{array}{l}\tan \frac{1}{2} a = \dfrac{1 - \cos a}{\sin a} = \dfrac{\sin a}{1 + \cos a} \\ = \operatorname{cosec} a - \cot a = \dfrac{1 + \sin a - \cos a}{1 + \sin a + \cos a}\end{array}\right\} \cdot \cdot (9)$$

$$\cot \frac{1}{2} a = \frac{1 + \cos a}{\sin a} = \operatorname{cosec} a + \cot a . . (10)$$

$$\left.\begin{array}{l}\sin (45^\circ + a) = \dfrac{\cos a + \sin a}{\sqrt{2}} \\ \sin (45^\circ - a) = \dfrac{\cos a - \sin a}{\sqrt{2}}\end{array}\right\} \cdot \cdot \cdot (11)$$

$$\left.\begin{array}{l}\tan (45^\circ + a) = \dfrac{1 + \tan a}{1 - \tan a}; \\ \tan (45^\circ - a) = \dfrac{1 - \tan a}{1 + \tan a}\end{array}\right\} \cdot \cdot \cdot (12)$$

$$\left.\begin{array}{l}\tan (45^\circ + \frac{1}{2} a) = \dfrac{1 + \sin a}{\cos a}; \\ \tan (45^\circ - \frac{1}{2} a) = \dfrac{1 - \sin a}{\cos a}\end{array}\right\} \cdot \cdot \cdot (13)$$

$$\sin (a \pm b) = \sin a \cos b \pm \sin b \cos a . . (14)$$

$$\cos (a \pm b) = \cos a \cos b \mp \sin b \sin a . . (15)$$

$$\tan (a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b} . \ . \ . \ . \ (16)$$

$$\cot (a \pm b) = \frac{\cot a \cot b \mp 1}{\cot a \pm \cot b} . \ . \ . \ . \ (17)$$

$$\left.\begin{array}{l}\sec (a \pm b) = \dfrac{1}{\cos (a \pm b)}; \\ \operatorname{cosec} (a \pm b) = \dfrac{1}{\sin (a \pm b)}\end{array}\right\} \cdot \cdot (18)$$

$$\sin (45^\circ \pm a) = \cos (45^\circ \mp a) = \frac{\cos a \pm \sin a}{\sqrt{2}} (19)$$

$$\left.\begin{array}{l}\tan (45^\circ \pm a) = \cot (45^\circ \mp a) \\ = \dfrac{\cos a \pm \cos b}{\cos a \mp \cos b} = \dfrac{1 \pm \tan a}{1 \mp \tan a}\end{array}\right\} \cdot \cdot \cdot (20)$$

$$\left.\begin{array}{l}\sin (45^\circ \pm \frac{1}{2} a) = \cos (45^\circ \mp \frac{1}{2} a) \\ \quad = \sqrt{\dfrac{1 \pm \sin a}{2}}\end{array}\right\} \cdot \cdot (21)$$

$$\left.\begin{array}{l}\tan (45^\circ \pm \frac{1}{2} a) = \cot (45^\circ \mp \frac{1}{2} a) \\ \quad = \dfrac{\cos a}{1 \mp \sin a}\end{array}\right\} \cdot \cdot (22)$$

$$\sin a + \sin b = 2 \sin \tfrac{1}{2} (a+b) \cos \tfrac{1}{2} (a - b). (23)$$

$$\sin a - \sin b = 2 \cos \tfrac{1}{2} (a+b) \sin \tfrac{1}{2} (a - b). (24)$$

$$\cos a + \cos b = 2 \cos \tfrac{1}{2} (a+b) \cos \tfrac{1}{2} (a - b). (25)$$

$$\cos a - \cos b = -2 \sin \tfrac{1}{2} (a+b) \sin \tfrac{1}{2} (a - b). (26)$$

$$\tan a + \tan b = \frac{\sin (a + b)}{\cos a \cos b} . \ . \ . \ . \ (27)$$

$$\tan a - \tan b = \frac{\sin (a - b)}{\cos a \cos b} . \ . \ . \ . \ (28)$$

$$\cot a + \cot b = \frac{\sin (a + b)}{\sin a \sin b} . \ . \ . \ . \ (29)$$

$$\cot a - \cot b = -\frac{\sin (a - b)}{\sin a \sin b} . \ . \ . \ . \ (30)$$

$$\sec a + \sec b = \frac{2 \cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{\cos a \cos b} (31)$$

$$\sec a - \sec b = \frac{2 \sin \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)}{\cos a \cos b} (32)$$

$$\sin (30^\circ + a) + \sin (30^\circ - a) = \cos a . (33)$$

$$\sin (30^\circ + a) - \sin (30^\circ - a) = \sin a \sqrt{3} (34)$$

$$\cos (30^\circ + a) + \cos (30^\circ - a) = \cos a \sqrt{3} (35)$$

$$\cos (30^\circ + a) - \cos (30^\circ - a) = - \sin a (36)$$

$$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\tan \frac{1}{2} (a + b)}{\tan \frac{1}{2} (a - b)} . \ . \ . \ (37)$$

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \tan \tfrac{1}{2} (a + b) . \ . \ . \ (38)$$

$$\frac{\sin a + \sin b}{\cos a - \cos b} = - \cot \tfrac{1}{2} (a - b) . \ . \ . \ (39)$$

$$\frac{\sin a - \sin b}{\cos a + \cos b} = \tan \tfrac{1}{2}(a - b). \quad . \quad . \quad . \quad (40)$$

$$\frac{\sin a - \sin b}{\cos a - \cos b} = -\cot \tfrac{1}{2}(a + b) \quad . \quad . \quad (41)$$

$$\left.\begin{aligned}\frac{\cos a + \cos b}{\cos a - \cos b} &= -\frac{\cot \frac{1}{2}(a + b)}{\tan \frac{1}{2}(a - b)} \\ &= -\frac{\cot \frac{1}{2}(a - b)}{\tan \frac{1}{2}(a + b)}\end{aligned}\right\} . \quad . \quad (42)$$

$$\frac{\sin a + \sin b}{\sin (a + b)} = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}. \quad . \quad . \quad . \quad (43)$$

$$\frac{\tan a + \tan b}{\tan a - \tan b} = \frac{\cot b + \cot a}{\cot b - \cot a} = \frac{\sin(a+b)}{\sin(a-b)}. \quad (44)$$

$$\tan^2 a - \tan^2 b = \frac{\sin (a + b) \sin (a - b)}{\cos^2 a \cos^2 b} \cdot \quad (45)$$

$$\cot^2 a - \cot^2 b = -\frac{\sin (a + b) \sin (a - b)}{\sin^2 a \sin^2 b} \quad (46)$$

$$\left.\begin{aligned}&\sin (a + b + c) = \sin a \cos b \cos c \\ &+ \sin b \cos a \cos c + \sin c \cos a \cos b\end{aligned}\right\} . \quad (47)$$

$$\left.\begin{aligned}&\cos (a + b + c) = \cos a \cos b \cos c \\ &- \sin a \sin b \cos c - \sin b \sin c \cos a \\ &- \sin c \sin a \cos b\end{aligned}\right\} . \quad (48)$$

$$\left.\begin{aligned}&\tan (a + b + c) = \\ &\frac{\tan a + \tan b + \tan c - \tan a \tan b \tan c}{1 - \tan a \tan b - \tan a \tan c - \tan b \tan c}\end{aligned}\right\} \cdot (49)$$

$$\left.\begin{aligned}&(\cos a \pm \sqrt{-1} \times \sin a)^m \\ &= \cos ma \pm \sqrt{-1} \times \sin ma\end{aligned}\right\} . \quad . \quad (50)$$

$$\left.\begin{aligned}&\cos a = \tfrac{1}{2}\left(e^{a\sqrt{-1}} + e^{-a\sqrt{-1}}\right); \\ &\sin a = \frac{1}{2\sqrt{-1}}\left(e^{a\sqrt{-1}} - e^{-a\sqrt{-1}}\right)\end{aligned}\right\} . \quad \cdot (51)$$

$$\left.\begin{aligned}&(-1)^m = \cos m (2n + 1) \pi \\ &+ \sqrt{-1} \times \sin m (2n + 1) \pi\end{aligned}\right\} . \quad . \quad . \quad . \quad (52)$$

For Trigonometrical series, see *Series*.

General formulas, expressing the relations between the sides and angles of a spherical triangle used in astronomical investigations:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad (53)$$

$$\cos A = \sin B \sin C \cos a - \cos B \cos C \quad (54)$$

$$\cot a \sin b = \cos b \cos C + \cot A \sin C \quad (55)$$

$$\sin A \cot B = \sin c \cot b - \cos c \cos A \quad (56)$$

The following is the analytical enunciation of what is usually known as Gauss' theorem:

$$\left.\begin{aligned}&\text{If } p = \cos \tfrac{1}{2} c \sin \tfrac{1}{2}(A + B); \\ &\quad q = \cos \tfrac{1}{2} c \cos \tfrac{1}{2}(A + B); \\ &\quad r = \sin \tfrac{1}{2} c \sin \tfrac{1}{2}(A - B); \\ &\quad s = \sin \tfrac{1}{2} c \cos \tfrac{1}{2}(A - B); \\ &\quad P = \cos \tfrac{1}{2} C \cos \tfrac{1}{2}(a - b); \\ &\quad Q = \sin \tfrac{1}{2} C \cos \tfrac{1}{2}(a + b); \\ &\quad R = \cos \tfrac{1}{2} C \sin \tfrac{1}{2}(a - b); \\ &\quad S = \sin \tfrac{1}{2} C \sin \tfrac{1}{2}(a + b); \\ &\text{Then is} \\ &\quad pq = PQ;\ pr = PR;\ ps = PS; \\ &\quad qr = QR;\ qs = QS;\ rs = RS;\end{aligned}\right\} \quad (57)$$

The following formulas are called Gauss' equations, and they result from considering the sides and angles of a spherical triangle, each less than 180° Retaining the same notation as in the last formulas, we have

$$p^2 = P^2;\ q^2 = Q^2;\ r^2 = R^2;\ s^2 = S^2. \ (58).$$

Gauss' Equations.

$$\left.\begin{aligned}\cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A+B) &= \cos\tfrac{1}{2}C \cos\tfrac{1}{2}(a-b) \\ \cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A+B) &= \sin\tfrac{1}{2}C \cos\tfrac{1}{2}(a+b) \\ \sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A-B) &= \cos\tfrac{1}{2}C \sin\tfrac{1}{2}(a-b) \\ \sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A-B) &= \sin\tfrac{1}{2}C \sin\tfrac{1}{2}(a+b)\end{aligned}\right\} (59).$$

From these, Napier's analogies may be at once deduced.

TRĪ-HĒ'DRAL, OR, TRĪ-Ē'DRAL ANGLES. [Gr. τρεις, three, and εδρα, a face]. A polyhedral angle of three faces. See *Polyhedral Angle.*

TRĪ-HĒ'DRON. [Gr. τρεις, and εδρα]. A name once given to a triangle.

TRĪ-LAT'ER-AL. [L. *tres*, three, and *latus*, a side]. Having three sides, as a triangle.

TRĪ-NŌ'MI-AL. [L. *tres*, three, and *nomen*, a name]. A polynomial having three terms, as $a^2 + ab + b^2$. See *Polynomial.*

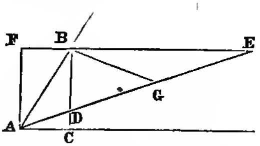
TRIP'LI-CATE RATIO [L. *triplicatus*, from *tres*, three, and *plico*, to fold]. The ratio of the cubes of two quantities; thus, the triplicate ratio of a to b is, $\frac{b^3}{a^3}$. Similar volumes are to each other in the triplicate ratio of their homologous lines.

TRĪ'POD. [Gr. τρειπους; from τρεις, three, and πους, a foot]. A stand with three legs used to support a theodolite, compass, level or other surveying instrument. The legs turn about hinges at their upper ends, by means of which they are attached to a brass

plate; they may be folded up or spread apart, so as to afford a firm base for the instrument to rest upon. The lower ends of the legs of the tripod are generally shod with metal, so that they may be easily planted in the ground: the upper plate terminates in a screw which serves to fasten it to the instrument that it supports.

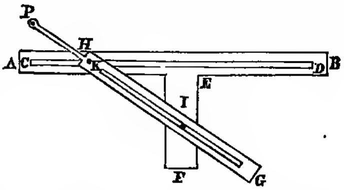
TRĪ-RECT-AN″GU-LAR TRIANGLE. A spherical triangle, whose angles are all right angles. It is equivalent to the eighth part of the surface of the sphere on which it is situated. See *Spherical Triangle*, *Triangle*.

TRĪ-SEC′TION. [L. *tres*, three, and *seco*, to cut]. The trisection of an angle is a problem of great celebrity amongst the ancient mathematicians. It belongs to the same class of problems as the duplication of the cube, and the insertion of two geometrical means between two given lines. Like them, it has hitherto been found beyond the range of Elementary Geometry.



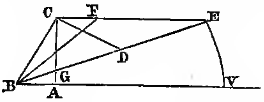
Let BAC be a plane angle, and suppose DAC to be one-third of it. From any point B, in the side AB, draw BC perpendicular to AC, and BE parallel to AC; produce AD to meet BE in E and complete the rectangle BCAF. Now, since DAC is one-third of BAC, BAD is twice DAC, or twice BEA. Draw BG, making the angle EBG equal to GEB; then BGA, being an exterior angle, is equal to twice BEA, or to BAG. Hence, the two triangles, EGB and BGA, are isosceles, or GE = GB = BA. The angle GBD is the difference between a right angle and EBG, and GDB, or its equal ADC, is the difference between a right angle and DAC, which is equal to EBG; therefore, GBD is equal to GDB; whence, GD = GB = AB. We see, then, that DE = 2AB. If, therefore, in a rectangle AFBC, a straight line AD could be drawn from the angular point A, so that on producing it to meet the opposite side FB, the produced part would be equal to a given straight line, the problem would be solved. The problem having been reduced to this condition, fails to yield further to Elementary Geometry. It may, however, be solved in a great variety of ways by means of the higher Geometry. We annex but a single solution, that, by means of the conchoid.

We shall first explain the mechanical method of constructing the Conchoid, by a continuous movement.



AB is a flat ruler, with a groove, CD, extending nearly through its length. Attached firmly to it is another ruler, EF, at right angles to it, in which there is a fixed pin, I. This pin passes into a groove of a third ruler, GK, which carries a second pin entering the groove CD. The system being thus adjusted, let a stem of any proposed length HP, be attached at H, carrying a pencil at P. The rectangular rulers being fixed, let HG be moved so that the pin K will move along the groove CD, the pin at I continuing in the groove GK: the pencil P will trace the superior conchoid. If another pencil were fixed to the ruler, at the same distance on the other side of K, it would trace out the inferior conchoid.

To apply the curve thus described to the trisection of an angle. Let ABC be the



angle to be trisected. Lay the instrument down so that the axis of the ruler EF shall coincide with BAV, the axis of the ruler AB being coincident with a line AC at right angles with BA. With an arm AV=2BC, let a portion of a conchoid EV be described. Draw CE parallel to BA, cutting the curve at E. Draw EB, and bisect the angle EBC by BF; then will the lines BE and BF trisect the given angle.

For, draw CD, making the angle DCE

DEC; then will CD = DE. The angle GCD = 90° − DCE, and CGD = 90° − GBA; but GBA = DEC = DCE; hence, GD = CD = DE; and CD = CB; therefore, the angle CBD is equal to the angle CDB, which is twice CED. But CED is equal to DBV, therefore, CDB is equal to twice DBV, and since CDB is bisected by BF, it follows that ABC is trisected by the lines BE and BF.

TRŌ'CHOID. [Gr. τροχος; from τρεχω, to run, and ειδος, shape generated by a wheel]. The same as cycloid. See *Cycloid.*

TROP'ICS [Gr. τροπη, a turning]. See *Spherical Projections.*

TRUNC'Ā-TED CONE, OR, PYRAMID. [L. *trunco*, to cut]. The portion of a cone or pyramid included between the base and a plane oblique to the base passed between it and the vertex. See *Frustum.*

TWIST'ED SURFACE. See *Warped Surface.*

U, the twenty-first letter of the English alphabet.

UL'TI-MATE RATIO. [L. *ultimus*, furthest]. If two varying quantities, which are functions of the same variable, vary in such a manner that their ratio continually approaches a fixed quantity, but cannot pass it, that quantity is said to be the ultimate ratio of the two quantities. It is nothing more than the limit of the ratio. As an example, consider the case of an arc of a circle less than 90°, and its chord. The arc is constantly greater than the chord; but, as the arc decreases, the length of the chord and arc approach equality, and their ratio approaches 1. Finally, when the arc becomes less than any assignable quantity, they become equal, and their ratio is 1. Hence, 1 is the ultimate ratio of a chord to its arc.

In like manner, if a regular polygon be inscribed in a circle, the area of the circle is always greater than that of the polygon; but, as the number of sides of the polygon is increased, the areas of the two magnitudes approach equality, and their ratio approaches 1; finally, when the number of sides of the polygon becomes greater than any assignable number, the difference of the areas becomes less than any assignable quantity, or their ratio differs from 1 by less than any assignable number. In this case, the ultimate ratio of the area of the polygon to that of the circle, is 1.

UN-DEC'A-GON. [L. *undecim*, eleven, and Gr. γωνια, angle]. A polygon of eleven angles or sides. See *Polygon.*

UN-DU-LĀ'TION, POINT OF. See *Singular Point.*

UN-Ē'QUAL QUANTITIES. [L. *inæqualis*, unequal]. Those which are not equal. See *Equality.* The algebraic symbol of the relation of inequality is $>$; it indicates that the quantity placed at its opening is greater than the one placed at the vertex. Thus, $a > b$ indicates that a is greater than b.

UN-Ē'VEN NUMBER. A number which is not exactly divisible by two; thus, 1, 3, 5, 7, &c., are uneven numbers. Unevenly uneven numbers are those which, being divided by 4, leave a remainder equal to 1; thus, 1, 5, 9, 13, &c. See *Number.*

UN''GU-LA. [L.] A segment of a solid. An ungula of a cone or cylinder is a portion of the cone or cylinder, included between a part of the base and a plane intersecting the base obliquely. A spherical ungula is a part of the sphere bounded by two semi-circles, meeting in a common diameter, and by a lune of the surface of the sphere.

U-NI-FORM-LY. [L. *uniformis*; from *unus*, one, and *forma*, a form]. Two dependent quantities are said to vary uniformly with regard to each other, when the ratio of their corresponding increments is constant. Thus, the co-ordinates of a straight line vary uniformly together.

U'NIT. [L. *unus*, one]. A single thing regarded as a whole. Numbers are collections of things of the same kind, each of which is a unit of the collection. Thus, 20 feet is a collection of 20 equal spaces, each of which is equal to one foot. Here, 1 foot is the unit or base of the collection.

ABSTRACT UNIT. The same as 1. The abstract unit 1 is the measure of the relation of equality of two quantities. It is the base of the system of natural numbers, and incidentally the base of all quantities.

UNITS OF CURRENCY. There are different units of currency in different countries. In

the United States, the different orders of units are arranged according to the scale of tens. The simplest unit, or that of the first order, is 1 *mill*. The units of the successive orders are 1 *cent*, 1 *dime*, 1 *dollar*, and 1 *eagle*, each of which is equal to ten units of the next lower order. In consequence of this law of relation, units of currency, in this country, may be treated in all respects as though they were abstract units, written in the decimal scale. This renders the currency of the United States the most convenient, in a practical point of view, of any currency of the world.

DENOMINATE OR CONCRETE UNIT. A unit, in which the kind of thing is named, embracing units of measure, units of weight, and units of time; as, 1 *foot*, 1 *pound*, 1 *hour*.

DUODECIMAL UNIT. A unit in the scale of 12's. See *Scale*.

FRACTIONAL UNIT. The unit of a fraction. In a fraction there are two terms, the numerator and the denominator. The denominator denotes the number of equal parts into which 1 is divided, and the numerator denotes how many of those parts are taken. The fractional unit is always equal to the reciprocal of the denominator. Thus, in the fraction $\frac{4}{5}$, the fractional unit is $\frac{1}{5}$, and four of them are taken or collected to form the fraction. In the fraction $\frac{a}{b}$, $\frac{1}{b}$ is the fractional unit. See *Fraction*.

INTEGRAL UNIT. The unit 1, the unit of integral numbers.

UNIT OF MEASURE. The unit of measure of any quantity is a quantity of the same kind, with which the quantity is compared; thus, the unit of measure of lines is a straight line of known or assumed length, as 1 *inch*, 1 *foot*, 1 *yard*, 1 *mile*, &c.. the unit of measure of surfaces is a square described upon the linear unit as a side, as 1 *square inch*, 1 *square foot*, 1 *square yard*, 1 *square mile*; the unit of measure of volumes is a cube, one of whose edges is the linear unit, as 1 *cubic inch*, 1 *cubic yard*, &c. The unit of weight is an assumed weight, the unit of measure of time is an assumed period of time, &c.

U'NI-TY. [L. *unitas*]. An entire collection, considered as a single thing. Thus, 20 feet, considered as a single distance, is unity; 1 foot is the unit of the expression. The number, 1, when unconnected with anything else, is generally called *unity*.

UN-LĪKE'. Dissimilar. See *Like*.

UN-LIM'IT-ED. A problem which admits of an infinite number of solutions. Thus, the problem to draw a tangent line to a circle is unlimited, because an infinite number of tangent lines may be drawn to any given circle. See *Indeterminate Problem*.

UN-KNŌWN'. The unknown quantity of a problem or equation is one whose value is not known, but is required to be determined. When there are just as many conditions given as there are unknown quantities to be determined, the problem is possible, and admits of a finite number of solutions. When there are just as many equations given as there are unknown quantities entering them, their values may be found. When there are fewer simultaneous equations than there are unknown quantities, they are indeterminate. In this case, the unknown quantities are variable.

V. The twenty-second letter of the English alphabet. As a numeral, it stands for 5; with a dash over it, it stands for 5000. Thus, $\overline{\text{V}} = 5000$.

VAL'UE. [L. *valor*; from *valeo*, to be worth]. The numerical value of an expression is the result obtained by making each quantity entering the expression equal to some number, and then performing the algebraic operations indicated. Thus, the numerical value of the expression,

$a^2(b-c)^4$, when $a = 4$, $b = 5$, and $c = 3$, is $\quad 4^2(5-3)^4 = 16 \times 16 = 256$.

VĀNE. The vane of a leveling staff is a piece of board or metal, arranged so that it can be raised or lowered at pleasure, to indicate the point of the staff at which the plane of apparent level through the axis of the telescope cuts it. In the common leveling staff, the vane is moved up and down by means of an endless cord attached to the vane, and passing over pulleys at the top and bottom of the staff. In the sliding staff the vane is attached to one branch of the staff, which branch itself slides in a groove of the other branch. In both, the vane is divided into

four parts by two lines at right angles to each other, and the alternate quarters are painted black and white, to show more distinctly the 0 line of the vane. The 0 line is the horizontal line through the middle of the vane. See *Leveling Staff*.

VAN'ISH-ING FRACTION. [L. *vanesco*, to disappear]. A fraction which reduces to the form of $\frac{0}{0}$ for a particular value of the variable which enters it, in consequence of the existence of a common factor in both terms of the fraction, which factor becomes 0 for this particular value of the variable. Every vanishing fraction which is a function of a single variable, may be reduced to the form

$$\frac{f(x)\,(x-a)^m}{f'(x)\,(x-a)^n},$$

which, when $x = a$, reduces to the form $\frac{0}{0}$. There may be three cases :

1st. When $\qquad m > n$.

In this case, if we divide both terms of the fraction by $(x-a)^n$, and then in the result make $x = a$, we shall have

$$\left(\frac{f(x)\,(x-a)^{m-n}}{f'(x)}\right)_{x=a} = \frac{0}{Q} = 0\,;$$

2d. When $\qquad m = n$.

In this case, if we divide both terms of the fraction by $(x-a)^n$, and make $x = a$ in the result, we shall have

$$\left(\frac{f(x)}{f'(x)}\right)_{x=a} = \frac{P}{Q}.$$

3d. When $\qquad m < n$.

In this case, if we divide both terms by $(x-a)^m$, and then make $x = a$, we shall have

$$\left(\frac{f(x)}{f'(x)\,(x-a)^{n-m}}\right)_{x=a} = \frac{P}{0} = \infty.$$

These are the only suppositions that can be made upon the common factor; whence, we conclude that the true value of a vanishing fraction, for the particular value of the variable, is either *zero, finite* or *infinite*. The discussion just made indicates the following rule for finding the true value of a vanishing fraction for the particular value of the variable :

Divide both numerator and denominator by the greatest common factor which enters them both, and in the resulting fraction make the particular supposition which reduced the given fraction to $\frac{0}{0}$: the result will be the true value of the fraction for that particular value of the variable.

In most cases, it will not be convenient to find the common factor and strike it out. The true value of the fraction may, however, be found as follows :

Substitute in the fraction for the variable that value which reduces it to $\frac{0}{0}$ plus an arbitrary quantity. Develop the resulting values of the numerator and denominator, and arrange the results according to the ascending powers of the arbitrary quantity; then strike out from both terms of the resulting fraction the highest power of this quantity, which is common to both, and in the resulting fraction make the arbitrary quantity equal to 0; the value obtained will be the value required.

There is another method depending upon the differential calculus, which may be employed in all cases in which the exponents of the powers of the factor $(x - a)$, in both terms of the fraction, are not fractional and contained between the same two consecutive whole numbers. The principle employed is expressed algebraically, as follows:

$$\left(\frac{f(x)(x-a)^n}{f'(x)\,(x-a)^n}\right)_{x=a} = \left[\frac{d(f(x)(x-a)^m)}{d(fx\ (x-a)^n)}\right]_{x=a}$$

$$= \left[\frac{d^2(f(x)\,(x-a)^m)}{d^2(f'(x)\,(x-a)^n)}\right]_{x=a} = \&c.$$

The rule is as follows: Differentiate the numerator of the given fraction for a new numerator, and the denominator of the given fraction for a new denominator; in the resulting fraction make the particular supposition; if the result is not $\frac{0}{0}$, it will be the true value of the fraction for the particular value of the variable. If the result is $\frac{0}{0}$, repeat the operation, and continue it till a result is found which does not become $\frac{0}{0}$; this will be the true value of the fraction. For example, let it be required to find the true value of the fraction

$$\frac{x^3 - a^3}{(x-a)^3}, \text{ when } x = a$$

By the rule

$$\left[\frac{3x^2\ dx}{3(x-a)^2dx}\right]_{x=a} = \frac{3a^2}{0} = \infty.$$

Let it also be required to find the value of

$$\frac{x^2-a^2}{(x-a)}, \text{ when } x = 0.$$

By the rule

$$\left[\frac{2xdx}{dx}\right]_{x=a} = 2a,$$

which is the required value.

VAN'ISH-ING LINE. The vanishing line of a plane, in perspective, is the intersection of the perspective plane with the visual plane parallel to the given plane. All lines parallel to the given plane, have their vanishing points in the vanishing line of the plane.

VANISHING POINT of a straight line, in perspective, is the point of intersection of the perspective plane, with a visual ray drawn parallel to the given line. The vanishing point of a line is the perspective of that point of the line which is at an infinite distance, and is one point of the indefinite perspective of the line.

To find the perspective of a line, find the point in which it pierces the perspective plane, and join it with the vanishing point; this will be the perspective required.

All lines parallel to the perspective plane, have their vanishing points at an infinite distance; hence, their perspectives are parallel. All lines perpendicular to the perspective plane, vanish at the centre of the picture.

VĀ'RI-A-BLE. [L. *vario*, to change]. Quantities which admit of an infinite number of sets of values, in the same expression. Thus, in the equation

$$x^2 + y^2 = R^2,$$

x and y are variables, for there are an infinite number of sets of values which satisfy it at the same time. When there are several variables in the same equation, it is customary to consider all but one as *independent variables*, or variables to which values may be assigned at pleasure: the remaining one is called a *function* of the others, its value being dependent upon the values attributed to them. In the equation of surfaces referred to rectangular axes, the variable z is generally taken as the function, x and y being independent variables. In the equation of lines referred to rectangular axes, y is taken as the function, x being the independent variable. In polar equations of magnitudes the radius vector is taken as the function, the angle or angles being the variables.

This is the general convention; any other variable of the equation may, however, be selected as the function. Whichever variable may be selected as a function of the others, its differential is always variable, whilst the differentials of the independent variables are constant.

The difference between the variables and the arbitrary constants which enter an equation is this: The variables admit of an infinite number of sets of values in the same expression; the arbitrary constants may admit of any one out of an infinite number of sets of values. To illustrate, take the equation of the circle,

$$(x-\alpha)^2 + (y-\beta)^2 = R^2.$$

In this equation, x and y are variables, α, β and R, are arbitrary constants. The latter may have any set of values assigned to them at pleasure, and this set of values determines the circle completely. In this circle x and y represent the co-ordinates of every point of it. In like manner, if any other set of values be assigned to α, β and R, x and y will represent the co-ordinates of every point of the new circle, and so on.

Every equation between two variables is the equation of some plane curve; every equation between these variables is the equation of a surface; the variables in both cases represent the co-ordinates of every point of the magnitude.

VĀ-RI-Ā'TION OF THE NEEDLE. In Surveying, the angle included between the true and magnetic meridians of the point at which the variation is taken. If the direction of the true meridian at the point were known, the variation of the needle would be found by simply taking the bearing of this line with the compass. If the bearing of the meridian is east of north, the variation is to the west; if the bearing is west of north, the variation is to the east. In order, therefore, to find the variation of the needle at any place, we first find the direction of the true meridian, or of some line which makes a known angle with

it; we then observe the bearing of this line; from this result the variation is easily computed. The line most usually employed is the line of greatest elongation of the pole star, either to the east or west. If we conceive a vertical plane to be passed at any point of the earth's surface through the pole star, this plane will move to the east and west about the vertical line at the place as an axis, whilst the star revolves about the pole. The two positions in which this plane makes the greatest angles with the meridian, both to the east and west, are those at which the star is at its greatest eastern and western elongation, and their intersection with the surface of the earth, form what are called the lines of greatest eastern and western elongation.

The angles which these lines make with the true meridian are equal to each other, and may be computed by means of the formula,

$$\sin E = \frac{\sin \Delta}{\cos l},$$

in which Δ is the polar distance of the star at the time of observation, and l the latitude of the place at which the observation is made. The value of Δ may be found from the Nautical Almanac, or it may be found from the following formula sufficiently near for ordinary purposes:

$$\Delta = 1^\circ\ 29'\ 24''\cdot 6 - n\ 19''\cdot 24,$$

in which n is the number of years and decimal parts of a year, from Jan. 1, 1850, to the time of making the observation.

The time of greatest eastern elongation, or of greatest western elongation, may be found by means of the formula,

$$\cos p = \tan l \tan \Delta,$$

in which p denotes the hour angle of the pole star at the time of greatest elongation. This formula makes known the hour angle which, converted into time, gives the number of hours from the time of the star's passage over the upper meridian till the time of greatest elongation; this, added to or subtracted from the time of the meridian passage of the star, gives the time of western or eastern elongation. The time of elongation and the values of the elongation, are computed and arranged in tables for different latitudes, and different epochs.

To find the variation of the needle for a particular time at a particular place, enter the tables and find the time of greatest eastern or western elongation, according to the season of the year, so that the elongation shall fall in the night time. A few minutes previous to the time, set up a theodolite at the station and level it, then bring the telescope so that the pole star shall appear to coincide with the intersection of the cross wires of the telescope, and clamp both the limb and vernier plate; follow the star by means of the tangent screw until it appears to stand still, and afterward returns in a contrary direction. In this position the plane of the vertical limb marks the plane of greatest elongation. Then direct the telescope by turning the vertical limb on its axis to some distant terrestrial object, made visible by a light attached to it, and mark that point and the axis of the instrument by stakes driven in the ground. These stakes mark out the line of greatest elongation. The cross hairs of the telescope may be made visible by light reflected into the barrel of the telescope from a lamp shining upon a white board set up in front of the telescope, and perforated so as not to impede vision in the direction of the axis.

In the morning, set up the compass at the station occupied by the theodolite, and take the bearing of the second stake.

Take from the table the elongation for the time and place; if the elongation is east, give it the sign $+$; if west, the sign $-$.

If the bearing of the line of elongation is east, affect it with the sign $+$; if west, with the sign $-$. Then, if we denote the elongation with its proper sign by E, and the bearing with its proper sign by B, and the variation by V, we shall have the relation,

$$\mathrm{V} = \mathrm{E} - \mathrm{B}.$$

If B exceeds E, numerically, the variation is to the west; if E exceeds B, the variation is to the east.

The line of elongation can be found by means of the compass sights and a plumb line suspended from a pole a few feet in advance of the station, and lighted from a candle held by an assistant.

The plumb line is suspended nearly in the direction of the star, and the compass sight

is moved upon a horizontal board till the star is at its greatest elongation. At this instant, the position of the compass sight and of the plumb line are marked, and thus the line of greatest elongation is determined. Then proceed as indicated above.

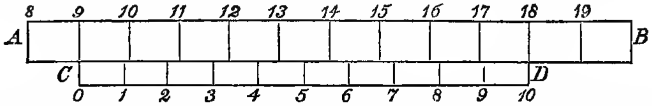
VĀ-RI-Ā'TIONS, CALCULUS OF. See *Calculus*.

VER-I-FI-CĀ'TION. [L. *verus*, true; and *facio*, to make]. The operation of testing or proving to be true. The verification of the roots of an equation, found from solving the equation, consists in showing that they are true roots of the equation. If we substitute for the unknown quantity each of the roots found, in succession, and find that the equation in each case is satisfied, that is, if the two numbers are equal to each other, the root found is correct, and is said to be verified.

Verification of an Equation. The operation of testing the equation of a problem, to see whether it expresses truly the conditions of the problem. Having solved the equation, we first verify the roots found, to see if they are the true roots of the equation. Having done this, we next perform upon them the operations indicated by the conditions of the problem; if these conditions are all fulfilled, the equation is the true equation of the problem, and is said to be verified.

VER'NIER. [Named from the inventor, Peter Vernier]. A contrivance for measuring fractional portions of one of the equal spaces into which a scale or limb is divided. The vernier consists of a graduated scale, so arranged as to cover an exact number of spaces on the primary scale, or limb, to which it is applied.

The vernier is divided into a number of equal parts, greater or less by 1, than the number of spaces which it covers on the limb. We shall take the former case as the most common, and best adapted to the illustration of the nature of the vernier. The vernier may be applied to any scale of equal parts. The modes of its application are extremely various; the principle, however, is the same in all, and may be illustrated by a simple diagram.



Let AB be any *limb* or scale of equal parts, one of which we will suppose equal to b. Let CD be a vernier equal in length to nine of these parts, and itself divided into ten equal spaces; each one of these will then be equal to nine-tenths of b. The difference between the length of a space on the limb, and on the vernier, is therefore equal to *one-tenth* of b, or $\frac{b}{10}$. This is the least space that can be measured by means of the vernier, and is called the *least count*. Hence, *the least count of a vernier is equal to one of the equal divisions of the limb divided by the number of spaces on the vernier.*

The true reading of the instrument for any position of the vernier, expresses the distance from the point where the graduation on the limb begins, marked 0, to the 0 point of the vernier. In the figure, that distance is expressed by 9 units of the scale, or 9. If, now, the vernier be moved along the scale till the division marked 1 coincides with the division marked 10 on the scale, or limb, the 0 point will have advanced along the limb a distance equal to $\frac{b}{10}$, and the reading will become $9 + \frac{b}{10}$. If we again move the vernier till the division 2 coincides with the division 11 of the scale, the 0 point will have advanced an additional distance of $\frac{b}{10}$, and the reading becomes $9 + \frac{2b}{10}$. When the division 3 coincides with 12, the reading becomes $9 + \frac{3b}{10}$, and so on, till finally, when the point 10 coincides with 19 of the scale, the distance will have been increased by $\frac{10}{10}b$, and will be 10, as it should be, since, in that case, the 0 point will have been moved a whole space, and should coincide with the division 10 of the limb. Hence, the follow-

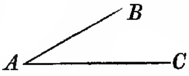
ing rule for reading an instrument which has a vernier:

Read the limb in the direction of the graduation up to the division-line next preceding the 0 point of the vernier; this is called the reading on the limb. Look along the vernier till a division-line is found, that coincides with one on the limb; multiply the number of the line by the least count of the vernier; this is the reading on the vernier: the sum of these two readings is the reading of the instrument.

VERS*E*D SINE. The distance from the foot of the sine of an arc to the origin of the arc. See *Trigonometry*.

VERT'EX. [L. *verto*, to turn,—primarily, around a point.

VERTEX OF A PLANE ANGLE. The point at which the sides of the angle meet. Thus, A is the vertex of the angle BAC.



VERTEX OF A POLYHEDRAL ANGLE. The common vertex of the plane angles which form its faces.

VERTEX OF A SPHERICAL ANGLE. The point at which the sides meet. See *Angle*.

VERTEX OF A CONE. Its apex; or, regarding a conic surface as generated by a straight line moving so as to pass through a fixed point, and constantly to touch a given curve, the fixed point is the vertex.

VERTEX OF A CURVE. The point in which the axis of the curve intersects it. The principal vertex of the conic sections is, in the *parabola*, the vertex of the axis of the curve; in the *ellipse*, the left-hand vertex of the transverse axis; and in the *hyperbola*, the right-hand vertex of the transverse axis.

VERTEX OF A DIAMETER OF A CURVE. The point in which the diameter intersects the curve.

VERTEX OF A PYRAMID. The common vertex of the lateral faces of the pyramid. See *Pyramid*.

VERTEX OF A SOLID OF REVOLUTION, or SURFACE OF REVOLUTION. The point in which the axis pierces the surface.

VERT'I-CAL. [L. *verto*, to turn]. Being in a position perpendicular to the plane of the horizon.

VERTICAL ANGLE. An angle, the plane of whose sides is vertical. If one side of the angle is horizontal, and the inclined side lies above it, the angle is called an angle of elevation. If the inclined side is below the horizontal line, the angle is called an angle of depression. See *Angle*.

VERTICAL LINE. In Surveying, the direction assumed by a plumb-line, with a weight attached to one extremity, when it is freely suspended from the other extremity.

The direction of a vertical is normal to the surface of a free fluid; as, for instance, the surface of still water. The vertical, being normal to the surface of the earth, passes very nearly through its centre, though not exactly, on account of the oblateness of the surface. Vertical lines, at different points on the earth's surface, are not parallel, but converge towards the centre.

VERTICAL LIMB of an instrument. A graduated arc used for measuring a vertical angle; that is, an angle. the plane of whose sides is vertical. See *Theodolite*.

VERTICAL PLANE. Any plane passed through a vertical line. One of the planes of projection is generally supposed to be vertical, the other one being horizontal; whence the names of the two planes of projection, *vertical* and *horizontal*.

VERTICAL PROJECTION of a *point*, *line*, or *surface*, in Descriptive Geometry. The projection of the point, line, or surface, upon the vertical plane of projection. See *Projection*, and *Descriptive Geometry*.

VINC'U-LUM. [L. *vinculum*, a bond of union]. In Algebra, a horizontal bar written over several terms, to show that they are to be considered together; thus,

$$\overline{a^2 + 2ab + c} \times \overline{a^2 - 4c}.$$

VIS'U-AL [L. *visus*, the sight]. Pertaining to sight.

VISUAL CONE. In Perspective, a cone whose vertex is at the point of sight. See *Perspective*.

VISUAL PLANE. Any plane passing through the point of sight. See *Perspective*.

VISUAL RAY. Any straight line passing through the point of sight. See *Perspective*.

VOL'UME. [L. *volumen*, a roll; bulk]. Dimensions: space occupied.

Volume of a Body. The number of cubic units which it contains. It is the same as the solidity. See *Solidity*.

VUL′GAR. [L. *vulgaris*]. Common.

Vulgar Fractions. Fractions, in which the denominators do not conform to the scale of tens, in contradistinction to *decimal fractions*, in which the denominator is conformable to that scale. Thus, $\frac{1}{8}$ is a vulgar fraction.

W, the twenty-third letter of the English alphabet. As an abbreviation in Surveying, W stands for *west*.

WARP′ED SURFACE. A surface which may be generated by a straight line moving so that no two of its consecutive positions shall be in the same plane.

Warped surfaces are divided inlo two clases: those having a plane directer, and those which have none.

Every surface of the first class *may* be generated by a straight line moving in such a manner, as constantly to touch two lines, and continue parallel to a fixed plane. The moving line is called the *generatrix;* the lines touched, the *directrices;* and the fixed plane, the *plane directer*.

Every surface of the second class *may* be generated by a straight line moving in such a manner, as constantly to touch three lines. The moving line is the *generatrix*, and the lines touched are the *directrices*. Warped surfaces may be generated in a great variety of other ways; but their generations may always be reduced to one of these methods.

In the first class of surfaces, if one of the directrices is a straight line, the surface is called a *conoid*. If this straight line is perpendicular to the plane directer, it is called a *right conoid*, and the rectilinear directrix is called the *line of striction*, because the elements are nearer together when measured on this line than at any of their other points.

If both directrices are straight lines, the surface is called an *hyperbolic paraboloid*, which is the most important surface of the class. The leading properties of the surface are: 1st, the elements divide the directrices proportionally; 2d, the surface has two generations. If any two elements of the surface, as described, be taken as directrices, and a plane directer be taken parallel to the directrices of the first generation, the same surface will be generated; this is called the surface of the second generation. 3d, through any point in the surface two straight lines can always be drawn, which shall lie wholly in the surface; these are the elements of the first and second generation. 4th, the plane of these two elements is tangent to the surface at their point of intersection and every plane parallel to such plane, cuts out an *hyperbola* whose asymptotes are parallel to these elements. 5th, all other planes cut from the surface are *parabolas*.

In the second class of surfaces, if all three of the directrices are straight lines, the surfaces are called *hyperboloids of one nappe*. If these lines are symmetrically arranged with respect to a fourth line, the hyperboloid becomes the hyperboloid of revolution of one nappe, the fourth line being the axis.

Hyperboloids of one nappe have two generations. If any three elements of the first generation be taken as directrices, and a straight line be moved so as constantly to touch them, the same surface will be generated, and this is called the surface of the second generation. Through any point of the surface two straight lines can always be drawn, which will lie wholly in the surface; these are elements of the first and second generation. The plane of these two elements is tangent to the surface at their point of intersection, and every plane parallel to this plane cuts from the surface an *hyperbola* whose asymptotes are parallel to these elements. All other planes cut *ellipses* from the surface.

To construct an element of any surface of the first class: Pass a plane parallel to the plane directer, and find the points in which it cuts the directrices; through these points draw a straight line; it will be an element of the surface.

To construct an element of any surface of the second class, take any point of one of the directrices, as a vertex of a cone, and a second directrix as a base; find the point in which this conic surface intersects the third directrix, and join this with the vertex by a straight line; this line will be an element of the surface.

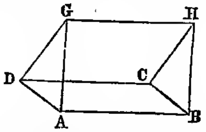
In either class of surfaces, if a plane passed through one element intersects the surface in a line cutting this element, the plane is a

tangent plane, and the point of intersection is the point of contact.

If two surfaces of the first class have a common plane directer, a common element, and two common tangent planes, the points of contact being on the common element, the two surfaces are tangent to each other all along the common element.

If two surfaces of the second class have a common element and three common tangent planes, the points of contact being on the common element, the two surfaces are tangent to each other all along the common element.

WEDGE. In Geometry, a solid bounded by five plane figures. The parallelogram, (usually a rectangle) ABGH is called the *back*; the two trapezoids DCGH and ABCD are called *faces*, and the two triangles ADG



and BCH are called ends of the wedge. The faces of the wedge meet in a line CD, parallel to the back, which is called the edge of the wedge. The distance from the edge to the back is the altitude of the wedge. If we denote the length of the back by L, the length of the edge by l, the breadth of the back by b, the altitude of the wedge by h, and its volume by V, we have

$$V = \frac{1}{6} bh (2L + l).$$

WEIGHT of a body, the resultant of the forces exerted by gravity upon all the different particles of the body. Under the same volume, different bodies have different weights. This is attributed to the fact that their heavy particles are closer together or more remote from each other, or as we may express it, are more or less dense.

The standard unit of weight, in this country, is the *pound*, and in order to acquire a correct notion of the pound, it is necessary to have a clear idea of the relation existing between the different units in the system of weights and measures. A pretty full description of this relation is given under the head of measures.

We subjoin some of the most important systems, showing the relation between the pound and other units of weight.

1. *Avoirdupois Weight.*

By this weight are weighed all coarse articles, as hay, grain, all the metals, except gold and silver, &c.

GROSS WEIGHT is the weight of the goods including the boxes, bags, &c., in which they are contained.

NET WEIGHT is what remains after deducting from the gross weight the weight allowed for the boxes, bags, casks, &c.

A hundred weight was 112 pounds; but it is now reckoned at 100 pounds.

16 *dr.* = 1 *oz.*; 16 *oz.* = 1 *lb.*; 25 *lb* = 1 *qr.*; 4 *qr.* = 1 *cwt.*; 20 *cwt.* = 1 *T.*

2. *Troy Weight.*

By this weight are weighed gold, silver, jewels, and some liquids.

24 *gr.* = 1 *pwt.*; 20 *pwt.* = 1 *oz.* 12 *oz.* = 1 *lb.*

3. *Apothecaries' Weight.*

This weight is used by druggists and apothecaries in weighing medicines. The pound and ounce are the same as in Troy weight; they differ only in the manner of subdivision.

20 *gr.* = 1 ℈; 3 ℈ = 1 ʒ;
8 ʒ = 1 ℥; 12 ℥ = 1 ℔.

Foreign Weights.

The French system of weights is one of the most perfect, as well as the most simple of all systems that have thus far been adopted. The unit of the system is the weight of a cubic decimetre of distilled water, and is called a kilogramme. The kilogramme weighs 2.204737 pounds Avoirdupois. The divisions are made decimally.

Table of Equivalents of the old French system:

1 livre	= 16 onces =	1.0780 lbs. Avoirdupois
1 once	= 8 gros =	1.0780 ozs. "
1 gros	= 72 grains =	58.9548 grs. Troy.
1 grain =		0.8188 " "

Comparison of the weights of different countries:

The standard avoirdupois pound of the United States, as determined by Mr. Hassler, is the weight of 27.7015 cubic

inches of distilled water, weighed in air at the temperature of maximum density (39°.83), the barometer being at 30 inches.

The imperial Avoirdupois pound of Great Britain is the weight of 27.7274 cubic inches of distilled water, weighed in air with brass weights, at the temperature of 62° Fahr., the barometer being 30 inches. Therefore

1 cubic inch of distilled water at 62° weighs 252.458 grains, or 0.003961 cubic inch weighs 1 grain; 22.815689 cubic inches weigh 1 Troy pound.

The pound of	Spain	weighs	1.0152	lbs. Av's.
" "	Sweden	"	0 9376	"
" "	Austria	"	1.2351	"
" "	Prussia	"	1.0333	"

The pound being determined according to the British standard.

Weight of Observations. If several observations, giving different results, are made, for the purpose of determining any required element, it may happen that some of them may be considered more reliable than others, and for this reason are said to have greater *weight*. We may then define the weights of observations to be, *numbers proportional to their relative goodness*. Let us suppose that n observations have been made for the purpose of determining a required element, giving the results A, A', A'', &c. Denote the weights of these respectively by c, c', c'', &c. If we employ the symbol Σ for the algebraic sum of homologous quantities, so that

$$\Sigma(cA) = Ac + A'c' + A''c'' + \&c. \quad \text{and}$$
$$\Sigma(c) = c + c' + c'' + \&c.,$$

it may be shown that the expression $\frac{\Sigma(cA)}{\Sigma c}$ is more likely to be the true value of the element sought than the expression $\frac{\Sigma A}{n}$, or a simple arithmetical mean of all the results.

Instead of c, c', c'', &c., any numbers proportional to them may be used, and in applying the results of the theory of probabilities, it has been found that a certain method of obtaining c, c', c'', &c., not only conforms to the above method of forming an average, but also renders them applicable to other important uses. We shall subjoin a sketch of the results of this method.

1. When a number of discordant observations are made, in which positive and negative errors are equally likely to occur, and which do not differ much from each other, and when it is exceedingly unlikely that the truth can differ much from the observations, it may be presumed that the chances of the error of any one of these observations, lying between x and $x + dx$, and between a and b, may be expressed by the terms

$$\sqrt{\frac{c}{\pi}}\, e^{-cx^2} dx \quad \text{and} \quad \sqrt{\frac{c}{\pi}} \int_a^b e^{-cx^2} dx,$$

in which c depends upon the relative goodness of the observation $\pi = 3.14159$, and $e = 2.71828$.

Even if this law of error does not exist, it is found that the treatment of a *considerable number of observations*, according to any reasonable law, is reducible to the same rules as derived from this law, which is now universally assumed by those observers who apply the theory of probabilities to their results.

2. The constant c is called the *weight* of the observation, and depends upon the various circumstances which determine the observation to have been good or bad. The greater it is the better is the class of observations to which it applies. It is approximately found for a given set of observations as follows: subtract each of the results from a mean of them all, and let e, e', e'', e''', &c., denote the remainders; then

$$c = \frac{n}{2\Sigma(e^2)}.$$

The sum of the squares of the departure from the average may be found by diminishing the sum of the squares of the results of observation by n times the square of the mean, and before doing this any convenient quantity may be deducted from each of the results of observation, provided the same be deducted from their mean.

3. The *probable error* is that within which, taken positively and negatively, there is an even chance that the error of an observation shall lie. Thus, if A is the true result and there is an even chance that the result of an observation shall be between it and $A + a$ or $A - a$, then is a the probable error of an observation. The probable error may be found by dividing .476936 by the square root of the weight.

4. The weight of the average of observation is the sum of the weights of the individual observations. If n observations are made, giving the results A', A'', A''', &c., all having the same weight c, the weight of the average is nc, and its probable error is

$$\frac{.476936}{\sqrt{nc}}.$$

But if the weights of the individual observations be different, as c', c'', c''', &c., then, $\frac{\Sigma\,(cA)}{\Sigma\,(c)}$ is the average, $\Sigma\,(c)$ its weight, and

$$\frac{.476936}{\sqrt{\Sigma\,(c)}}$$

its probable error. In the former case the probable error of the average may be found by formula

$$p = \frac{.67449\sqrt{\Sigma(e)^2}}{n},$$

in which p denotes the probable error, and e', e'', e''', &c., the departures from the average; the average being taken for the truth, their departures taken for the errors.

5. Generally, other things being equal, the probable error of an average will not be inversely as the number of observations, but as the square root of that number. If p denotes the probable error of an observation, and P that of the average of n such observations, then

$$p = \sqrt{n}\,.\,P.$$

An observer who takes such a method as gives the probable error of an observation twice as great as it need be, must not hope to indemnify himself for his carelessness by making twice as many observations as would otherwise be necessary, but must take *four times* as many

6. If p denote the probable error of an observation, an average or other result, the following table will be sufficient to connect the probable error with other errors for rough purposes of estimation :

Odds.	Against	For.	Odds.	Against	For.
$1\frac{1}{2}$	.79	1.25	$7\frac{1}{2}$	.27	2.32
2	.64	1.43	8	.21	2.36
$2\frac{1}{2}$	.54	1.58	$8\frac{1}{2}$	.20	2.40
3	.47	1.71	9	.19	2.44
$3\frac{1}{2}$	.42	1.81	$9\frac{1}{2}$	.18	2.47
4	.38	1.90	10	.17	2.50
$4\frac{1}{2}$	.34	1.98	20	.09	2.94
5	.31	2.05	30	.06	3.17
$5\frac{1}{2}$	.29	2.11	40	.05	3.34
6	.27	2.17	50	.04	3.50
$6\frac{1}{2}$	.25	2.22	100	.02	3.90
7	.23	2.27	1000	.002	4.90

The table is used as follows : Let p denote the probable error above mentioned ; it is $1\frac{1}{2}$ to 1, or 3 to 2, *against the error being less* than .79 p, and it is $1\frac{1}{2}$ to 1 *that the error is less* than 1.25 p.

It is 8 to 1 *against the error being less than* .21 p, and 8 to 1 *for its being less than* 2.36 p. It is 1000 to 1 *against its being less than* .002 p, and 1000 to 1 *for its being less than* 4.90 p.

WESTING. In Surveying, the departure of a course, when the course lies to the west of north. See *Departure*.

WIDTH. Breadth. See *Volume*, *Magnitude*.

WYES. The supports of the telescope in the theodolite and level. They are named from their shape, which is like the letter Y. There is a loop turning on a hinge at the top of the extremity of one of the upper branches of the Y, and which may be fastened to the top of the other branch by a pin passing through a hole in the branch and the loop. See *description of the Level and the Theodolite*.

X. The twenty-fourth letter of the English alphabet. As a Roman numeral character, it stands for 10 ; with a dash over it, for 10,000, thus, $\overline{\text{X}} = 10{,}000$.

Y. The twenty-fifth letter of the English alphabet. As a numeral, it has been used to denote 150 ; with a dash over it, 150,000, thus, $\overline{\text{Y}} = 150{,}000$.

YĂRD. A unit of measure, equal to three feet.

YĒAR. A unit of time, marked by the revolution of the earth in its orbit.

The year is either *astronomical* or *civil.*

The astronomical year is determined by astronomical observation, and is of different lengths, according to the point of the heavens to which the revolution is referred.

When the earth's motion is referred to a fixed point in the heavens, as a fixed star, the time of revolution is the time which elapses from the moment when the star, the sun, and the earth, are in a straight line, till they again occupy the same position : this is called a *sidereal* year. If the revolution is referred to one of the equinoctial points, the year is somewhat shorter than the sidereal year, on account of the precession of the equinoxes

that is, the retregression of the equinoctial points aleng the ecliptic. This is called the equinoctial, tropical, or solar year. The length of the sidereal year is 365.2563612 mean solar days, or $365^{d.}\ 6^{h.}\ 9^{m.}\ 9^{s.}6$.

The length of the solar or equinoctial year is 365.2422414 mean solar days, or $365^{d.}\ 5^{h.}\ 48^{m.}\ 49^{s.}7$.

The difference between these two years is $19^{m.}\ 19^{s}.9$ mean solar time, that being the time required for the earth to advance in its orbit a distance of $50''.1$ of arc.

The civil year is the year of the calendar. It contains a whole number of days, beginning always at midnight of some day. According to the present system, or, according to the Gregerian calendar, every year the number of which is not divisible by 4, also every year which is divisible by 100, and net by 400, are common years, and contain 365 days. All other years are called leap years, and contain 366.

Z. the twenty-sixth letter of the English alphabet.

ZĒ'NITH. The point of a plane in which a vertical, at the place produced, pierces the heavens. The opposite point of the heavens is called the Nadir of the place.

ZĒ'RO, in common language, means no *thing*; in Arithmetic, it is called naught, and means *no number*; in Algebra, it stands for *no quantity*, or *for a quantity less than any assignable quantity*.

If we take the fraction $\frac{a}{b}$, and suppose b to remain constant whilst a continually diminishes, the value of the fraction becomes smaller and smaller, and finally when a becomes less than any assignable quantity, the value of the fraction becomes less than any assignable quantity, or 0. In the same fraction, if we suppose a to remain constant, whilst b continually increases, the value of the fraction continually diminishes; when b becomes very great in comparison with a, the value of the fraction becomes very small; finally, when b becomes greater than any assignable quantity, or ∞, the value of the fraction becomes less than any assignable quantity, or 0; hence,

$$\frac{a}{\infty} = 0;$$

this kind of 0 differs analytically from the absolute 0, obtained by subtracting a from a; $a - a = 0$. It is in consequence of confounding the 0, arising from dividing a by ∞, with the absolute 0, that so much confusion has been created in the discussions that have grown out of this subject. About the absolute 0 there can be no discussion: all absolute 0's are equal. But the other 0's are nothing else than infinitely small quantities, or infinitesimals, and there is no incompatibility in supposing that they differ from each other, and that the ratio of two such zeros may be a finite quantity.

Let us consider the two fractions, $\frac{a}{b}$ and $\frac{2a}{b}$ and suppose that a remains constant, whilst b increases without limit: now, it is plain, that for every value of b, the second fraction is twice as great as the first; finally, if b passes to its superior limit in both fractions, by becoming ∞, the two fractions will still have the same ratio, 2, to each other. But, in this case, we conventionally call both 0.

Logical accuracy would seem to require that some other name should be given to the result, in this case; but if the two meanings of the term 0 are fully understood, no trouble need arise in retaining that nomenclature, which has been sanctioned by the custom of centuries.

ZŌNE. The portion of the surface of a sphere included between two parallel planes The area of a zone is equal to the circumference of a great circle of the sphere, multiplied by the altitude of the zone; that is, the distance between the parallel planes which form its bases.

ZONE OF ANY SURFACE OF REVOLUTION, is that portion of the surface which is included between two planes perpendicular to the axis.